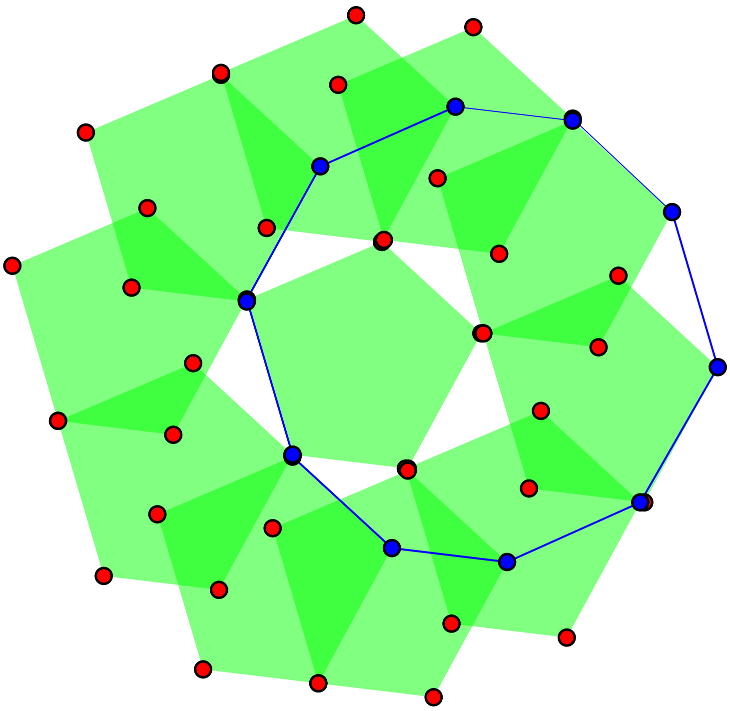
TRANSLATION GLISSETTES



"A mathematician who can only generalise is like a monkey who can only climb up a tree, and a mathematician who can only specialize is like a monkey who can only climb down a tree. In fact neither the up monkey nor the down monkey is a viable creature. A real monkey must find food and escape his enemies and so must be able to incessantly climb up and down."

George Polya, *How to Solve It*

Definition

We shall define a ***translation glissette*** as the locus C_3 of a point P fixed with respect to a convex, closed curve, C_2, which makes a complete circuit of another convex, closed curve, C_1, maintaining at least one point of contact with it and keeping parallel to itself.

Present this definition to the children with whatever is to hand. For example, hold one copy of their text book against the whiteboard and slide a second part-way round it, inviting the children to watch what happens to the first letter of the title.

Specialising and Generalising

Homework is to choose their own C_1, C_2 and P and determine C_3 by whatever means:

- Using ATM mats or cutting the shapes from card with a hole for P to take a pen,
- Showing each stage in the circuit on squared or isometric paper,
- Using software. Since copy-&-paste and click-&-drag are functions of basic word processing packages, let alone dynamic geometry software, this is the obvious candidate, particularly as, in the case of a polygon, a C_2 vertex may be chosen for P.

The children should return having made some observation to share with the class. This is the stage of specialising: starting from a definition and examining a whole range of special cases. In the stage which follows, their observations may lead to 'local' generalisations, and those in turn to a 'global' statement.

In the table I try to anticipate the students by picking out 22 cases. They will have chosen specific values for the variables but we shall cut to the 'local' generalisations of C_3 which may emerge when they pool their observations. These will require proof at an appropriate level. What we require of a proof is that it be an explanation good enough to convince classmates. What constitutes a proof at KS3 will differ from that offered at KS5.

You will not be surprised to learn that the list of cases did not spring into existence so formed. The pattern common to all cases where C_2 is similar to C_1 and set parallel to it, discussed in the 'Comment' section, prompted a journey down Polya's tree again to confirm what happens when the shapes are *anti*-parallel (cases **20**, **21**). Notice that this finding is hidden in result **4** because the shapes already have half-turn symmetry. **22** was also an afterthought.

Teaching strategies

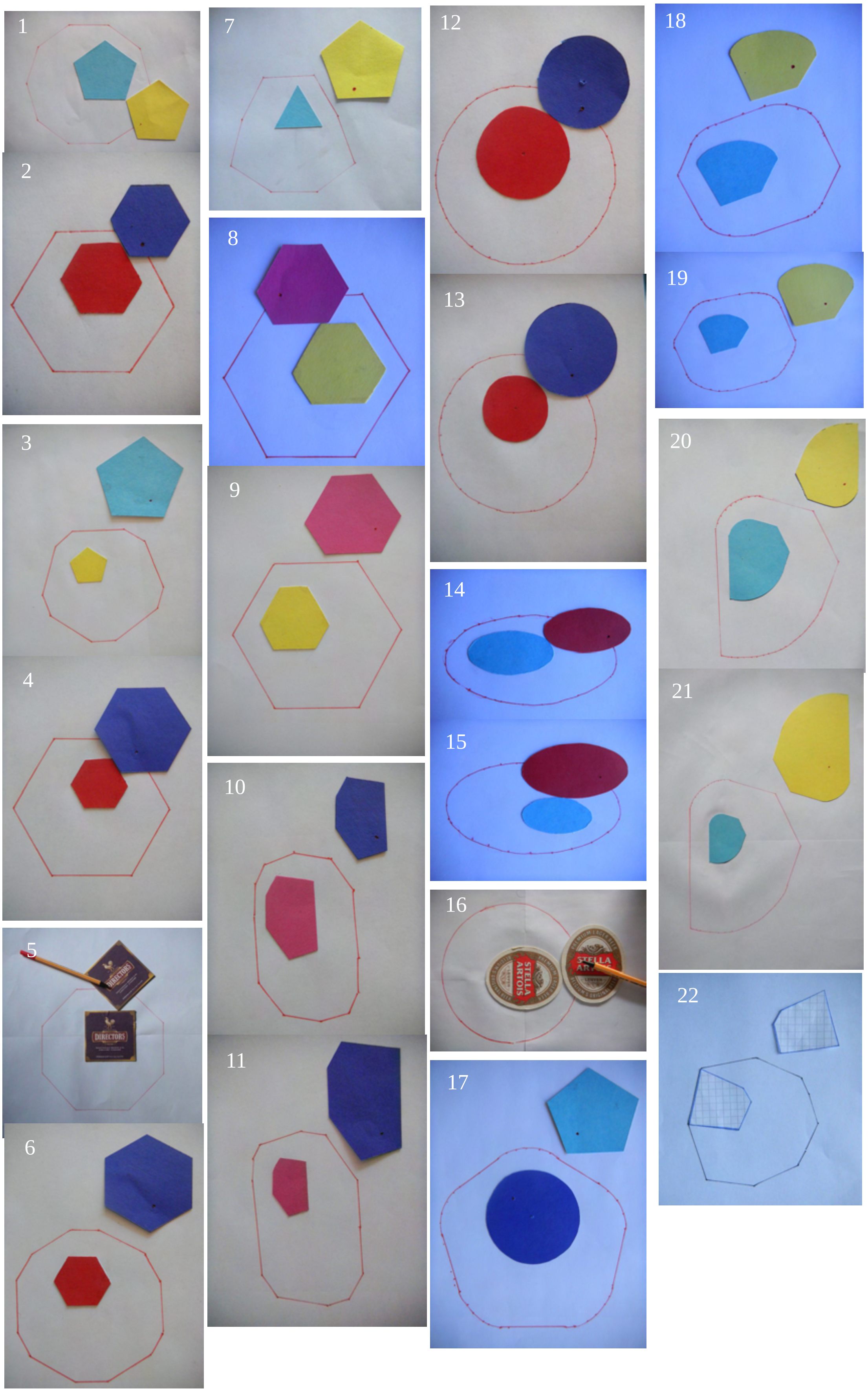
I talk about "anticipating" the students. There is the danger of using the material to head off pupils who (one feels) are going in unproductive directions. Bart Simpson's profile and Wallace's face are non-convex. Should I 'steer' or leave the artist to discover the snags?

Students who choose cases close together on my chart may collaborate with advantage. They may take on specific roles in their team: see nrich.maths.org/7011 for what Jo Boaler calls 'complex instruction'.

When you feel the investigation is reaching its natural end, lay out all the records the children have made and encourage them to design a wall display which shows how special cases are subsumed under more general headings. For example, **12** is a special case of **13** when $k = 1$, which is in turn a special case of **15** when $a = b$, which is in turn the special case of **19** where the shapes are ellipses.

If you've read so far, that's all you need do before the children return. You will only want to pick up on the cases they've dealt with. I hope you will find most of them discussed in the pages which follow, consisting of the chart, accompanying thumbnails, and notes bringing out links between the individual cases and hinting at proofs. (For a more complete analysis of case **19** request the .pdf ***Parallel translation glissettes***.)

C_1, C_2	ARRANGEMENT				n	'LOCAL' GENERALISATION	
$n-$gons	regular, side s	the same	parallel	congruent	odd	regular $2n-$gon, side s	**1**
					even	regular $n-$gon, side $2s$	**2**
				similar, C_2 scaled k	odd	$2n-$gon, angles equal, sides alternately s,ks	**3**
					even	regular $n-$gon, side $(k+1)s$	**4**
			opposed vertex-side	congruent	even	regular $2n-$gon, side s	**5**
				similar, C_2 scaled k	even	$2n-$gon, angles equal, sides alternately s,ks	**6**
		$C_1 : n_1$ sides, $C_2 : n_2$ sides	single sides parallel			one symmetry axis	**7**
	equal angles, $2n$ sides alternating s,t	the same	parallel	congruent	odd	regular $2n-$gon, side $(s+t)$	**8**
				similar, C_2 scaled k	odd	equal angles, $2n$ sides alternating $(s+kt),(ks+t)$	**9**
	irregular	the same	parallel	congruent		figure with half-turn symmetry	**10**
				similar, C_2 scaled k		See 'Comment'.	**11**
circles, radius r				equal		circle, radius $2r$	**12**
				C_2 scaled k		circle, radius $(k+1)r$	**13**
ellipses, semi-major axis a, semi-minor axis b			parallel	congruent		similar ellipse, scaled 2	**14**
				similar, C_2 scaled k		similar ellipse, scaled $(k+1)$	**15**
			at right angles	congruent		circle, radius $(a+b)$	**16**
circle, regular $n-$gon						circle, separated symmetrically into equal sectors by C_2	**17**
figure with straight and curved segments		the same	parallel	congruent		figure with half-turn symmetry	**18**
				similar, C_2 scaled k		See 'Comment'.	**19**
			anti-parallel	congruent		similar shape, scaled 2	**20**
				similar,		similar shape, scaled $(k+1)$	**21**
		mirror forms	set either side of a mirror line			one symmetry axis	**22**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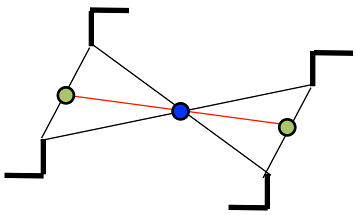
Comment

Two children who have chosen the same shapes but a different position for P should find that it makes no difference (other than to displace the curve bodily).

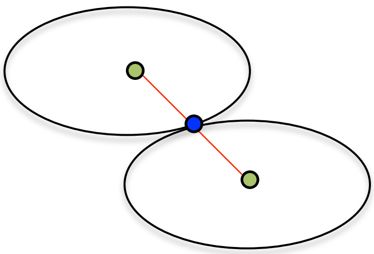
In ‘Rolling ellipses and their symmetries’ we took advantage of the fact that the roller and the rolled-upon could be swapped. The children may do the same here.

4 is a generalisation of **2**, **9** of **8**, **13** of **12**, **15** of **14**, **21** of **20** when $k = 1$.

If we translate a shape with half-turn symmetry anywhere in the plane, the composite figure of original + image itself has half-turn symmetry:



This explains an observation about **14**. There we have such a figure. The centre of the ellipse pair is their point of contact, which therefore lies on the line joining their individual centres:

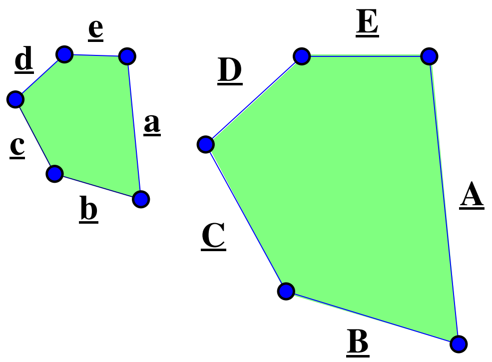


5 = **1** and **6** = **3**: when two polygons with an odd number of sides are set parallel, a vertex of one automatically comes opposite a side of the other.

In 3-D analogues, every point of a closed surface C_1 meets some point of a closed surface C_2. In cases **2** and **4** we can swap cubes for squares. Corresponding to a rectangle, sides s, t (or a parallelogram with those sides and a particular angle) we have a cuboid, sides s, t, u (or the corresponding parallelepiped). In cases **12**, **13** we can swap spheres for circles; in cases **14**, **15**, ellipsoids with semi-major axes a, b, c for ellipses with semi-major axes a, b. *Taking cases* ***1*** *to* ***6****, are there other 3-D analogues easy to visualise?*

17 has an analogue in 3 dimensions. Moving a sphere round a polyhedron gives it rounded edges and corners. For a Platonic solid with v vertices, the edge profile is this fraction of a circle: 2π / the supplement of the dihedral angle, and the sectors surrounding a vertex bound a 3-D sector of $4\pi/v$ steradians.

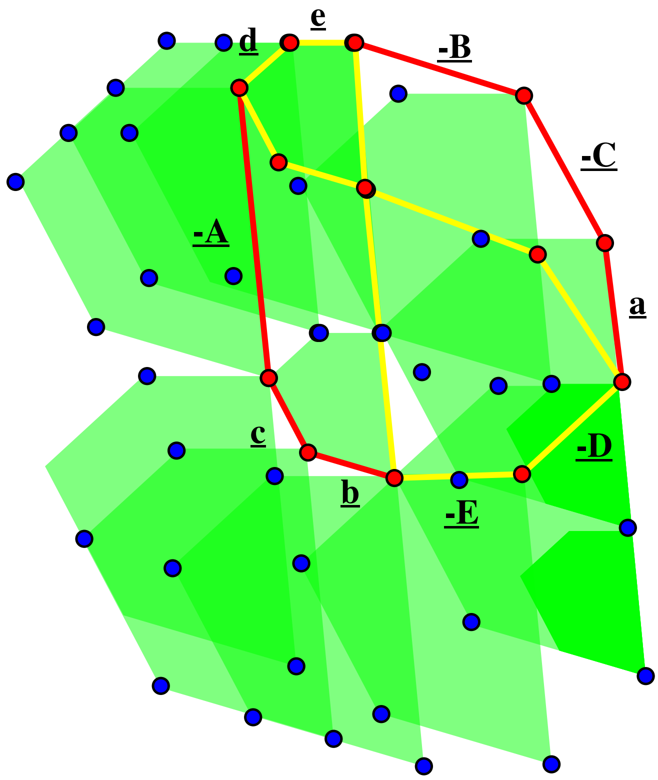
19 exhibits all the properties of cases where the shapes are similar and set parallel (**1** to **4**, **8** to **15**, **18** to **19**) but suggest to the children that they take case **11**. Leave them to label how they like the features they consider important. Here again we jump ahead to reveal how C_1 and C_2 fit within C_3. We use a vertex for P:



The vectors as labelled are to be read clockwise.

$\underline{\mathbf{A}} = k\underline{\mathbf{a}}$, etc.

Use the fact that
$\underline{\mathbf{A}} + \underline{\mathbf{B}} + \underline{\mathbf{C}} + \underline{\mathbf{D}} + \underline{\mathbf{E}} = \underline{\mathbf{a}} + \underline{\mathbf{b}} + \underline{\mathbf{c}} + \underline{\mathbf{d}} + \underline{\mathbf{e}} = \mathbf{0}$
to check that all circuits, internal and external, have a zero sum.

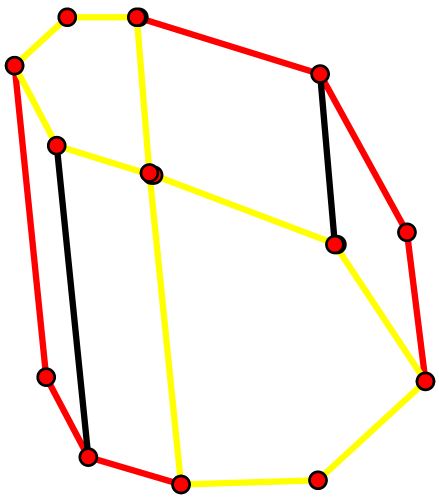


There are several things to observe in the completed circuit, C_3:

- Each side of C_1 and C_2 is represented.
- Their contributions occur in alphabetical (cyclic) order.
- They occur in complementary positions, thus:

$$\begin{array}{cccccccc} -\underline{\mathbf{A}} & * & * & -\underline{\mathbf{B}} & -\underline{\mathbf{C}} & * & -\underline{\mathbf{D}} & -\underline{\mathbf{E}} \\ \underline{\mathbf{a}} & * & * & \underline{\mathbf{b}} & \underline{\mathbf{c}} & * & \underline{\mathbf{d}} & \underline{\mathbf{e}} \end{array}$$

- In keeping with this pattern we find, within C_3, C_1 and a copy of C_2 given a half-turn, arranged as shown in yellow. Start from one vertex and finish at the opposite one, taking 4 routes:
  1. round the right-hand side,
  2. round the right-hand side of the yellow figure-of-eight,
  3. round the left-hand side of the yellow figure-of-eight,
  4. round the left-hand side.
  You should of course find the vector sums are the same. But notice how you can turn the sequcnce of terms in **1** & **2** into that for **3** & **4** by swapping upper case letters for lower case.
  Notice also how you can make C_3 by translating C_1 or C_2 along the sides of the other.
- As a further consequence, the area of the region to the right of the yellow part equals the area to the left. To see why this is so, dissect the regions into parallelograms, express their areas as vector products and pair them off:



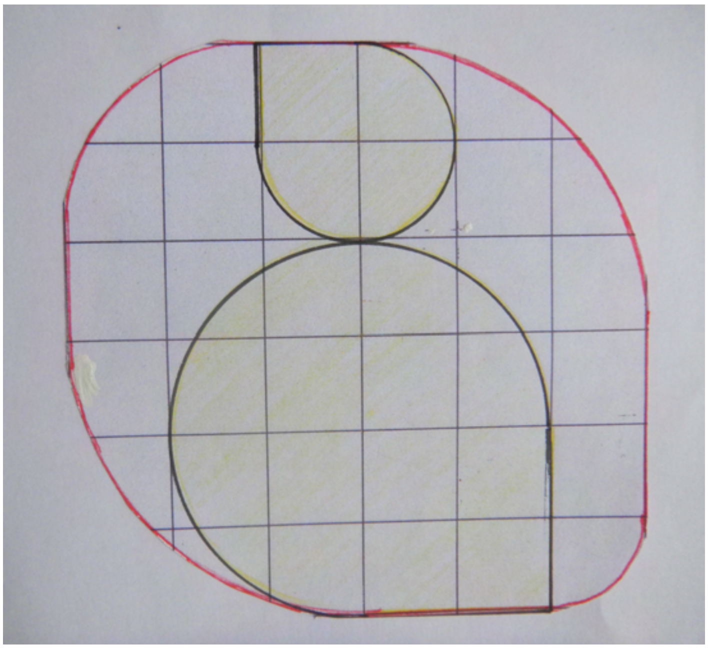
Left: **Right**:

$\underline{\mathbf{c}} \times \underline{\mathbf{-A}} = k(\underline{\mathbf{c}} \times \underline{\mathbf{-a}})$ $\underline{\mathbf{-C}} \times \underline{\mathbf{a}} = k(\underline{\mathbf{-c}} \times \underline{\mathbf{A}}) = k(\underline{\mathbf{c}} \times \underline{\mathbf{-a}})$

$\underline{\mathbf{b}} \times \underline{\mathbf{-A}} = k(\underline{\mathbf{b}} \times \underline{\mathbf{-a}})$ $\underline{\mathbf{-B}} \times \underline{\mathbf{a}} = k(\underline{\mathbf{-b}} \times \underline{\mathbf{a}}) = k(\underline{\mathbf{b}} \times \underline{\mathbf{-a}})$

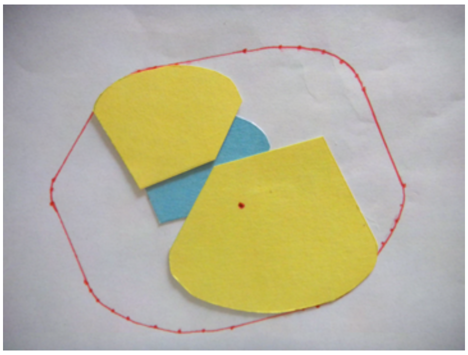
Though the language is appropriate to KS5, the argument can be appreciated by KS3 children. It depends only on the fact that, if you take a parallelogram and, keeping the angles the same, swap a short side for a long side, you don't change the area.

In moving to case **19** you're inviting the children to think of a shape with curves as a polygon with an infinite number of sides and concede that the same argument would apply. Psychologically that's not very satisfactory. They can at least check a representative case. In the example on the right C_1, C_2 have a symmetry axis and (therefore) C_3 also has a parallel one, but the symmetry is nicely broken once we fit C_1 and C_2 inside C_3. (Would C_3 have a symmetry axis if C_1 and C_2 were *not* similar?)

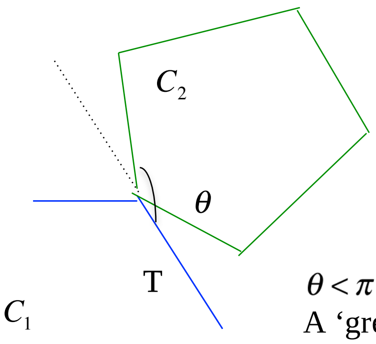


The children can work out the areas to left and right by adding and subtracting unit squares, and halves and quarters of circles radius 1, 2 and 3.

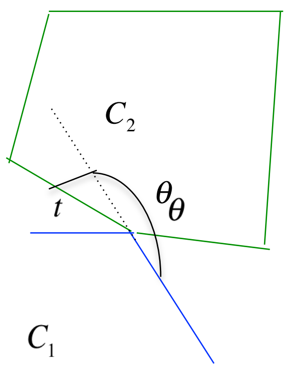
19 is more complicated than case **11** because the curved sections are split up, (symmetrically in case **17)**, but we can always fit C_1 and C_2 into C_3 in the same way:



The angle θ tells us whether C_1 or C_2 will contribute the next side, and therefore whether the next letter in the sequence will be upper case or lower case:



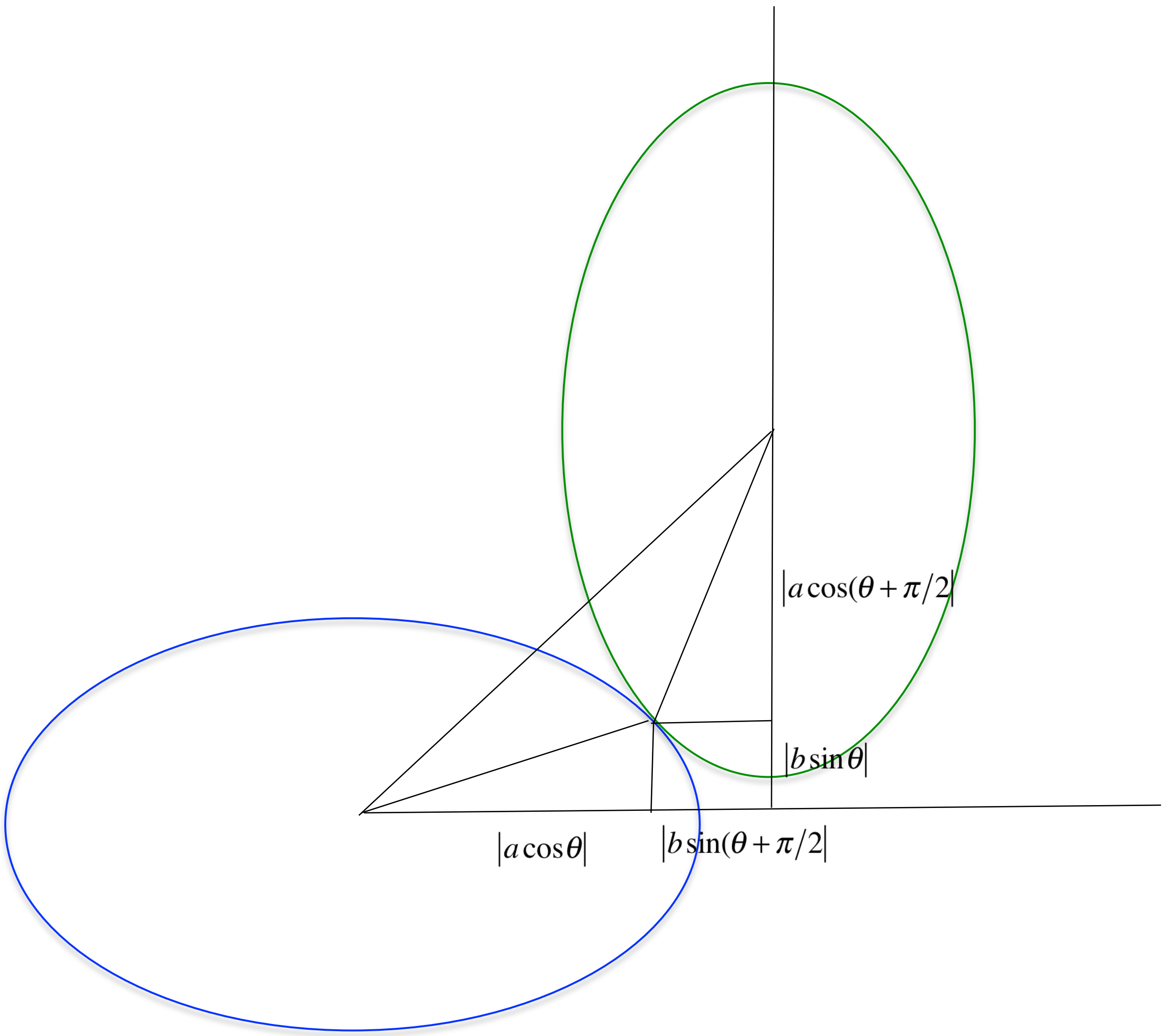
$\theta < \pi$.
A 'green' vertex traces the side T .



$\theta > \pi$.
A 'blue' vertex traces the side t .

In discussing case **11** we noted that upper and lower case letters occur in complementary positions. If C_1 has n sides, we find the side **s** on one side of C_3 and the side –**S** directly opposite, n sides later. Thinking in terms of the above diagram, when C_2 moves round to the other side of C_1 it presents a blue vertex to C_1's green. The vertex figure is therefore identical and so θ is the same. The next side in the C_3 circuit is **s** in the one case, -**S** in the other.

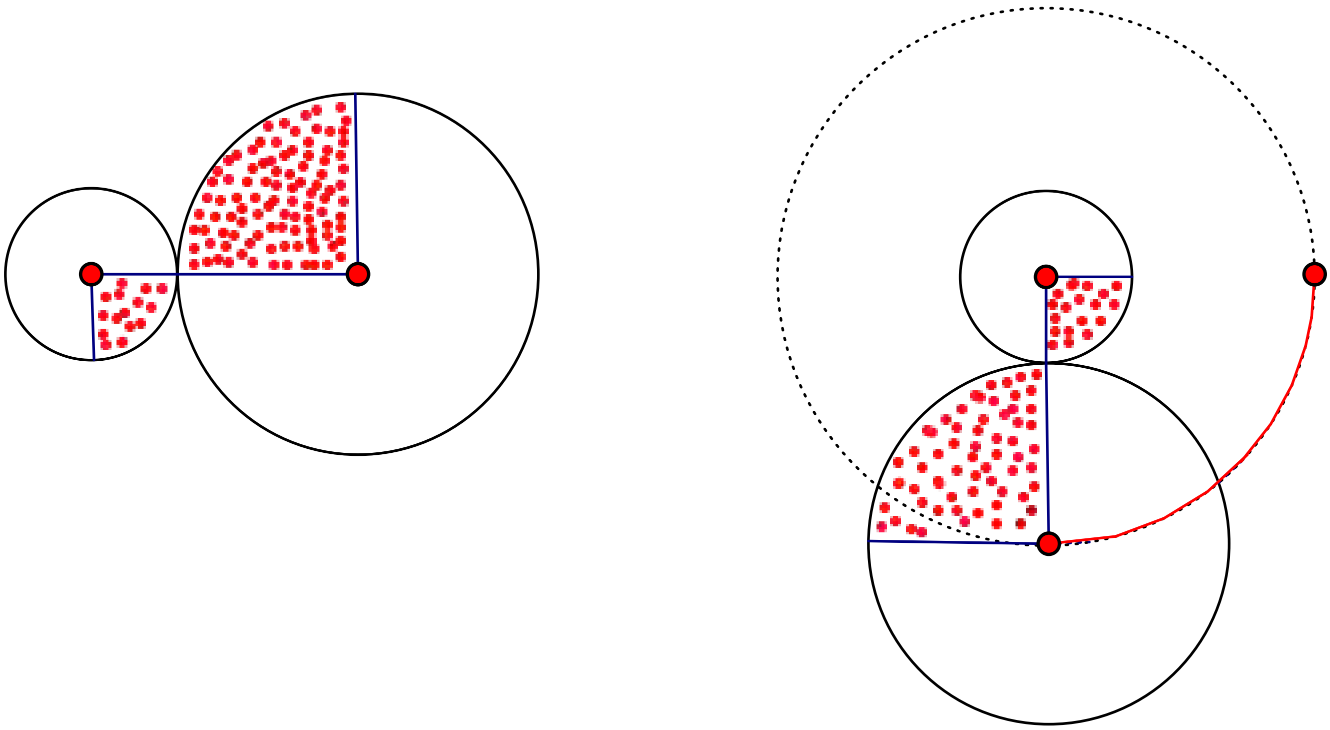
16 invites comparison with 'Rolling ellipses and their symmetries', **5b**, where we started with the same arrangement but rolled the outer ellipse round the inner. The locus of the centre was again a circle of radius $a+b$. Advanced students may derive result **16** by applying Pythagoras' theorem in this diagram:



(N.B. θ is the angle measured at a point on the auxiliary circle of each ellipse. We are only using the fact that the angles for the two ellipses remain $\pi/2$ out of phase.)
Note the algebraic symmetry. To read the vertical labels from the horizontal ones we just swap

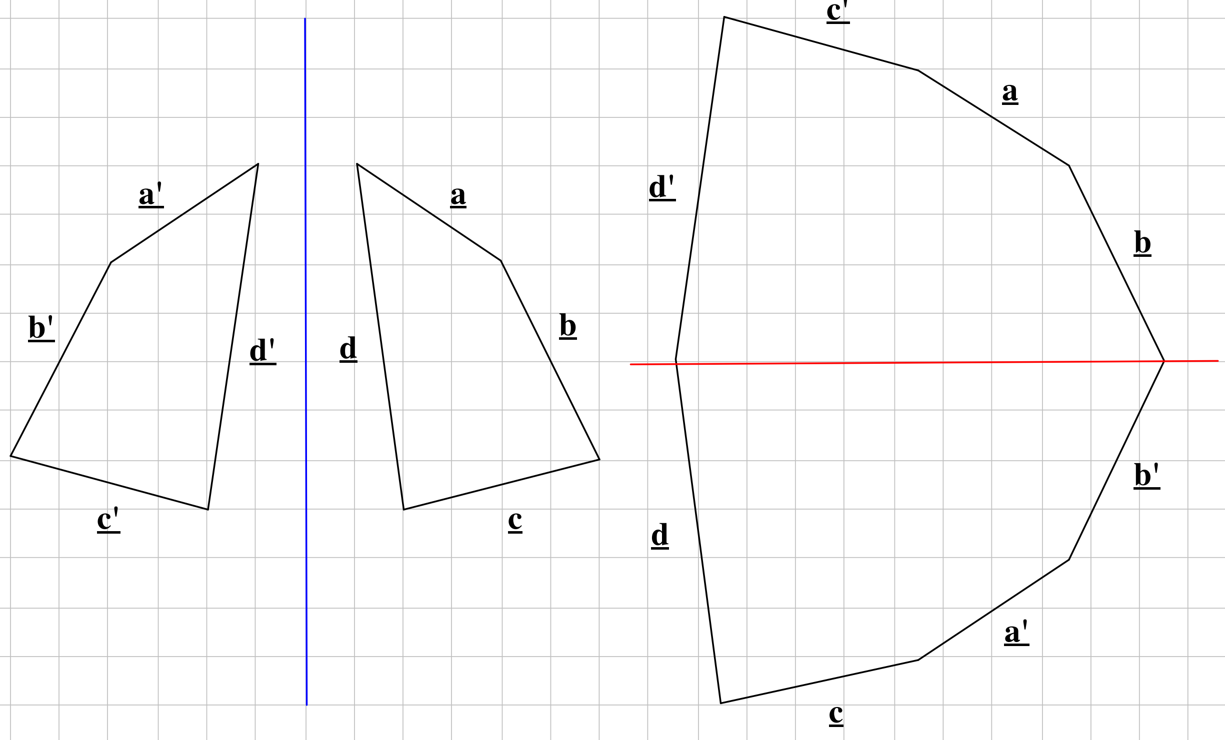
θ for $(\theta + \pi/2)$. There is an analogy in the plane with two congruent rectangles, sides s and t, set at right angles, where the locus is a square of side $(s+t)$.

Something about **18** to **21** which **12** and **13** prepare us for: when two curves coincide, the radii sum:



20 and **21** yield enlargements because, by giving C_2 a half-turn, we bring corresponding sides into contact.

In **22** C_1 is an asymmetric shape. C_2 is its reflection in a chosen line. This fixes its orientation. In the figure below we have labelled edges to indicate mirror pairs. (The sense of the vectors doesn't matter long as it's consistent within C_1 and C_2.)



a'
b'
d'
c'
d
a
b
c
c'
a
d'
b
b'
d
a'
c

In our discussion of **11** we noted that in C_3 the edges of C_1 and C_2 occur in complementary positions. The same is true here:

$$\begin{array}{cccccccc} \underline{\mathbf{a}} & \underline{\mathbf{b}} & * & * & \underline{\mathbf{c}} & \underline{\mathbf{d}} & * & * \\ * & * & \underline{\mathbf{b'}} & \underline{\mathbf{a'}} & * & * & \underline{\mathbf{d'}} & \underline{\mathbf{c'}} \end{array}$$

Corresponding edges have an equal and opposite inclination to a line normal to the mirror line, which is therefore a symmetry axis.

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