DISSECTION

IRRATIONAL NUMBERS FROM REGULAR POLYGONS, SURDS WITHOUT PYTHAGORAS

Practising the ancient art of dissection with the two set squares in a modern school geometry set

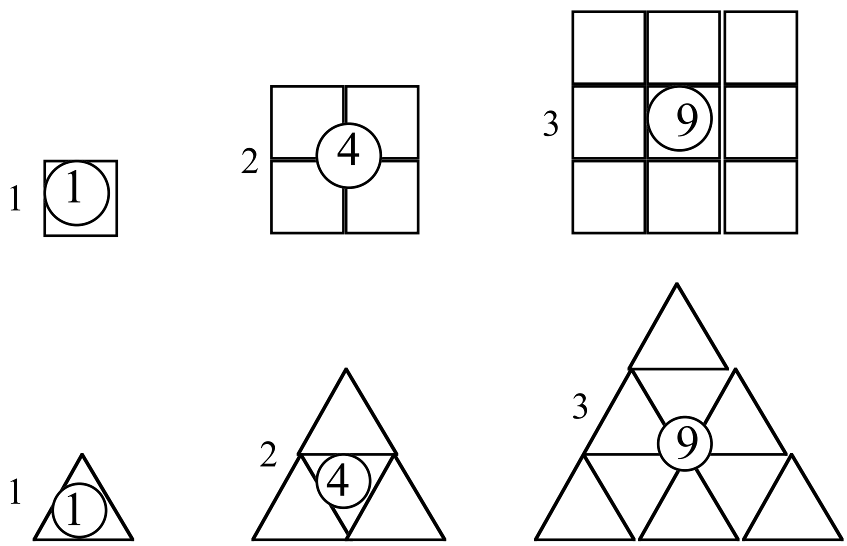
For the Ancients it was easy to find a number whose square was 4, but not a number whose square was 2. They did - and we shall – represent a number by a length, and the product of two numbers (in particular the product of a number and itself, the square of a number) by an area. What can we find by cutting up that area and rearranging the pieces?

You have a large magnetic whiteboard, and the pairs of children corresponding boards A2 size. All have dry-wipe pens and there is a stock of the magnetic shapes described.

materials needed	*activity (Teacher demonstration/Pupil experiment)*
Magnetic equilateral triangles, squares, as required	**E1** *Teacher demonstration*

Introduce the topic in the following terms:

"Here are two families of *similar* shapes, that is to say, figures whose shape is the same though their size may be different. We measure some feature of the shape – here the length of a side – and find we must square this number to find how many times as big the area is. Thus if a single square or triangle has area *A,* one scaled up by 2 has area $2^2 A = 4A$; one scaled up by 3, $3^2 A = 9A$. We see this simply by putting identical copies together:



We shall make a logical jump and claim that, for *any* two-dimensional shapes, not just equilateral triangles and squares,

'*Area' factor = 'Length' factor squared*.

We shall call this *the Principle of Similarity*.

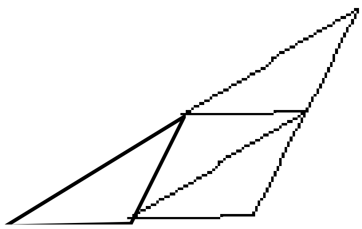
We can say the square and the equilateral triangle are *rep-4* , meaning that the smallest number of copies we can fit together to make a similar shape is 4."

Scalene triangles, as required.

E2 *Pupil experiment*

"These are no special kind of triangle. What's the smallest number you need to combine to make a similar one?"

They should find that the general triangle is also rep-4:



Right-angled isosceles triangles, as required

E3 *Pupil experiment*

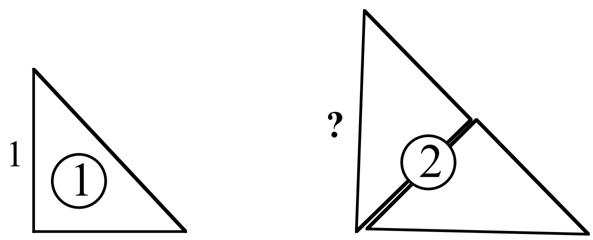
"Cut a square in two by slicing along a diagonal and you have one of the two set squares you find in an ordinary school geometry set: the right-angled isosceles triangle.

What's the smallest number of these you need to combine to make a similar one?"

The children now know that they can do no worse than 4. They should find the answer is 2.

Above

E4 *Teacher demonstration*



"So, the area factor is 2, but what is the scale factor?"

Discussion should clarify the following inverse relations.

2 squared is 4. The number you square to make 4, the *square root* of 4, is 2. '?' squared is 2. The number you square to make 2, the square root of 2 is '?' …, i.e. we don't know what so we leave it as '$\sqrt{2}$'.

" Is $\sqrt{2}$ perhaps $\frac{7}{5}$?

$\left(\frac{7}{5}\right)^2 = \frac{49}{25} = 1\frac{24}{25}$. No.

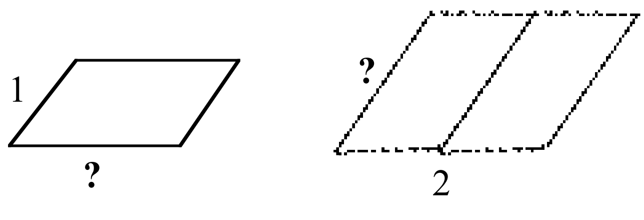
"To cut a long story short – in fact the thousand years of mathematical history from around 1,500 B.C. to around 500 B.C., there is no such fraction. We can call $\frac{49}{25}$ a fraction or write it 49:25 and call it a *ratio*. Either way, it is a *ratio*nal number. $\sqrt{2}$ is not. It is an *ir*rational number. We can never find two natural numbers whose ratio is $\sqrt{2}$. The proof that this is so is one of the most famous achievements of the ancient Greeks. The proof depends on showing that, if you insist on $\sqrt{2}$ being a ratio, it leads to a nonsense. This method of proof is called *proof by contradiction*.

"There's nothing to stop us finding $\sqrt{2}$ *approximately*, for example by measuring the length of the hypotenuse of our triangle and comparing it with the length of a shorter side. But we'll leave it as $\sqrt{2}$. A number written in this way is called a *surd* and it's called that because people did indeed consider such numbers to be *ab*surd.

Dry-wipe pen

E5 *Teacher demonstration*

"So, we know one rep-2 triangle. Is there a rep-2 *quadrilateral*? We need a parallelogram with sides in the right ratio. Here it is:



"What is the ratio?"

Ensure this relation emerges from the discussion:
$\frac{?}{1} = \frac{2}{?}$, leading to: $(?)^2 = 2$, i.e. $? = \sqrt{2}$.

"Look around this room and tell me where you can see a parallelogram with sides in the ratio $\sqrt{2}:1$?"

[The tease assumes the children have come with A4 pads.]

Sheet of plain A4, x 2, x 15

E6 *Pupil experiment*

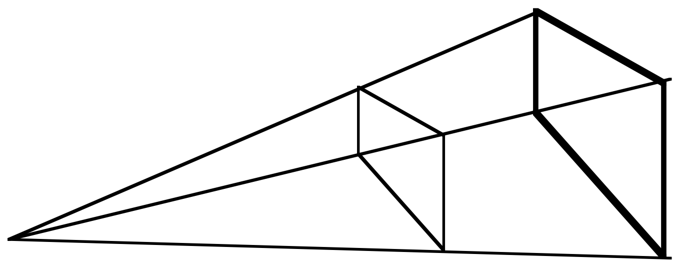
Instruct the children as follows.

"Take one of your two sheets. Fold it in two widthways and tear it in two down the fold.
" Take one of the two halves and turn it through a right-angle.
"Your partner now holds up a complete sheet in 'landscape' orientation.
"Hold your own sheet in the same orientation, parallel to his/hers so that his/hers sticks out round the side of yours to form a border.
"Bring your sheet slowly towards his/hers.
"What happens?"

[Every part of the border disappears at once.]

"Could someone draw a diagram on the board of what happens?"

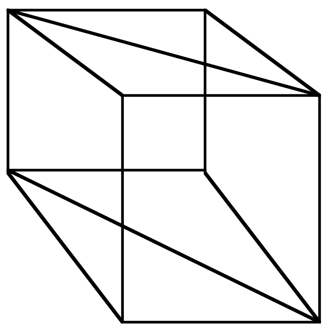
Expect a figure suggesting the operation of enlargement in some way: Lines traced through corresponding points on the two rectangles meet in a point.



Model as text

E7 *Teacher demonstration*

Point out to the children that they can obtain this special rectangle by sectioning a cube like this:



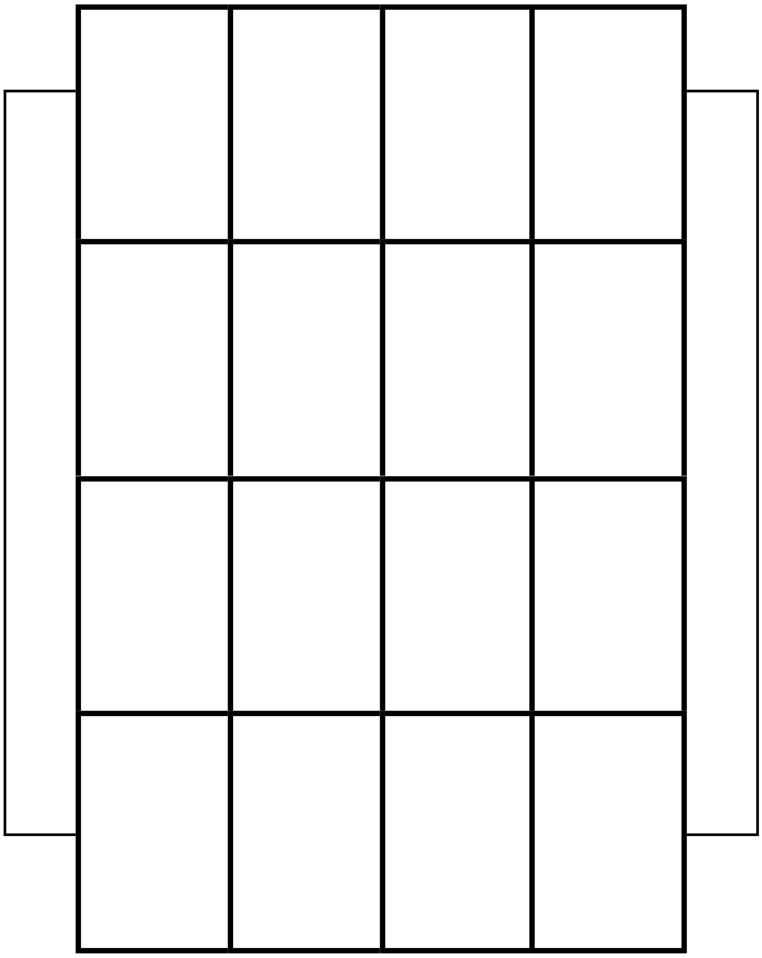
Plain A4 sheets, as required, metre rule, 'fridge' magnets,

E8 *Teacher demonstration (Collaborative experiment)*

Draw a metre square on the board.
Ask a pair of volunteers to arrange paper sheets over the top

as required, calculator

to make a rectangle of the same area.



Ask a volunteer to run his/her finger around:

a sheet of A3,
a sheet of A2,
a sheet of A1,
a sheet of A0.

Ask how many square mm there are in 1 square m [1,000,000].
Remind the children that on the front of their pads it says that one sheet of A4 is 297 mm x 210 mm. Hand a volunteer a calculator and ask him/her to work out 16 x 297 x 210. Ask the class why the answer is not 1,000,000 [Dimensions approximate, only given to nearest mm]. Ask them to find the fraction $\frac{297}{210}$ in lowest terms [$\frac{99}{70}$]. Point out that $\frac{99}{70}$ is closer to $\sqrt{2}$ than $\frac{7}{5}$.

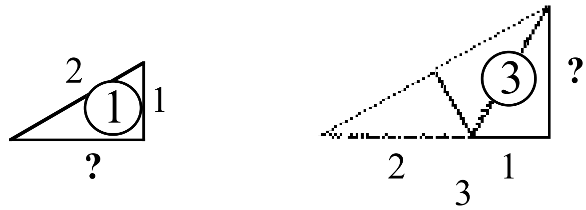
1-2-$\sqrt{3}$ triangles, as required

E9 *Pupil experiment*

"We've dealt with one of the two set squares in a geometry set: what we may call the 1-1-$\sqrt{2}$ triangle [Hold up specimen, indicating corresponding sides]. Slice an equilateral triangle down a symmetry axis and we have the other [Hold up specimen].

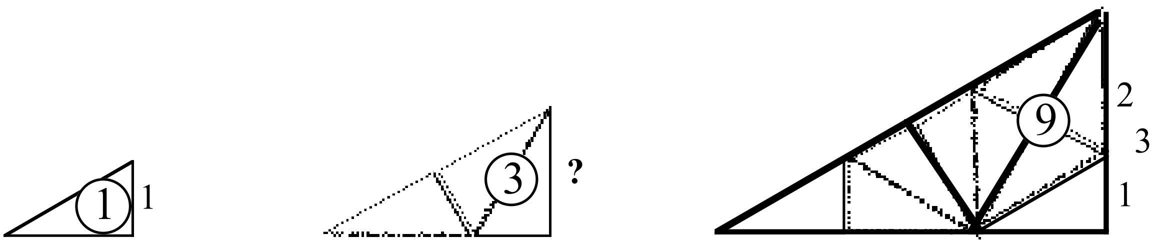
"We know that all triangles are rep-4 or less. What's the smallest number of *these* special triangles you need to combine to make a similar one?"

Expect the children to have some difficulty finding the following arrangement:



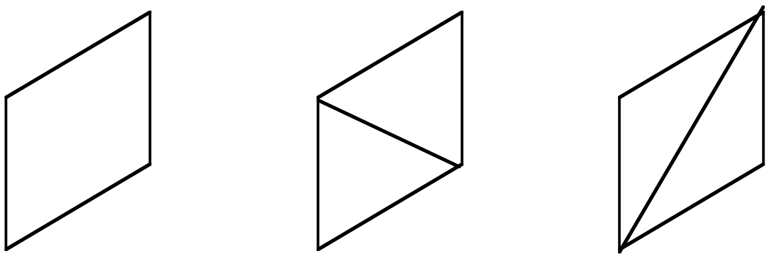
"We now know better than to try to find a number for '?': we just write '$\sqrt{3}$'.

"So, what we may call the 1-2-$\sqrt{3}$ triangle is rep-3. Take 9 of them and try to make a triangle-of-triangles."



Point out that, by scaling up twice, we've made our new shortest side $\sqrt{3} \times \sqrt{3} = 3$ units long.

Draw attention to the rhombus made from 4 1-2-$\sqrt{3}$ triangles.



"Split by a short diagonal, we have our original equilateral triangles.

Split by a long diagonal, we have two triangles of the same area but a different shape. How do you describe it? [An isosceles triangle with apex angle $(2\times 60)° = 120°$ / a triangle with sides in the ratio $1{:}1{:}\sqrt{3}$] We shall call this shape, which we shall need for the next experiment, the $1-1-\sqrt{3}$ triangle."

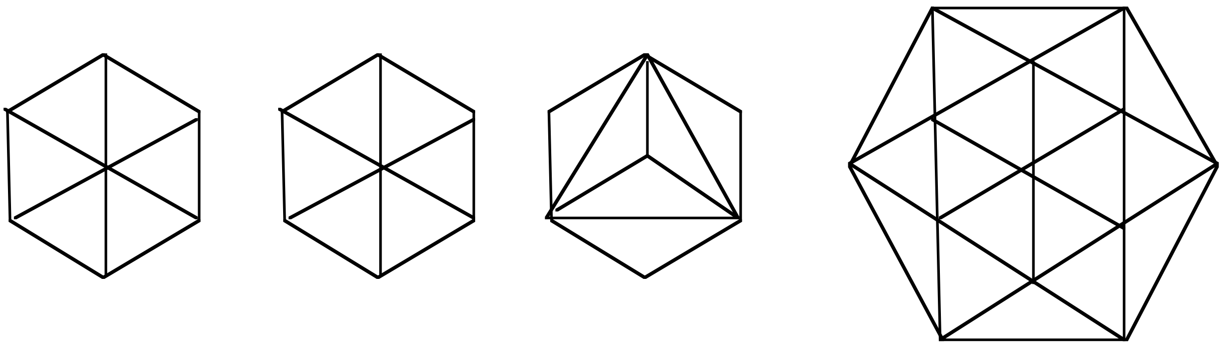
Equilateral triangles, 1-1-$\sqrt{3}$ triangles, as required

E10 *Pupil experiment*

"Take 12 equilateral triangles and 6 $1-1-\sqrt{3}$ triangles and make 3 *congruent* regular hexagons (*congruent* = identical in shape and size).

"Now use the same pieces to make a single regular hexagon."

The children should soon manage this arrangement:

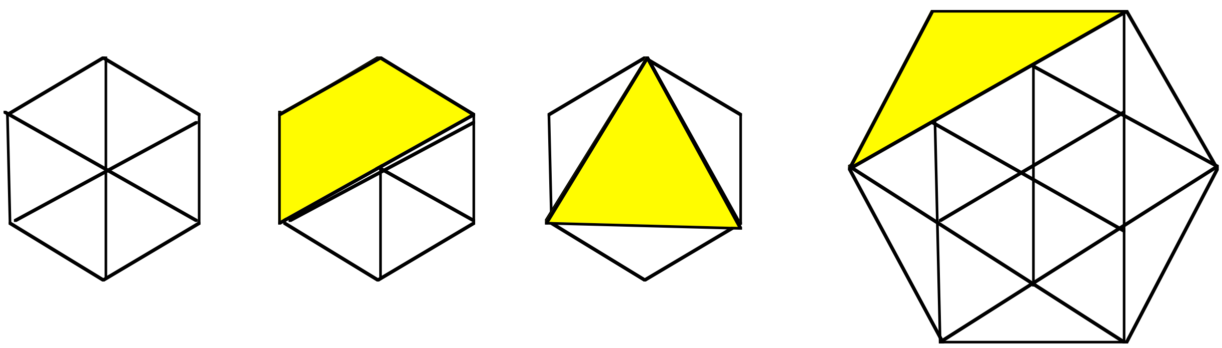


"Because we know about our new triangle, we know that the ratio between the edge of the big hexagon and the edge of a small hexagon is $\sqrt{3}:1$. Can someone tell me how we would know that to be true even if we knew nothing about the new triangle?" [By the principle of Similarity area goes up x 3, therefore any linear dimension goes up by $\sqrt{3}$]

Magnetic shapes as text

E11 *Teacher demonstration*

"What can you say about the 3 yellow areas?" [All the same]



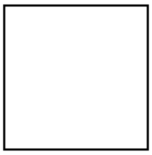
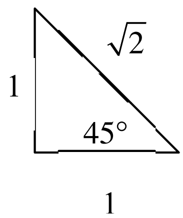
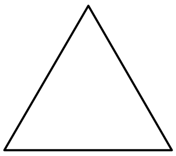
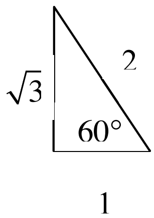
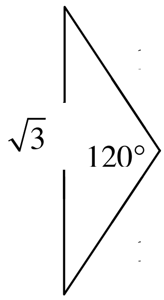
"We can be sure of this even though we've measured no lengths and no angles. All we've done is what the Indians and

Chinese did as long ago as 1,000 B.C.: cut shapes up and recombine them. We call this process *dissection*."

A supply of all 5 shapes shown in table following, projected as acetate

E12 *Pupil experiment*

"This table shows the shapes we've studied so far. You may use any or all of them to solve the next set of problems.

Shape	**S** **'Square'** Square (regular 4-gon)	**HS** **'Half-square'** 90° isosceles triangle	**T** **'Triangle'** Equilateral triangle (regular 3-gon)	**HT** **'Half-triangle'** 60° right-angled triangle	**DHT** **'Double-half-triangle'** 120° isosceles triangle
Angles, Side ratios					
Area	s	$\frac{s}{2}$	t	$\frac{t}{2}$	t

"s stands for the area of our unit square; t stands for the area of our unit equilateral triangle. We can express all the other areas we need in terms of these two letters.

"Your challenge is to build 4 regular 12-gons [Hold up specimen]:
a small one,
one with 2 x its area,
one with 3 x its area,
one with 4 x its area.

This chart uses the 5 symbols heading the above table to show some of the solutions the children may find. It need not be displayed to the children but they should confirm that the area of their particular solution is a multiple of $6(s+2t)$.

‘Area’ factor		1	2	3	4
Area in terms of s and t units		$6s+12t$ $=6(s+2t)$	$12s+24t$ $=12(s+2t)$	$18s+36t$ $=18(s+2t)$	$24s+48t$ $=24(s+2t)$
Growth sequences showing what pieces are added at each stage	A	6**S**, 12**T**	+ 12**HS**, 12**T**	+ 12**HS**, 12**DHT**	
	B	6**S**, 12**T**		+ 12**S**, 12**T**, 12**DHT**	
	C	6**S**, 12**T**			+18**S**, 36**T**

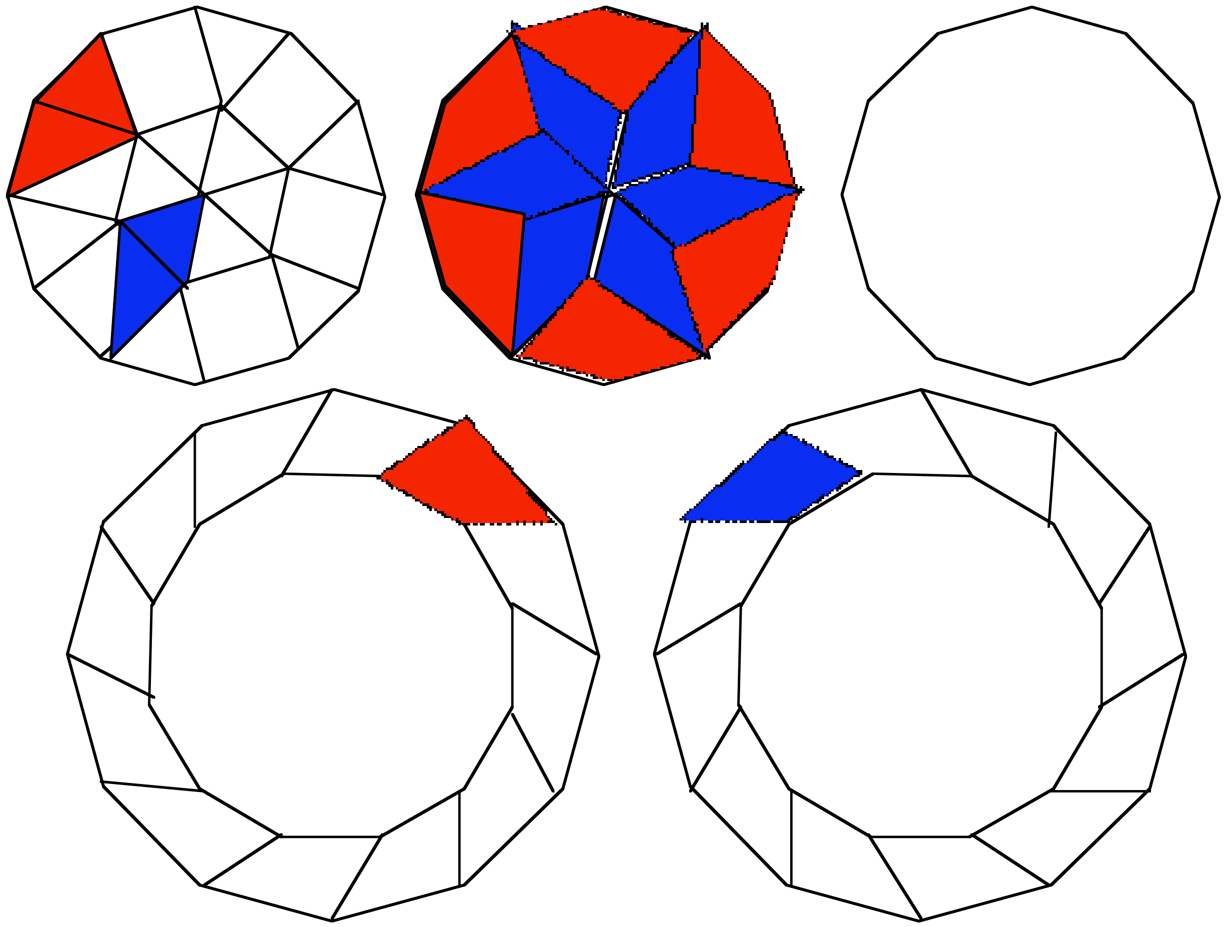
Magnetic shapes, as text

E13 *Teacher demonstration*

“So far you’ve studied the *science* of dissection. This was important in the early history of mathematics. Since then, dissection has become a mathematical *art*. Anyone can excel at this art: yourself as an able young mathematician, a professor of mathematics 5 times your age, anyone.

“ The art is to change one shape into another by cutting it into as few pieces as possible. For example, in **E10** we would not have split up the small hexagon we used in the centre of the big one. And look how we can save on pieces by joining an **HS** to a **T** in **E12.** This enables us to dissect 2 ‘size 1’ 12-gons into 1 ‘size 2’ 12-gon in just 13 pieces. Note that a red piece is just a blue piece turned over and that the outer ring can be made in either a ‘left-handed’ or a ‘right-handed’ orientation.

Invite a volunteer to complete what you have started.

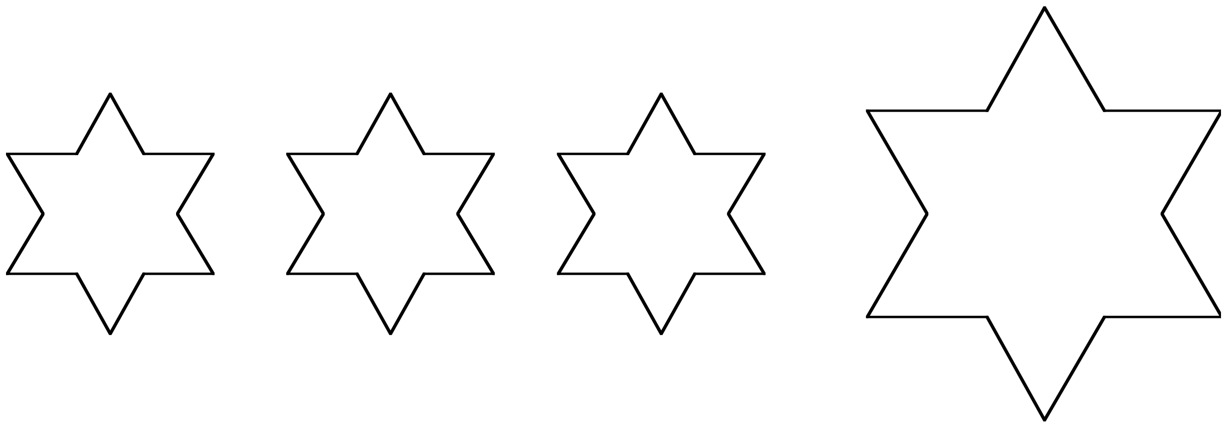


Shapes as **E12**, sellotape as needed, figure as acetate

E14 *Pupil experiment*

"Your challenge is this.

"In **E10** you changed 3 small hexagons into 1 big one. This time you're to change 3 hexa*grams* into 1 big one:

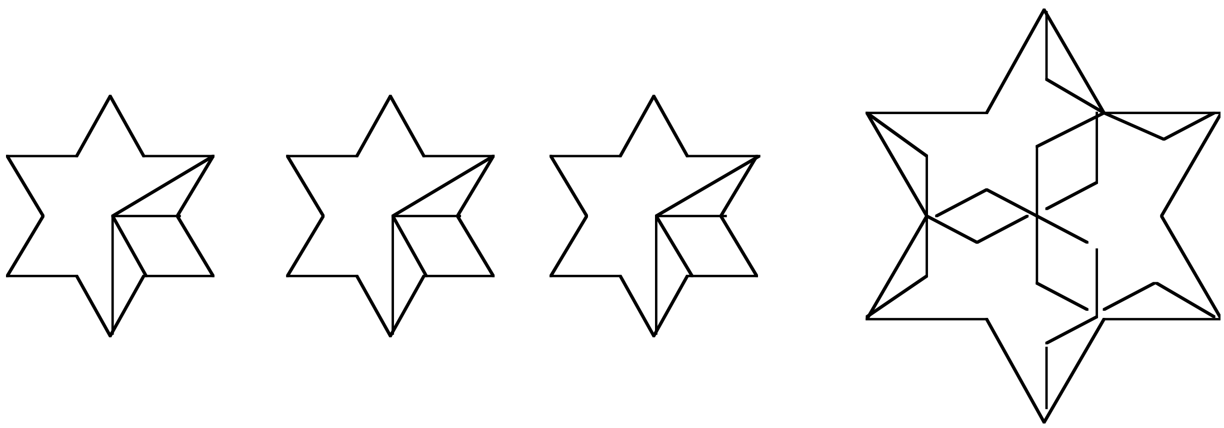


"As you can see, a hexagram is a 6-pointed star. In what other ways can you think of it? To put it another way, how would you draw it yourself?"

[Suggestions may include:
A regular hexagon with an equilateral triangle drawn outwards on each side;
two equilateral triangles, one given a half-turn/a sixth-turn with respect to the other, superposed]

"Of the 5 shapes on our chart, you only need two – I won't say which. And a lot can be stuck together, as we did in **E12**. In fact the dissection can be done in 12 pieces."

[Here, for your reference, is the optimal solution:]



P.S. 8.3.12