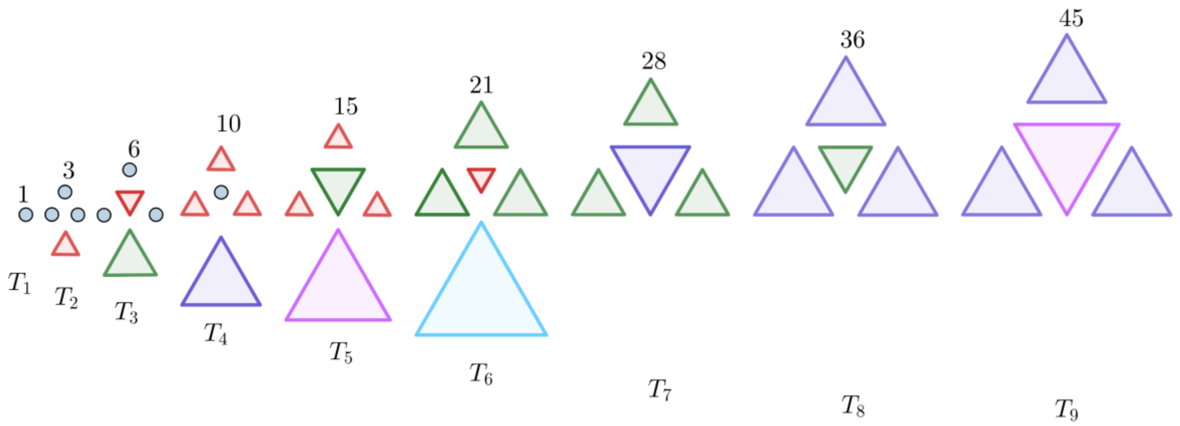
What does it say?

In what follows, T_k is the k^{th} triangle number, $\frac{k(k+1)}{2}$, S_k is the k^{th} square number, k^2, CS_k is the centred square, $S_{k-1} + S_k$, k is a natural number. Note that $S_{2k} = 4S_k$.

When we show a number type as an array of dots, as the Ancients did, we are stating a visual definition certainly but our hope is that the symmetry of the figure will reveal a property. For example, when we draw the squares in sequence, we see that they grow by the addition of an odd number. The equilateral triangle is immediately informative. The symmetry reveals three smaller triangles of one size and one of another nested inside the original. When we draw the triangles in sequence, the rule becomes clear: For a triangle with an odd suffix the fourth triangle is a size bigger than the other three; for a triangle with an even suffix it's a size smaller.



Having stated the rule through pictures, then in words, we can be more concise and express it with symbols:

$$\begin{aligned} T_{n-1} + 3T_n &= T_{2n} \\ 3T_n + T_{n+1} &= T_{2n+1}. \end{aligned}$$

We can confirm these identities by direct substitution using our defining equation for T_k. But the sequence of triangles also reveals a cycle of length 4 beginning on a triangle with odd suffix and involving triangles of three consecutive sizes:

$$\begin{aligned} 3T_1 + T_2 &= T_3 \\ T_1 + 3T_2 &= T_4 \\ 3T_2 + T_3 &= T_5 \\ T_2 + 3T_3 &= T_6 \end{aligned}$$

We add: $4T_1 + 8T_2 + 4T_3$

take out a factor: $= 4(T_1 + 2T_2 + T_3)$

regroup: $= 4[(T_1 + T_2) + (T_2 + T_3)]$

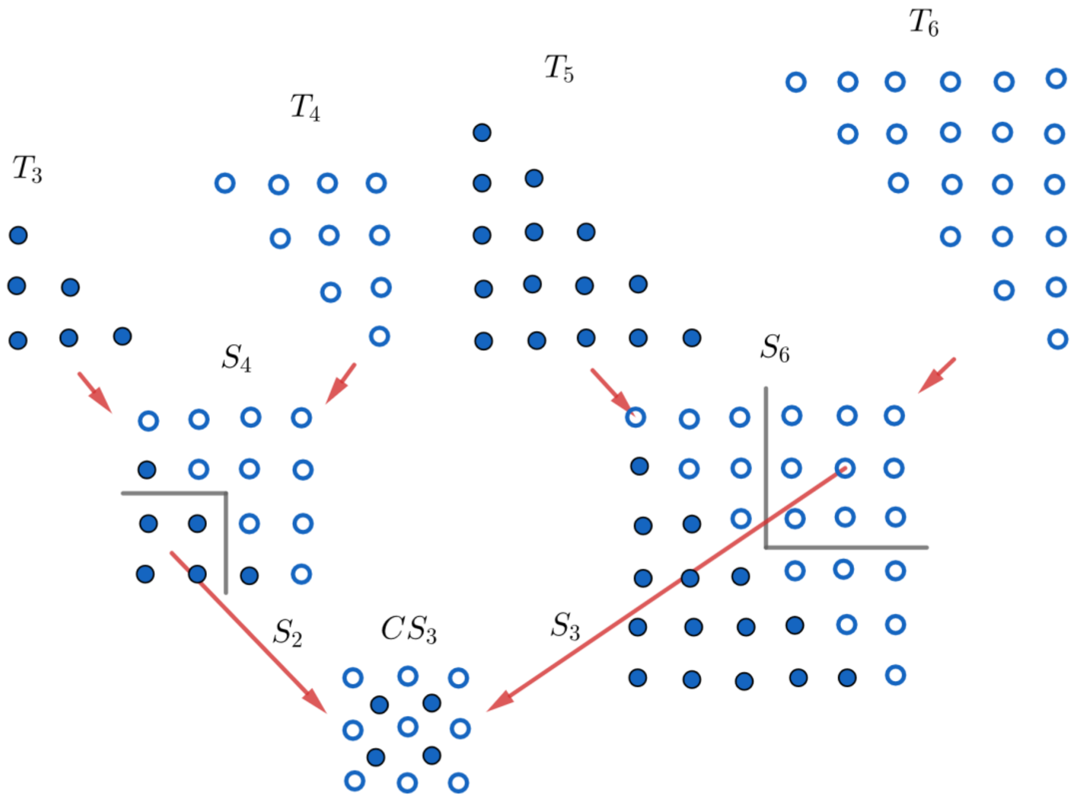
use $T_{k-1} + T_k = S_k$: $= 4(S_2 + S_3)$

use $S_{k-1} + S_k = CS_k$: $= 4CS_3$

$= T_3 + T_4 + T_5 + T_6.$

If the dots are arranged, not in an equilateral triangle but in an isosceles right triangle, what we lose in symmetry we gain in compatibility with composite numbers most easily visualised using a square grid.

In the following sequence of figures we reverse the above derivation..



We'll use algebra again and generalise our result:

$$T_{2n+1} + T_{2n+2} + T_{2n+3} + T_{2n+4} = 4CS_{n+2} \, .$$

Would we have discovered the last identity had we started with a square grid and not a triangular one? The only thing we can say is, translating between geometric, verbal and algebraic statements increases the chance that we will discover a result of that (admittedly not very important!) kind. Without this possibility, we are like Egyptologists trying to read hieroglyphs without the aid of the Rosetta stone.