The Scottish Bubble and the Irish Bubble
A Maths Club Project
The Elementary Mathematics in an Advanced Problem

Overview

The contents of this CD are intended for you, the leader of the maths club.
The material is of 3 different kinds:

history: *the problems in their historical setting: how they arose, how people first tackled them. Many of you will be specialists in a science other than maths. I hope you find something to interest you personally among the references I have cited. I also hope you take this project as an opportunity to work with colleagues from across the curriculum.*

calculation: *what students of school age can work out for themselves; something not for club-time but as a private challenge for the keen. This work is pitched at the 16-18 age range. However, elementary algebra, Pythagoras' Theorem, trigonometric identities and their use in solving triangles is almost all the maths the students will need.*

experiment: *what the students can do and make. This is the project core and can involve students from right across the secondary age range. With a weekly session of an hour and a half, the work will take a full term.*

As you see, these headings appear as possible bookmarks down the left side of the screen.

Introduction

When scientists turn to mathematicians for answers, they expect them to have ready-made solutions, something they can pull down off the shelf. In one area, packing, this has not been the case. The reason lies in the problems' untidiness. To identify a packing as the best, one must exclude all others possible. This invariably involves an exhaustive process, using hours of computer time, and the results are difficult to check manually. On the other hand, the problems are usually easy to state and understand and your students, the mathematicians of the future, may come up with simple insights which will make the computer algorithms more efficient. It also happens that those who are authorities on the subject are also good communicators [**1**].

Our subject is the 'Kelvin' problem. But we enter it via two related problems: the 'Tammes' problem and the 'Kepler' problem. All three require us to think about spheres in some way. We devote a chapter to each. Connecting these are the all-important 'links'.

Contents

Notes

[1] Aste, T. & Weaire, D. (2008), *The Pursuit of Perfect Packing*, (2nd edition), Taylor & Francis.

Chapter 1

history 1: *Thomson and Tammes*

By the end of the nineteenth century it was known that atoms contained positive and negative electrical charge in equal measure and that the negative charge was concentrated in the tiny packets we now call ‘electrons’. It was not known how they were distributed through the atom, and it would be 5 years before it was confirmed that they all had equal charge, but in 1904 their discoverer, J.J.Thomson, considered the following problem:

How would n identical point charges distribute themselves on the surface of a sphere so that the total potential energy was minimal?

This total was proportional to the sum of the reciprocals of the distances between all the pairs of charges. It was therefore least when the charges were as far from each other as possible. Thinking of the point charges as vertices of a polyhedron – necessarily one inscribable in a sphere – the question was therefore:

What polyhedron will 2, 3, 4, … point charges form?

He confessed “I have not as yet succeeded in getting a general solution” – and no one yet has!

In 1930 the Dutch botanist Tammes was studying the distribution on the surface of (more or less) spherical pollen grains of the pores through which the pollen contents issue on fertilisation [**1**]. The first thing he noticed was their number: there were 4 and 6 (but never 5), 8 and 12 (but rarely 7, 9, 10 and almost never 11). The second was that they tended to be distributed regularly over the surface. He took a rubber ball and a pair of compasses and drew circles on them centred on the imaginary pores. He asked the question:

How would n pores arrange themselves so that the smallest distance between any two was as large as possible?

The question is often called the Problem of the Friendly Dictators. *n* dictators agree to divide up their planet so that the border of each circular empire is an equal circle [**2**].

The ‘Thomson’ question and the ‘Tammes’ question are not quite the same. Take the case *n =5.*

experiment 1: *work on the Lénárt sphere*

Take a Lénárt sphere [**3**] and compasses and repeat what Tammes did for this case.

You should find that you can make 3 circles of angular radius 60^0 centred on points around the equator. But that leaves you with circles of only 30^0 centred on each pole.
It turns out that this is the best solution to the Thomson problem. But now centre 4 points around the equator and one on a pole. Each circle will have an angular radius of 45^0 , an angle greater than 30^0. You will realise you could also have used the other pole. In other

words, for Tammes' purpose, the $n = 5$ solution is no better than the $n = 6$ solution: in both cases the points define a regular octahedron.

You can find tables of the best solutions found to the Thomson problem [**4**]. Your students can use Polydron ('Frameworks') [**5]** to build their own best guesses. Because all Polydron edges are the same, only the regular deltahedra will accurately fit a sphere. But the object is only to distinguish the forms topologically – to realise, for example, that the solution for $n = 5$ is a triangular dipyramid (2 congruent triangular pyramids stuck base-to-base), not a square-based pyramid.

Here is a table showing the dozen most interesting shapes. These are the *n* values the students should try to model.

n	Regular polygon	Regular deltahedron	Dipyramid	Approximation to this semiregular solid	Antiprism - between -pyramids	Other shape
2	'digon'					
3	triangle					
4		tetrahedron = digonal antiprism				
5			triangular			
6		octahedron = triangular antiprism				
7			pentagonal			
8				square antiprism		
9						triangular prism with pyramid on each square face
10					square antiprism	
12		icosahedron = pentagonal antiprism between pyramids				
14					hexagonal antiprism	
24				snub cube		

Notice how many antiprisms or antiprisms-between-pyramids there are. If the solution for $n = 8$ was a cube, we could add a triangular antiprism-between-pyramids. (The triangular faces would be right-angled isosceles.) Surprisingly, it's not. Using a couple of formulae from spherical trigonometry, older students can do the calculation to compare the Thomson energies for the cube and Archimedean square antiprism **[6]**. Though the best solution to the Thomson problem for $n = 8$ is not exactly Archimedean, the Tammes solution is. The students can test this on the model sphere.

A. Draw 8 circles which each fill 1/8 of the sphere so that their centres define a cube. Measure their radial angles.
B. Start again. The 4 circles in the northern hemisphere stay in contact but move a short way down their meridia ($0°$, $90°$E, $90°$W, $180°$) so that they extend over the equator. They will have grown a little. The 4 circles in the southern hemisphere move E (say) $45°$ and their centres move up their meridia till they touch their northern neighbours. Draw the new situation. You will have to keep rubbing out and redrawing till you're happy the northern and southern circles are the same size. Measure the radial angles. You should find that, within experimental accuracy, the new circles are bigger than the old ones. (The angular radii are respectively around $35°$ and $37\frac{1}{2}°$.)

Ask the students what is special about the solutions for $n = 4, 6$ and 12.

The answer to this question is important. In terms of equal Tammes circles centred on the vertices, these solutions are the spherical analogy of close packing in the plane:

Figure	Number of nearest neighbours
Tessellation 3^6	6
Solid 3^5 (icosahedron)	5
Solid 3^4 (octahedron)	4
Solid 3^3 (tetrahedron)	3

These arrangements of points are uniform in the following sense. You will know the dandelion clock from childhood. The parachute ball of flowers of the genus Taraxacum consists of more than a hundred tiny arrows ('achetes'), their tails ('pappi') forming a feathery white ball, their heads (the single-seeded fruit) stuck together at the centre:

Think of the shafts of these arrows as rays emanating from the centre of the inner sphere, their tails as points on the outer sphere. Any two arrow shafts (other than a pair directly opposed) define a great circle and form an angle in that particular plane. If there is no third shaft in this plane between two chosen ones, we shall call them ‘adjacent’.

Can all pairs of adjacent achetes make the same angle?

Answer: No. Consider the shafts of 3 arrows adjacent in pairs. If the pairs form the same angle, the tails form an equilateral triangle. Over the whole sphere, therefore, our condition can only be met if the tails define one of the regular deltahedra listed above. There would have to be only 4,6 or 12 achetes for this to be possible and we’ve got more than a hundred to accommodate.

This means, among other things, that all pairs of dimples on a golf ball cannot be equally spaced. How then do you make a ‘fair’ golf ball, meaning that its flight is not affected by how you sit it in the tee when driving off? Manufacturers have chosen certain numbers of dimples (our *n* values) and patented designs for them which they claim are the best. On the Tammes criterion – that the smallest angle between two should be as large as possible for that number – it turns out that certain values of *n* allow better arrangements than others. We shall investigate these, the ‘magic’ numbers, shortly. Incidentally, as well as the pores on a pollen grain which Tammes was studying, there are dimples and spikes which serve the same function as the dimples on a golf ball: to extend its flight [**7**, **8**].

In the case of pollen seeds it seems nature has tried to give the pores as much space as possible; the same applies to dandelion clocks and their seed parachutes. How has it solved the problem? As it always does: by letting rival models fight it out over millions of generations, a principle programmers have copied under the name ‘genetic algorithm’.

Another example: manned space exploration is expensive and arguably less productive scientifically than unmanned exploration. Consider a further flight to one of the larger moons of Saturn or Jupiter. We wish to determine the internal structure of the satellite by placing seismometers on the surface. If we are to devote 2, 3, 4, 5, … to the project, how should they be placed?

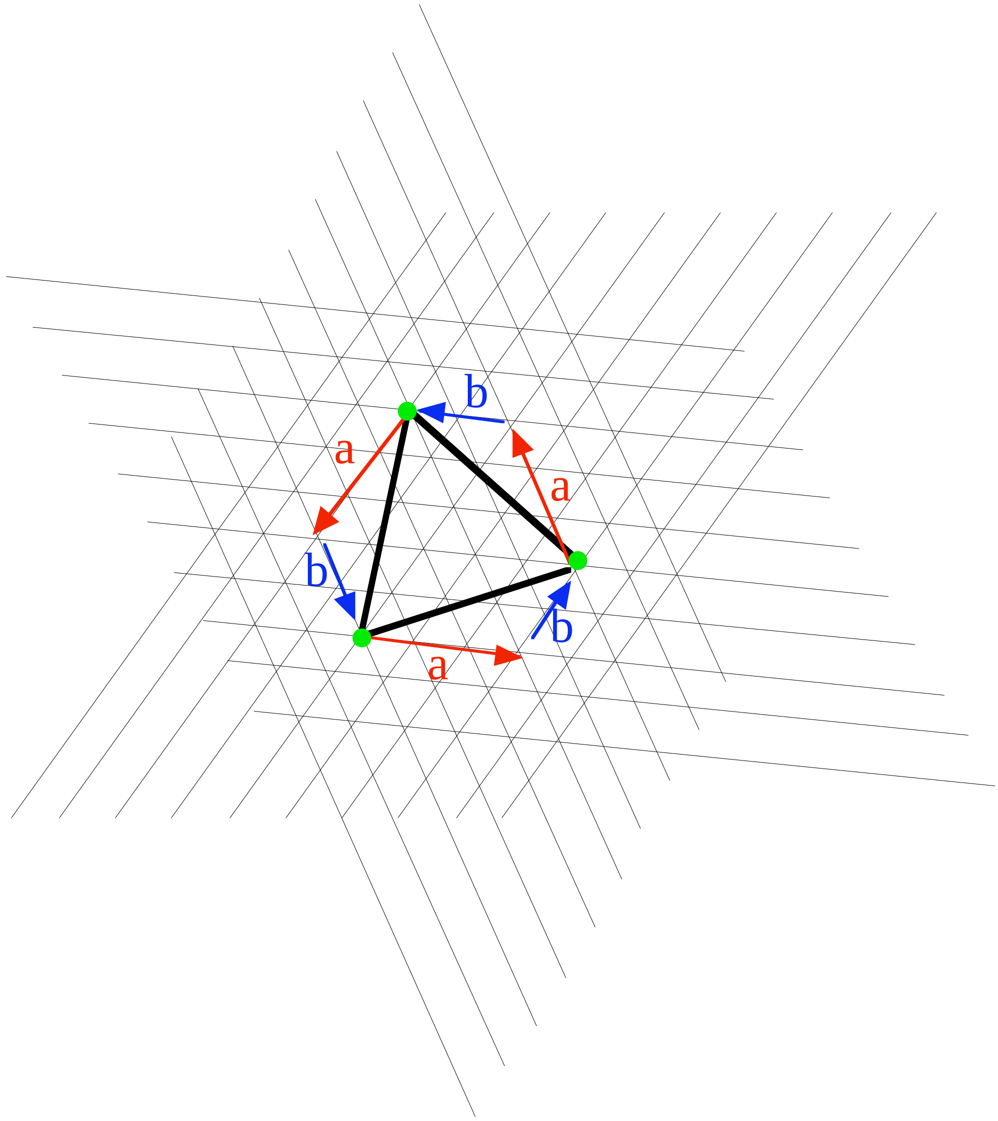
The magic numbers were found by people studying the structures of viruses in the early 1960s. (See references to ‘CK’ (Caspar & Klug) in [**9**]. My own treatment is based on [**10**].)

experiment 2: *work on isometric paper*

Take a sheet of isometric paper. You could use it to draw the net for a regular icosahedron. There would be 20 faces and, when you folded it up, you would find that you had 12 vertices. But instead you’re going to use it to create a *pseudo*-icosahedron – in fact many.

In **Chapter 2** we shall make a Kepler packing, a shell of 12 spheres packed tightly around a central one. Each sphere marks a vertex – in this case of a cuboctahedron. Now imagine packing a second shell around that. This time we find we have used 32 spheres. We continue. Our polyhedron – a pseudo-icosahedron - will approximate a sphere more and more closely. We shall not attempt to prove this but, with the exception of the first shell, it turns out that if we inscribe this polyhedron in a sphere and project the vertices on to it from the centre we shall have points as uniformly distributed as we can achieve for that number. The shell numbers are the ‘magic’ numbers. You are going to use a construction to reveal these numbers first suggested by the geometers H.S.M. Coxeter and his collaborator Michael Goldberg in the 1930s. Again, we shall not prove that the resulting arrangement is indeed the most uniform for that number, though we shall show how it appeals to our intuition.

To draw the face of the general pseudo-icosahedron of type (a,b), choose a vertex, move a edges in one direction, b edges at 120°, and mark another. Do so symmetrically:



However you choose a and b, there will be the vertices shown in green, but every lattice point on the isometric paper will also be a vertex. In the case here, $(3,2)$, there are 9 vertices on the interior of the triangle and none on the edges. We know that the complete icosahedron has 20 such faces, 30 edges and 12 vertices like the 3 shown. Since there are 9 vertices within each face but none on the edges, the total will be $20(9)+12=192$.

Draw a triangle for every (a,b) pair up to, say, (5,5) and make a table.

When do you expect there to be points on the edge? (Answer: when a,b are coprime or b is 0.)

Here is part of such a table, going from (2,0) to (2,2):

a	b	20 x interior points	+ 30 x edge points	+ 12	= magic number, N
2	0		30 x 1	12	42
2	1	20 x 3		12	72
2	2	20 x 4	30 x 1	12	122

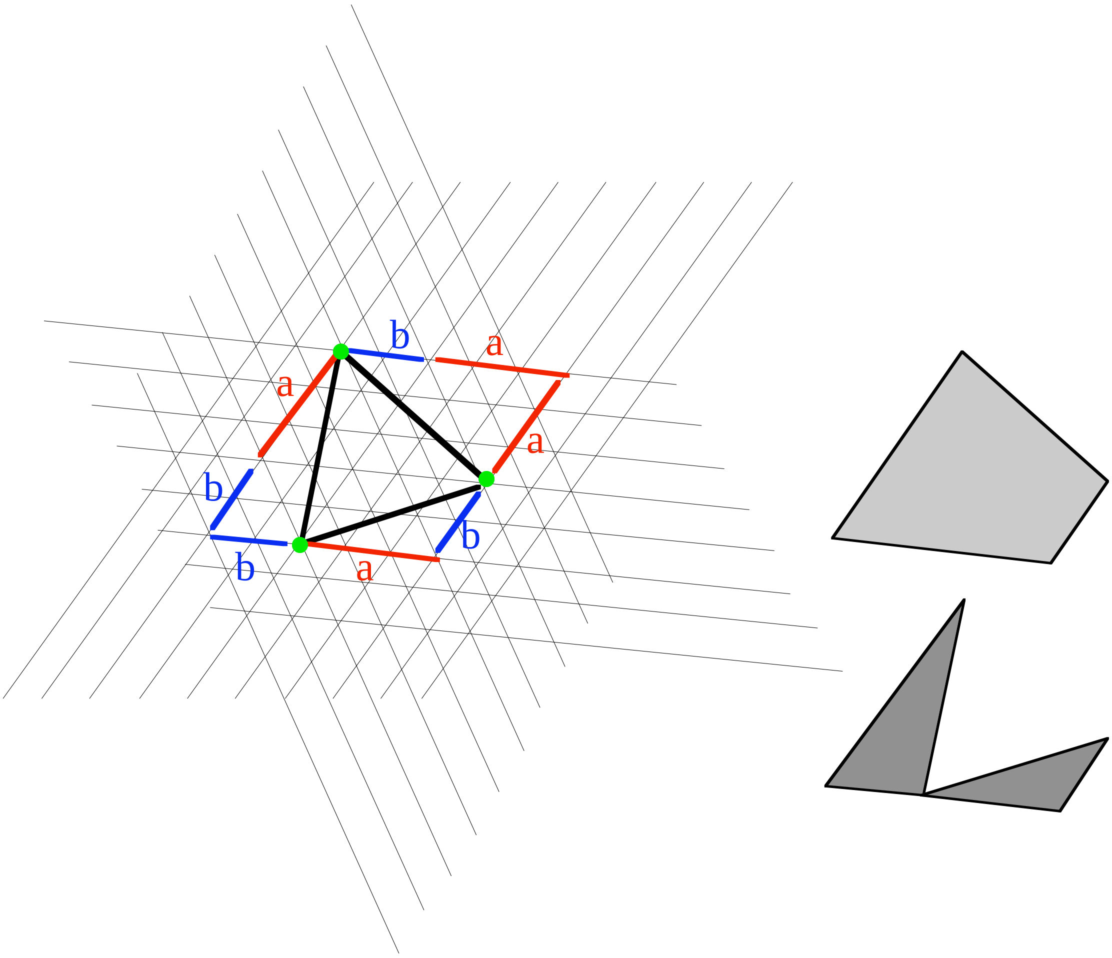
What does the Goldberg-Coxeter construction do? If we triangulate the faces of a regular icosahedron in the simplest way, i.e. keep the b values zero, we shall produce a sequence of magic numbers, but widely spaced. By twisting the isometric grid with respect to a face we preserve as far as possible the uniform environment of each point while allowing intermediate values for the number of points per face.

calculation 1: *swapping computation for construction*

You might wonder if there's a formula which will save you doing all this drawing. There is. $N_{(a,b)} = 10(a^2 + ab + b^2) + 2$.

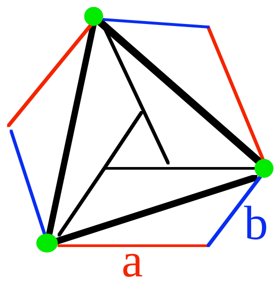
We can derive the formula using a simple discovery by the Austrian Georg Pick [**11**, **12**].

First we work out the area of our triangle using parallelograms as units:



Subtracting the two triangles from the trapezium, we have area $A = \dfrac{a^2 + ab + b^2}{2}$.

(Paul Gailiunas suggests this alternative dissection, which preserves the rotational symmetry of the figure:



We have 3 half-parallelograms and an equilateral triangle of side $(a-b)$.)

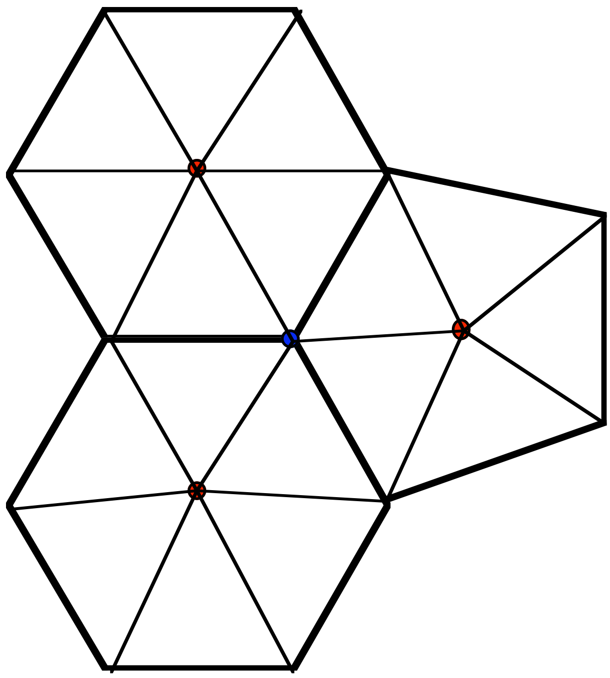
Pick's Theorem says that the area A of such a figure drawn on a parallelogram lattice $=\dfrac{d}{2}+i-1$, where d is the number of grid points on the boundary, i the number in the interior. Setting these two expressions equal, we have $a^2+ab+b^2=d+2i-2$.
In the case of every triangle, d is made up from two kinds of points: the 3 vertex points and p_e edge points. So we can substitute (p_e+3) for d, giving $a^2+ab+b^2=p_e+2i+1$.

The number of points on a single triangle edge is $\dfrac{p_e}{3}$.

For $N_{(a,b)}$ we have to add the points on all 30 edges, in all 20 face interiors and at all 12 vertices. Thus $N_{(a,b)}=30\left(\dfrac{p_e}{3}\right)+20i+12=10p_e+20i+10+2=10\left(p_e+2i+1\right)+2$
$=10\left(a^2+ab+b^2\right)+2$.

The magic numbers also arise when we make a 'geodesic' dome of the kind designed by the architect Buckminster Fuller. In the classic example you start with pentagons and hexagons inscribed in a sphere, then join their vertices to points on the sphere to form shallow pyramids and a structure composed entirely of triangles. (In cases where 3 hexagons meet in a vertex, those hexagons must be irregular if all the vertices are to lie on a sphere.)

Here is part of one:



calculation 2: *magic numbers and geodesic domes*

We start by deriving a remarkable result about polyhedra consisting only of hexagons and pentagons. Let the numbers of vertices, faces and edges be v, f, e respectively. From Euler's formula we have $v + f = e + 2$.

Let there be f_6 hexagons and f_5 pentagons, so $f = f_6 + f_5$.

We can work out the number of vertices by summing them for all the separate polygons then dividing the total by 3 since 3 faces share a vertex:

$$v = \frac{6f_6 + 5f_5}{3}.$$

We can obtain e in the same way, this time dividing by 2 since 2 faces share an edge:

$$e = \frac{6f_6 + 5f_5}{2}.$$

If we substitute these expressions in the Euler formula and simplify, we are left with the naked fact:

$$f_5 = 12.$$

This tells us that, however many hexagons our solid has, it must have exactly 12 pentagons, no more, no less.

We now sum the vertices of our geodesic dome. These are of two kinds:

v_a for the apices of the shallow pyramids (red) – one for each hexagon and pentagon:

$$v_a = f_6 + f_5 = f_6 + 12,$$

v_b for the vertices of the generating solid (blue):

$$v_b = \frac{6f_6 + 5f_5}{3} = \frac{6f_6 + 5(12)}{3} = 2f_6 + 20.$$

So $v = v_a + v_b = (f_6 + 12) + (2f_6 + 20) = 3f_6 + 32$, which we shall write $3(f_6 + 10) + 2$.

Since there are 6 triangular faces for every hexagon and 5 for every pentagon:

$f = 6f_6 + 5f_5 = 6f_6 + 5(12) = 6f_6 + 60 = 6(f_6 + 10)$.

Each face has 3 edges, each shared by 2 faces, so:
$e = \frac{3}{2} f = 9(f_6 + 10)$.

(You can check that our expressions for v, f and e are consistent with the Euler formula.)

To summarise:
$v = \left[3\left(f_6 + 10\right) + 2\right]$,
$f = 2\left[3\left(f_6 + 10\right)\right]$,
$e = 3\left[3\left(f_6 + 10\right)\right]$.

It turns out that the forms which give the most even distribution of points on the sphere – ‘even’ in the sense we have been investigating – are those where the expression in square brackets = $10T$, where T, the ‘triangulation number’, is our familiar expression $a^2 + ab + b^2$.

We have, then, $3\left(f_6 + 10\right) = 10T$, i.e. $f_6 = \frac{10\left(T - 3\right)}{3}$. This expression only gives an integral value for f_6 when T is a multiple of 3, implying in turn that f_6 is a multiple of 10. It’s instructive to compile the following table – to be continued as far as you like, the last column of which yields magic numbers:

a b	$T = a^2 + ab + b^2$	$f_6 = \dfrac{10(T-3)}{3}$	$v = \left[3(f_6 + 10)\right] + 2$
1 0	1		
1 1	3		
2 0	4		
2 1	7		
2 2	12	30	122
3 0	9	20	92
3 1	13		
3 2	19		
3 3	27	80	272
4 0	16		
4 1	21	60	212
4 2	28		
4 3	37		
4 4	48	150	482
5 0	25		
5 1	31		
5 2	39	120	392
5 3	49		
5 4	61		
5 5	75	240	752

summary:

We have examined two problems, the ‘Tammes’ problem and the ‘Thomson’ problem. Though distinct, we would expect the solutions to converge as the number of points n became large.

The ‘Tammes’ problem is difficult because it deals with the sequence of natural numbers on the one hand, and with the symmetries of 3-dimensional solids on the other. The latter generate their own integer sequence, from which many of the natural numbers are missing.

We have looked at special values of n, the so-called ‘magic’ numbers, and discussed 3 separate ways in which they arise:

(**a**) We begin with a unit sphere and form shells of like spheres about it. We project from the centre on to an enclosing sphere through the centres of the spheres in a given shell.

(b) We begin with a regular icosahedron and use a particular construction to impose an isometric grid upon it. Again, we project the points which arise from the centre on to an enclosing sphere.

(c) We begin with a polyhedron consisting of pentagons and hexagons inscribed in a sphere and add points corresponding to the apices of pyramids with those polygons as bases.

We note a correspondence between **(b)** and **(c)**: in both cases 12 pentagons occur; in **(b)** around the 12 vertices of the original solid, in **(c)** as a consequence of the Euler condition.

In all three cases a rule is applied which plausibly distributes points on the surface of the sphere as evenly as possible *but we have offered no mathematical justification for this*.

Notes

[1] Tammes, P.M.L. (1930) 'On the Origin of Number and Arrangement of the Places of Exit on Pollen Grains', Recuil d. trav. Bot néerlandais **27**, 1.

[2] In Isaac Asimov's novel 'Foundation and Earth' (Grafton, 1987) these dictators, the 'Solarians', are hermits since their subjects are robots.

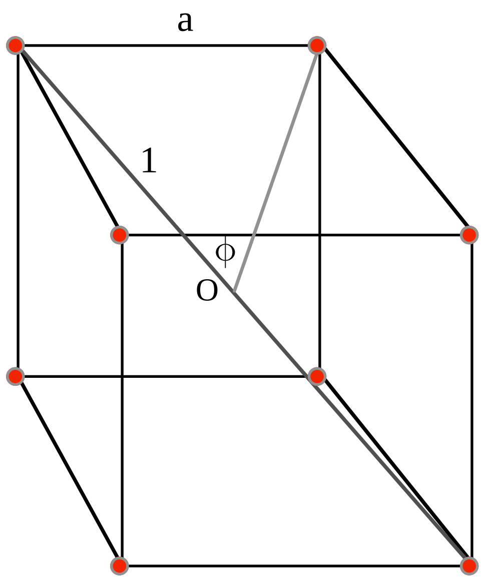
[3] Go to http://www.keypress.com/x5883.xml .

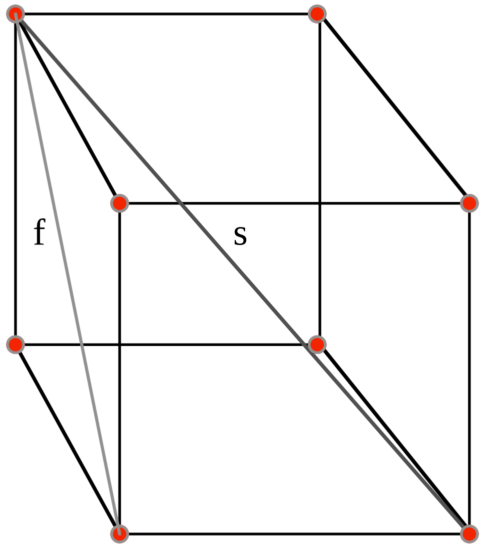
[4] Go to http://en.wikipedia.org/wiki/Thomson_problem .

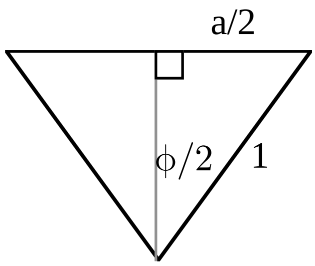
[5] Go to http://www.polydron.com . Turn to p.11 in the downloadable catalogue. You will need equilateral triangles (code 10-F300), isosceles triangles with an apical angle smaller than 60° (code 10-F301) and squares (code 10-F400).

[6] calculation 3: *the Thomson solution for 8 electrons*

First, the cube.



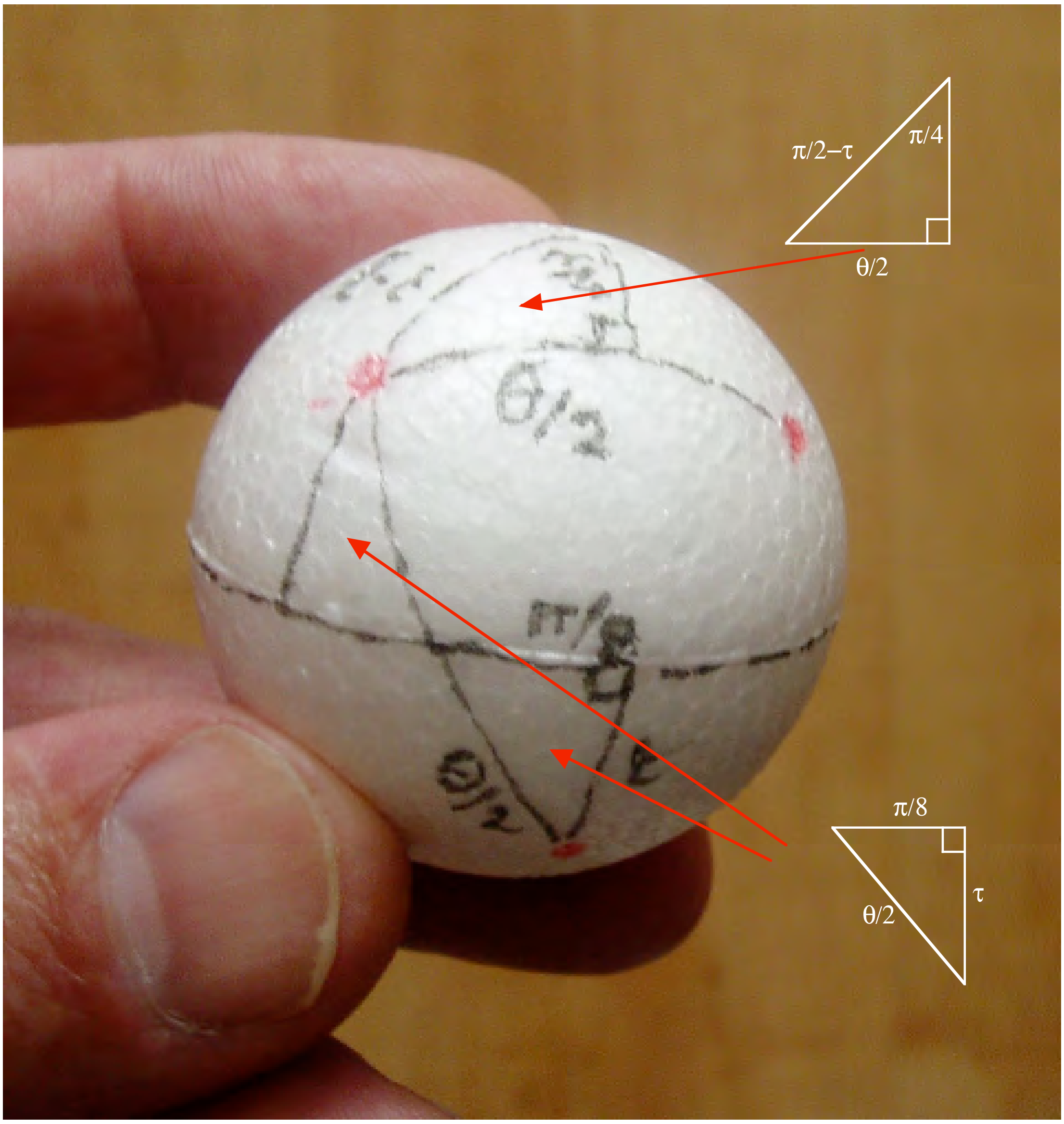




O is the centre of the sphere, radius 1. s, the space diagonal of the cube and the diameter of the sphere, is $\sqrt{3}a$. Thus $a = \frac{2}{\sqrt{3}}$, whence $\phi = 2\,\mathrm{arcsin}\,e\left(\frac{a}{2}\right) = 2\,\mathrm{arcsin}\,e\left(\frac{1}{\sqrt{3}}\right) \approx 70\tfrac{1}{2}°$.

Choosing one of the 8 points, we note that there are:
3 at distance a from it,
3 at distance $f = \sqrt{2}a$ from it,
1 at distance $s = \sqrt{3}a$ from it.

Now, the square antiprism. We need to know the angle θ corresponding to ϕ, the angle at the centre of the sphere subtended by an edge of the solid, before we can work out its length, b. We have drawn the necessary 'spherical triangles' on the surface of this little polystyrene sphere. The sides of such triangles are great circles, curves where planes through the sphere's centre cut the surface. The 'length' of a triangle side is the angle it subtends at the sphere's centre. The 'angle' between its sides is the dihedral angle between those planes. It turns out that the trigonometric formulae for such triangles can be as simple as, or simpler than, those for plane triangles.



The big red dots are 3 vertices of the square antiprism. They lie at the same latitude, τ, north and south of the equator. The apex of the upper triangle is the north pole. We've taken one of the two congruent lower triangles.

To the top triangle we apply the spherical equivalent of the sine rule: the ratios of the *sines of* the sides to the sines of the opposite angles are all the same:

$\dfrac{\sin\left(\frac{\pi}{2}-\tau\right)}{\sin\frac{\pi}{2}} = \dfrac{\sin\left(\frac{\theta}{2}\right)}{\sin\frac{\pi}{4}}$. This simplifies to $\sin\left(\frac{\theta}{2}\right) = \sin\frac{\pi}{4}\cos\tau$ [I].

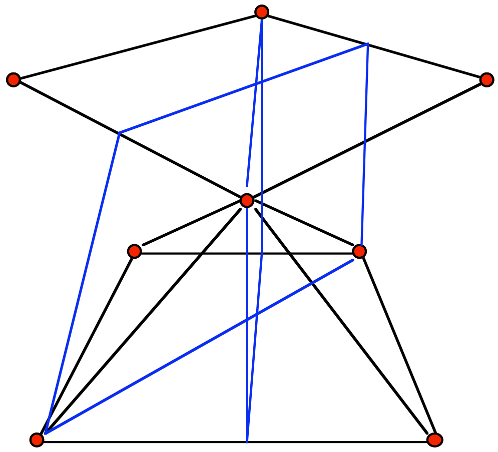
To the bottom triangle we apply the spherical equivalent of the cosine rule, which is particularly simple in the case of a right-angled triangle: the cosine of the hypotenuse is the product of the cosines of the other sides:

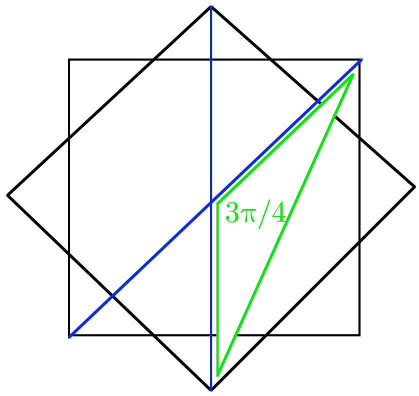
$$\cos\left(\frac{\theta}{2}\right) = \cos\frac{\pi}{8}\cos\tau \text{ [II].}$$

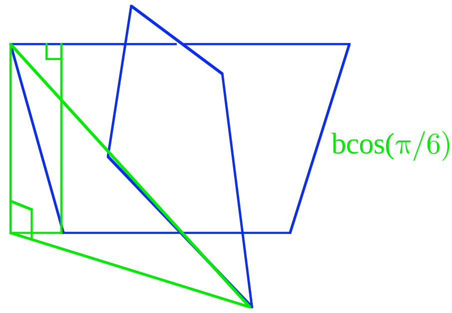
We divide [I] by [II] to obtain $\tan\left(\dfrac{\theta}{2}\right)=\dfrac{\sin\dfrac{\pi}{4}}{\cos\dfrac{\pi}{8}}=\dfrac{2\sin\dfrac{\pi}{8}\cos\dfrac{\pi}{8}}{\cos\dfrac{\pi}{8}}=2\sin\dfrac{\pi}{8}$, whence

$\left(\dfrac{\theta}{2}\right)=\arctan\left(2\sin\dfrac{\pi}{8}\right)$, giving $\theta\approx 75^{\circ}$, and, splitting an isosceles triangle into 2 right-angled ones, as we did for the cube, $b=2\sin\dfrac{\theta}{2}$.

Here is our square antiprism shown out of its sphere but complete:







These views allow us to work out all the distances we need. Two regular trapezia are shown in green, one inverted and turned $\dfrac{3\pi}{4}$ with respect to the other. A slant side is the altitude of an equilateral triangle face of our antiprism. A shorter side has length b, a longer side $\sqrt{2}b$, so we can use the small green right-angled triangle to work out the height of the antiprism. We can also work out the length of the base of the big green isosceles triangle because an equal side has length $\dfrac{\sqrt{2}b}{2}$. We now have what we need to work out the hypotenuse of the big green right-angled triangle, which turns out to be $\left(\sqrt{2}+1\right)b$.

Choosing one of the 8 points, we note that there are:
4 at distance b from it,
1 at distance $\sqrt{2}b$ from it, (the one at the opposite corner of the square),
and 2 at a distance $\left(\sqrt{2}+1\right)b$ from it, (the distance we've just shown how to work out).

Now we can do the Thomson calculation, which is to add the reciprocals of all the distances. To compare the two solids we need only do so for 1 point and its 7 neighbours since all 8 points are identically situated with respect to the others:

The cube: $\left(\frac{3}{1}+\frac{3}{\sqrt{2}}+\frac{1}{\sqrt{3}}\right)\frac{1}{a}$.   The square antiprism: $\left(\frac{4}{1}+\frac{1}{\sqrt{2}}+\frac{2}{\left(\sqrt{2}+1\right)}\right)\frac{1}{b}$.

Substituting the values we've found for a and b, we find the second represents an improvement of around 0.2%. (As noted in the main text, the best solution is not Archimedean: you can obtain a further improvement by making the square-triangle edge slightly shorter than the triangle-triangle edge, so that the triangles remain isosceles but are not equilateral.)

[7] For the comparison go to http://www.inwit.com/inwit/writings/dimpledgolfball.html .

[8] For scanning electron micrographs of pollen grains go to http://en.wikipedia.org/wiki/Pollen and click on the enlargement ikon.

[9] Zandi, R. et al (2004), 'Origin of Icosahedral Symmetry in Viruses', UCLA article.

[10] Stewart, I. (1989), *Game, Set and Math*, Penguin, pp.79-81.

[11] Stephenson, P. (2009), 'Pick's Theorem', Symmetry+, Issue **39**, pp. 2-4.
[12] Richeson, D.S. (2008), *Euler's Gem*, pp. 126-7. [**12**] is more useful than [**11**] because Richeson derives the theorem from Euler's polyhedron formula.

Link 1-2

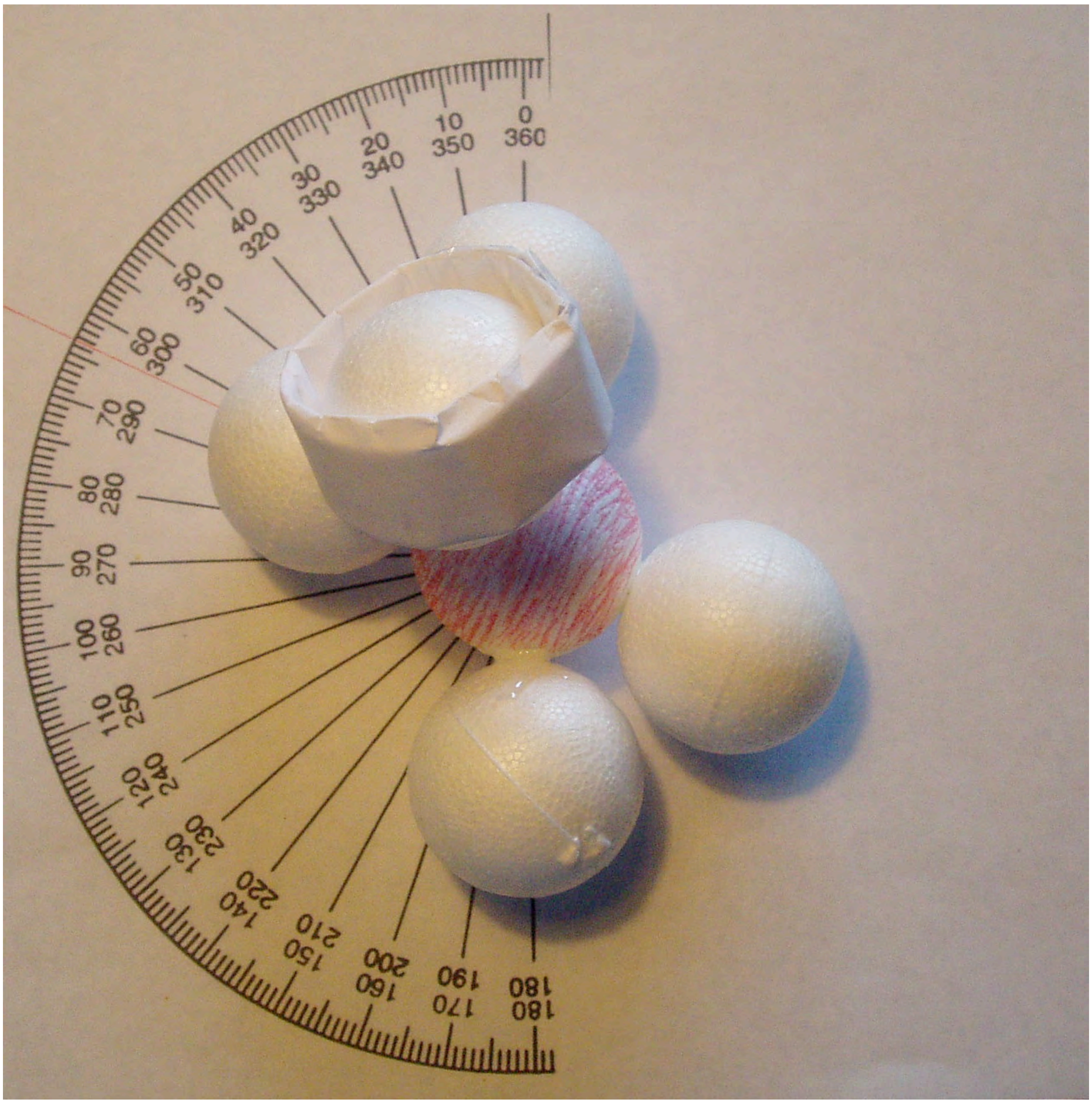
experiment 3: *a sphere and its icosahedral shell*

We are going to stick 12 polystyrene spheres, as widely separated as possible, around an equal sphere. We know that their points of contact (and indeed their centres) lie at the vertices of a regular icosahedron. We must work out how to make the model [**1**].

In the first picture we've started by sticking spheres at the north and south pole of the central one then added a pair in antipodal positions at the correct angle [**2**].

As an aside, the centres of the 4 outer spheres form a golden rectangle.

In the second picture we've added a 5th sphere. We've given it a paper collar to separate it from the pair it lies on:

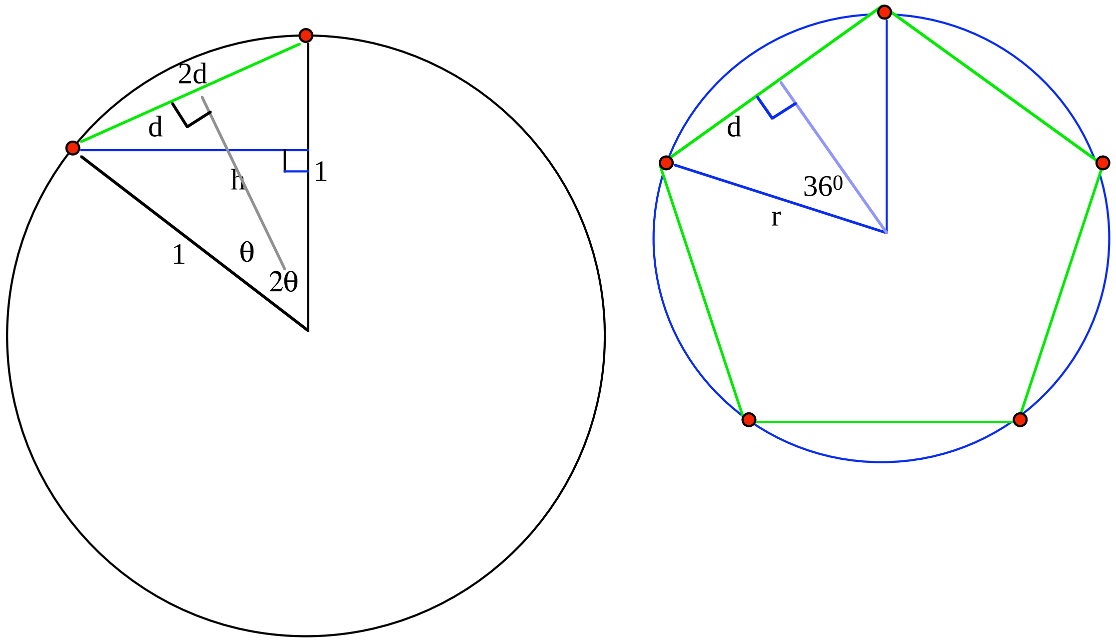


As we continue to add spheres, the collars get in the way and it's easier just to hold them in position while the glue dries. The idea is to average out errors by equalising the separation from immediate neighbours. This is the final configuration:

The gaps are what will interest us in **Chapter 2**.

Notes

[1] calculation 4: *the central angle adjacent sphere centres subtend*



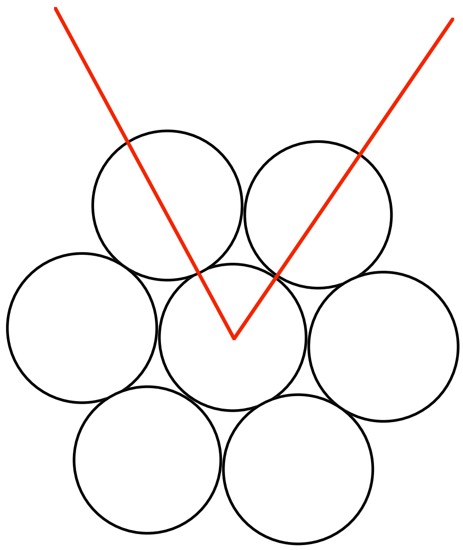
The left-hand figure shows a meridian with one point at the north pole and another at the latitude shown by the blue line. The length of this blue line, r, is the radius of the small circle shown in the right-hand figure. The length of the green line, $2d$, is the distance between the points. What we need is the angle, 2θ, this subtends at the sphere's centre.

From the right-hand figure we have $r = d\cos ec 36^{\circ}$.
From the left-hand figure we have $d = \sin\theta$.
So $r = \sin\theta \cos ec 36^{\circ}$.
We can write the area of the isosceles triangle in the left-hand figure in two ways and set the expressions equal:

Area = $dh = \frac{1}{2} r.1$

From the left-hand figure, $h = \cos\theta$.
Substituting, we have:
$\sin\theta\cos\theta = \frac{1}{2}\sin\theta\cos ec 36^{\circ}$, whence $\cos\theta = \frac{\cos ec 36^{\circ}}{2}$ and $2\theta = 2\arccos\left(\frac{\cos ec 36^{\circ}}{2}\right)$,
which, to the accuracy we can attain in our model, is about $63\frac{1}{2}^{\circ}$.

From the symmetry of this figure we realise that, if the spheres were touching, their angular separation would be 60° so we can expect a gap.



[2] For polystyrene spheres or wooden balls use an epoxy glue (Araldite or other) which sets in 3-5 minutes. For table tennis balls use a petroleum-based adhesive (Evostik or other), though an epoxy resin will work there as well.

Chapter 2

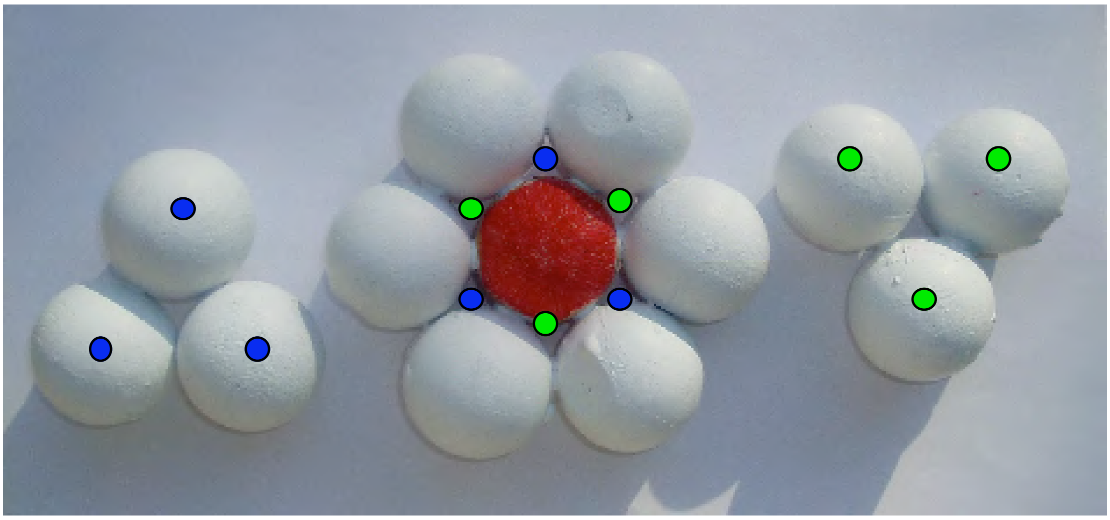
history 2: *Kepler*

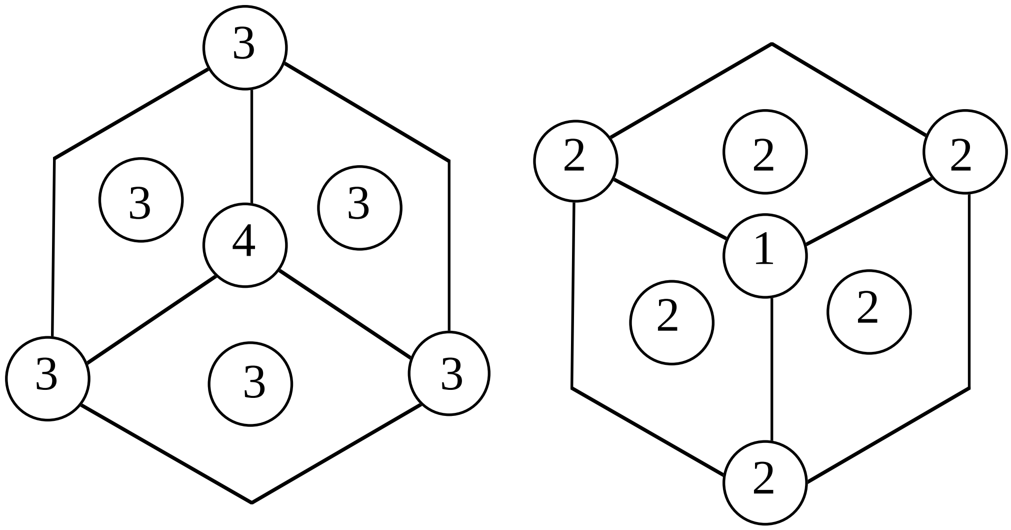
Sir Walter Raleigh asked the mathematician Thomas Harriot to study the stacking of cannonballs. Harriot wrote to Kepler. Kepler thought that in the most efficient packing one ball would be surrounded by 12, arranged as follows [**1**]. The upper picture shows 3 layers; the coloured spots indicate how they locate in the assembled model in the lower picture.

The diagram beneath shows why crystallographers, metallurgists and other solid state physicists call this arrangement ‘face-centred cubic’ (f.c.c.). You’re looking down on the corner of a cube which has been split in two: the lower 3 faces, meeting in the bottom-most vertex, are shown on the right; the upper 3 faces, meeting in the top-most vertex, are shown on the left. The circles show the positions of spheres in the packing. The numbers label the layers from the bottom.

Note incidentally that the f.c.c. arrangement is merely the most symmetrical of an infinite number of arrangements in which the constituent layers are hexagonally packed. In the f.c.c. arrangement the layers form the repeated positional sequence ABC; in the hexagonal close packed (h.c.p). arrangement the sequence runs AB; but in silicon carbide, for example, it is random.

There’s an interesting link with the Thomson table in **Ch.1** [**2**].





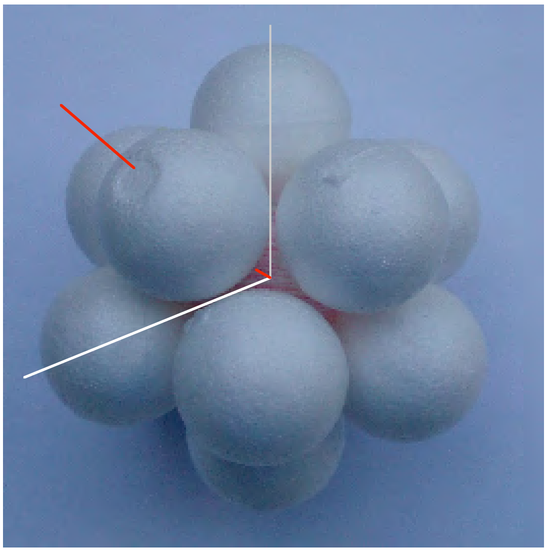
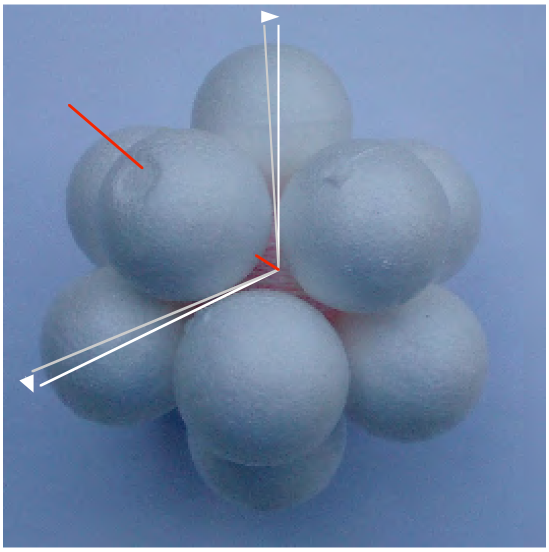
Kepler admitted, however, that he could not prove this arrangement was indeed the best.

It was more than 2 centuries before any progress was made on the problem when in 1831, following the work of L.A.Seeber, C.F. Gauss established that if the spheres were centred at the vertices of a lattice – that is to say, were arranged in a pattern which repeats 'periodically', i.e. by translation - the Kepler arrangement has the highest possible packing density (the ratio between the volume of space enclosed by spheres and the total volume).

But what if an irregular arrangement is allowed?

calculation 5: *the near neighbour number*

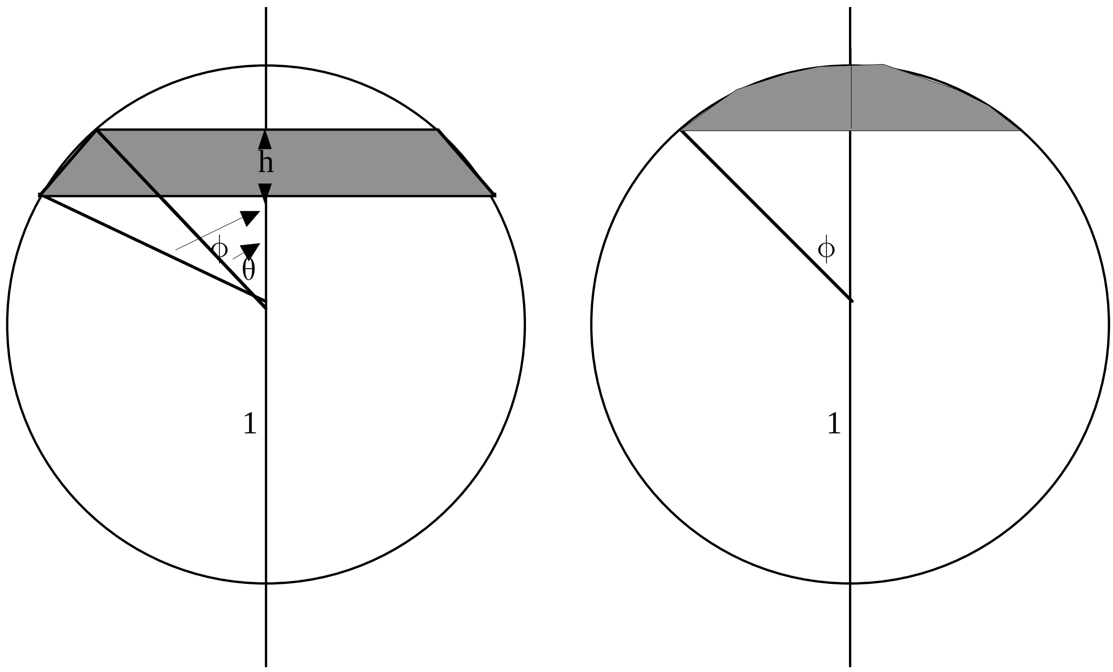
Here again is our icosahedral shell of 12 spheres. We picture each sphere sitting in a cone centred on the inner sphere. In the left-hand picture the cone touches the sphere and we know that the angle of this cone is the same as the angle between the centres of spheres that touch, 60°. In the right-hand picture we imagine our unit spheres swell until they touch. These sit in bigger cones. From our calculation on the separation of spheres in this arrangement we know the angle of these too: $2\arccos\left(\frac{\operatorname{cosec}36^\circ}{2}\right)$. In a moment we shall halve these values as we shall need the cones' *semi*apical angles.

These cones cut the sphere in little spherical caps. We can work out their area. Using calculus we can derive this formula for the area $A_{\theta\phi}$ of any spherical zone like the one shaded on the left in the next diagram: $A_{\theta\phi} = 2\pi(\cos\theta - \cos\phi)$.

As an aside, we note something interesting about this formula.
$\cos\theta - \cos\phi = h$, so the area is directly proportional to the height of the zone.
This fact, known to Archimedes, has simple consequences:
1) Cut a tomato into slices of equal thickness and the slices all have the same amount of skin;
2) Project a transparent globe from a strip light down its axis on to a surrounding cylinder and the countries have their true areas; (since $\cos 60° = \frac{1}{2}$, the areas north and south of latitude $30°$ are equal). This is the basis of the Lambert cylindrical equal-area global map projection.

On the right θ has become zero, and the zone a cap. The formula reduces to
$A_{\phi} = 2\pi(1 - \cos\phi)$.



The question is this. What if, statistically speaking, we could trade the number of unit spheres around the central sphere for the areas they command? How many spheres would we then have?

The answer is 12 x the ratio of areas of the two cap sizes: $12\left(\frac{1-\frac{\cos ec 36°}{2}}{1-\cos 30°}\right)=13.4$ to 3 significant figures. We'll call this the 'near neighbour number'.

Now we know that no more than 12 unit spheres can all touch a central one – the 'kissing' number for spheres in 3 dimensions - but might we manage more in a slightly bigger shell with the result that if we took a volume filled with thousands of spheres all jumbled up the mean packing density would be slightly higher than for the 'Kepler' arrangement?

history 3: *Scott and Hales*

The physicist G.D.Scott filled a container with ball bearings, shaking it periodically to let them settle into the densest possible arrangement [**3**]. In no experiment did he achieve a density greater than 0.64 (2 S.F.). The value for the Kepler arrangement is 0.74.

On August 9 1998 Thomas Hales of Michigan University completed a proof showing that no arrangement as good as Kepler's was possible. The proof showed that 'only' 5,000 different configurations have to be examined and required the writing of 3 gigabytes of computer program. Hales is presently attempting to reduce the computer's role to that of a checker of results produced manually. He estimates that this project will take him a further 20 years [**4**].

Notes

[1] Kepler, J. (1611), *On the Six-cornered Snowflake*, reissued by Oxford University Press in 1966.

[2] The best shape for *n*=9 is a triangular prism with a pyramid on each prism face. We might ask whether the best shape for *n*=12 is a *square* prism with a pyramid on each (rectangular) prism face. But this is just our Kepler model. Here it is appropriately reorientated. (We shall study the arrangement further in **Link 2-3**. There we shall see that the pairs of triangles sharing a base here are right-angled isosceles and make squares.)

[3] Scott, G.D. et al (1969), 'The Density of Random Close Packing of Spheres', J.Phys.D: Appl.Phys. **2**, pp. 863-866.

[4] http://en.wikipedia.org/wiki/Kepler_conjecture has an extensive list of references.

Link 2-3

history 4: *Bernal, Marvin and Matzke*

In the liquid state, the molecules are weakly attracted by the slight excess of electrical charge which results from the internal charges not quite cancelling out (‘van der Waals’ forces) but they are essentially free to move around in the bulk liquid like a mass of ball bearings. In 1959 the physicist J.D.Bernal wanted to know the mean number of nearest neighbours a molecule has in the liquid state. Back in 1727 the botanist Stephen Hales was studying the packing of cells in undifferentiated vegetable tissue. He squashed a random packing of peas in a pot and examined the resulting polyhedra, i.e. the duals of the solids defined by the spheres’ points of contact [**1**]. Bernal did the same with balls of plasticine, rolled in chalk dust so that they separated easily after compression. He found the mean number of faces was 13.3 [**2**].

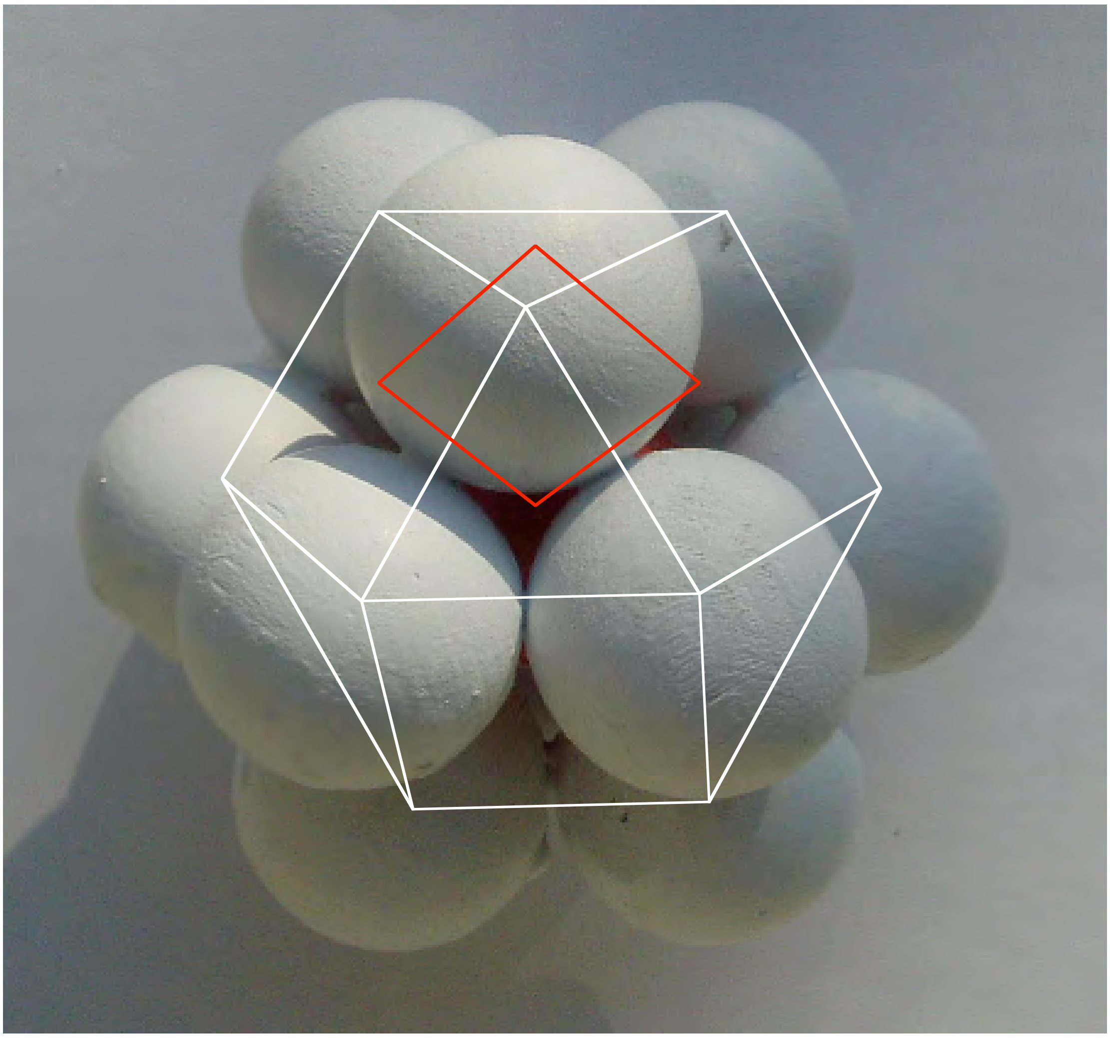
The agreement with our near neighbour number is good. But the Scott result quoted in the last chapter suggests that the mean separation between the spheres before they got squashed was high.

In 1939 the biologists J.W.Marvin and E.B.Matzke used lead shot in a steel cylinder with a plunger. In one experiment they arranged the shot randomly, as Scott had done, but in another they used the Kepler packing. The typical resulting shape was the rhombic dodecahedron [**3**]. This was to be expected and had been predicted by an observation of Kepler himself [**ch.2, 2**]. Kepler observes that the loculi in a pomegranate have this form. How does it arise? “For the loculi to begin with, when they are small, are round, so long as there is enough space for them inside the rind. But later as the rind hardens, while the loculi continue to grow, they become packed and squeezed together.”

We can think of an example where the mechanism is different but the result the same: the honeycomb. Be aware, however, that, when the bees secrete a honeycomb, they don’t make a space-filling system. One lot of bees pack together sideways, each bee defining a hexagon; another lot, similarly packed, push in from the opposite side, so that the centre of a hexagon on one side corresponds to a vertex on the other. Thinking now in 3 dimensions, and looking into the hexagonal prism occupied by a bee, what you see at the end of the cell are 3 coincident rhombi forming part of a rhombic dodecahedron [**4**]:

Here in white we have traced the shape formed by the centres of the 12 spheres surrounding a given one, and also their points of contact with it, an Archimedean solid, the cuboctahedron.

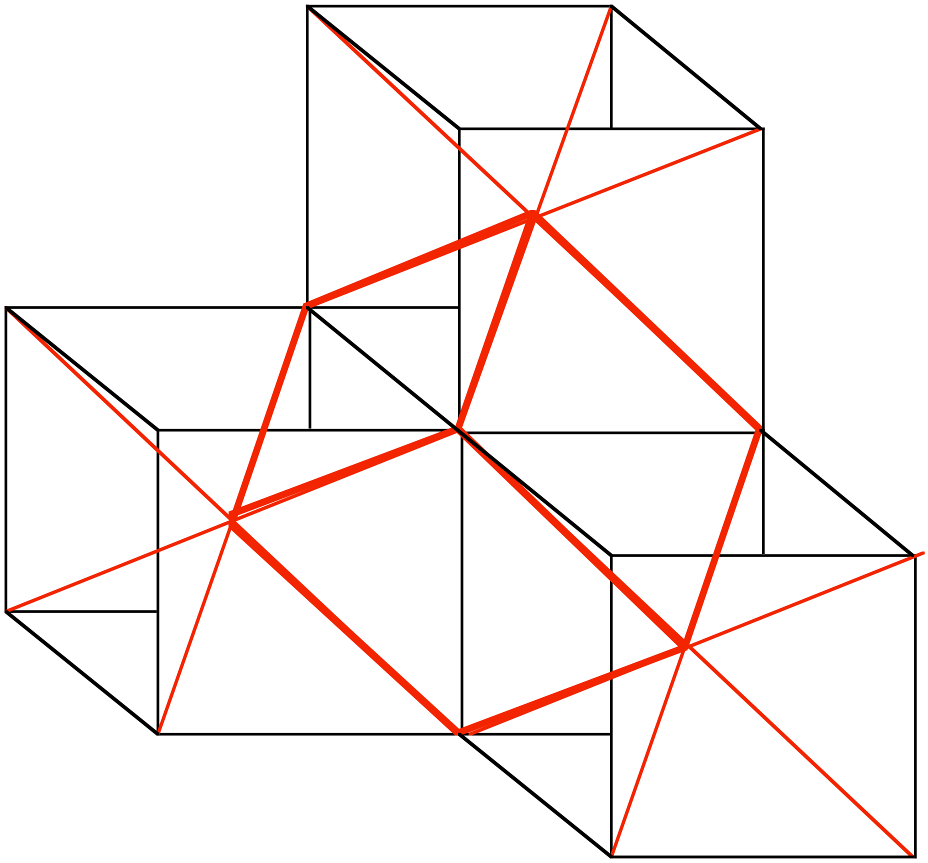
As stated above, since the points of contact become face centres when the spheres expand, the solid which results is the dual of this, the rhombic dodecahedron. We show one face of this in red.



In a regular ('Platonic') solid all the faces are regular polygons and they're all of the same kind. The cube is the only Platonic solid that packs. Is there another solid – of any kind – whose faces are all the same and which packs? Yes: the rhombic dodecahedron.

The cube and the rhombic dodecahedron are simply related.

On a chessboard black and white squares touch along an edge. Black squares touch at corners. Imagine the analogue in 3-D. In place of black squares imagine cubes. In place of white squares imagine gaps. Cubes touch only at edges and a cube face is bordered by empty space. 6 cubes border each cubic hole and 6 holes border each cube. Now let each cube claim a pyramid with its apex at the centre of each bordering hole. You have a rhombic dodecahedron. This solid has 6 x 4 = 24 triangular pyramid faces but each pair combine to make one of the 12 rhombuses, as shown in this diagram:



The diagram will be important when we come to examine Kelvin's thought process in the next chapter.

Notes

[1] Hales, S. (1727), *Vegetable Staticks*.

[2] Bernal, J.D., 'A Geometrical Approach to the Structure of Liquids', Nature, **183** (1959), pp.141-147.

[3] Matzke, E.B., 'In the twinkling of an Eye', Bulletin of the Torrey Botanical Club, **77** (1950), pp. 222-227.

[4] Photograph taken at The Honey Farm, Cheswick, Berwick-on-Tweed, Northumberland.

Chapter 3

history 5: *Kelvin's problem*

William Thomson, Lord Kelvin, the most prolific [**1**] mathematical physicist of the nineteenth century, was born in Belfast but adopted by Glasgow, to whose university he devoted his life [**2**].

Nowadays a 'model' usually means a set of equations. To Kelvin and his contemporaries it meant a physical system whose properties matched those of the thing you were interested in. But to Kelvin himself it also meant something you *made*.

"I am never content until I have constructed mechanical models of the subjects I am studying. If I succeed in making one, I understand; otherwise I do not."

Earthquake waves travel in rock, sound waves in air, water waves in water. But light waves pass through the vacuum of space so in what medium do *they* travel? All physicists of the time could do was give it a name, the aether. The only property they could agree on was that it had to be static with respect to a fixed point in space. Kelvin modelled it with pitch. If you apply a prolonged force to pitch, it flows like water, modelling the passage of the earth through space as it orbits the sun [**3**]. If you give it a sharp tap, it shatters like glass, modelling the aether's ability to transmit rapid vibrations. But he also constructed a second model. If you want to propagate a vibration in one dimension, you use a string; if you want to do so in two dimensions, but leave plenty of empty space, you use some sort of mesh or net. What do you need in three? The mesh would form the walls of compartments into which space was divided. Kelvin envisaged identical compartments and sought the shape which would have the greatest volume for its surface area.

Before we study the paper he wrote on the problem [**4**] and the model he built, we shall review what we have already learned. But first we must decide how to rank solids in terms of volume and surface area.

calculation 6: *the isoperimetric quotient*

We shall start in 2 dimensions.

Until the 1920s the fact that the circle is the shape which encloses the greatest area for its perimeter was itself an unsolved problem – though no one doubted that it was the case. If we want to know how good other shapes are in this respect we need a measure. We want our measure to be dimensionless so we must take the first power of the area and the square of the perimeter. For the circle we have:

$$\frac{area}{circumference^2} = \frac{\pi r^2}{(2\pi r)^2} = \frac{1}{4\pi}.$$

To give our measure the value '1' for the circle we must therefore multiply by 4π, giving us the 'isoperimetric quotient' (I.Q.) $4\pi \dfrac{area}{perimeter^2}$.

Find this for the square and the regular hexagon. You should get values to 3 significant figures of 0.785 and 0.907 respectively.

Now we move to 3 dimensions.

We must now relate $volume^2$ and $area^3$.

Our ideal solid is the sphere, for which this ratio is $\dfrac{\left(\frac{4}{3}\pi r^3\right)^2}{\left(4\pi r^2\right)^3} = \dfrac{1}{36\pi}$.

To give the sphere the value 1 our 3-dimensional I.Q. therefore becomes $36\pi \dfrac{volume^2}{area^3}$.

review:

Our ideal solid is the sphere. Unfortunately these do not ‘pack’, i.e. fill space. When packed as tightly as possible there are still spaces between them.

In **Ch.2** we learned that the best packing of spheres we can achieve is the ‘Kepler’ or f.c.c. arrangement.

In the **Linking section 2-3** we learned that compression of the spheres in this arrangement deforms them into rhombic dodecahedra.

history 6: *Kelvin’s solution*

Now we can turn to Kelvin’s paper.

Because the tension forces in the surface of a soap film minimise its surface area, he knows that his problem is solved in a soap foam.

Without observing an actual foam, he also knows that it cannot consist of rhombic dodecahedral cells for the following reason. Two conditions apply to a foam:
1. Walls meet in 3s, at the angle dictated by symmetry: $2\pi / 3$ [**5**].
2. Wall edges meet in 4s, at the angle dictated by symmetry, that subtended at the centre of a regular tetrahedron by an edge, the angle familiar to chemistry students as the bond angle in the methane molecule: $\arccos\left(-\dfrac{1}{3}\right)$, about $109\tfrac{1}{2}^\circ$ [**6**].
The rhombic dodecahedron passes the first test but fails the second since at 6 of its 14 vertices 4 edges already meet, leaving none to extend into the surrounding space.

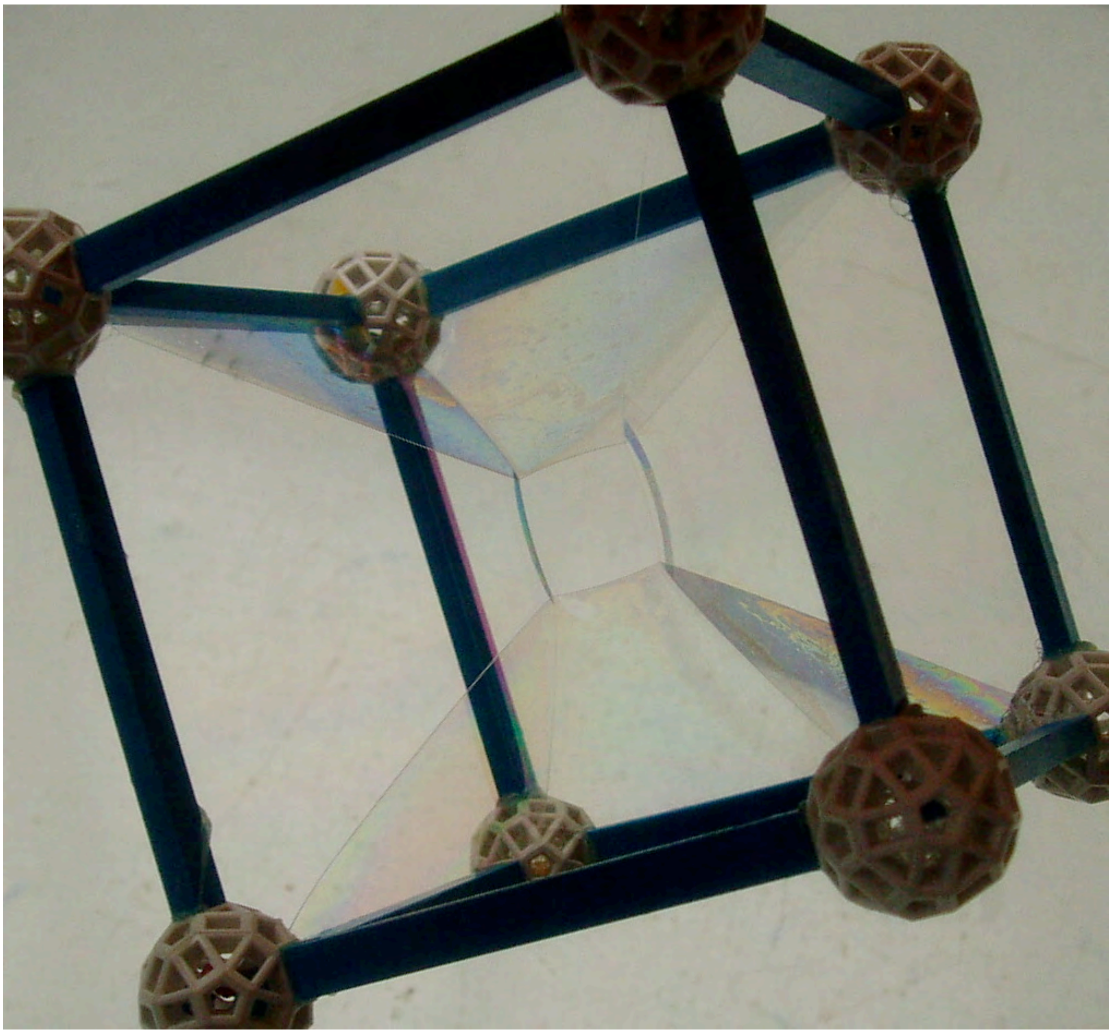
However, he sets up the following thought experiment, where we begin by creating just such a foam.

He imagines a cubic lattice of wire – the 3-D chessboard of **Link 2-3 –** with tiny metal globes at the centres of the empty cells, the ‘holes’.

He now imagines a soap film to connect the edges of the wire lattice and the globes.

Each foam cell will we know be a rhombic dodecahedron.

Before he makes his next move, he reminds us what happens in Joseph's Plateau's experiment where you dip a cubic wire fame in soap solution and withdraw it [**7**]:

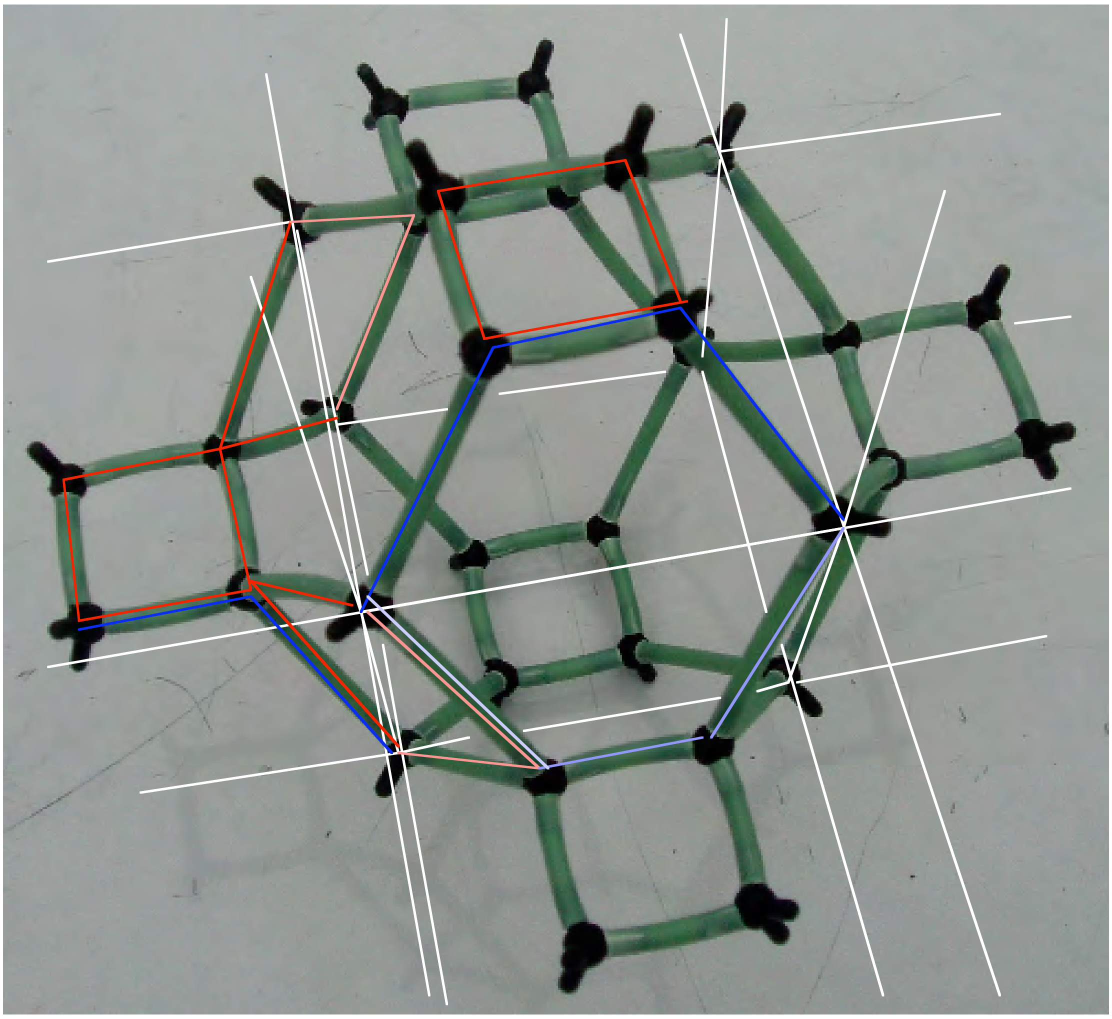


We now annul the tiny central globes. Every 'hole' in the 3-D chessboard jumps to the Plateau configuration.

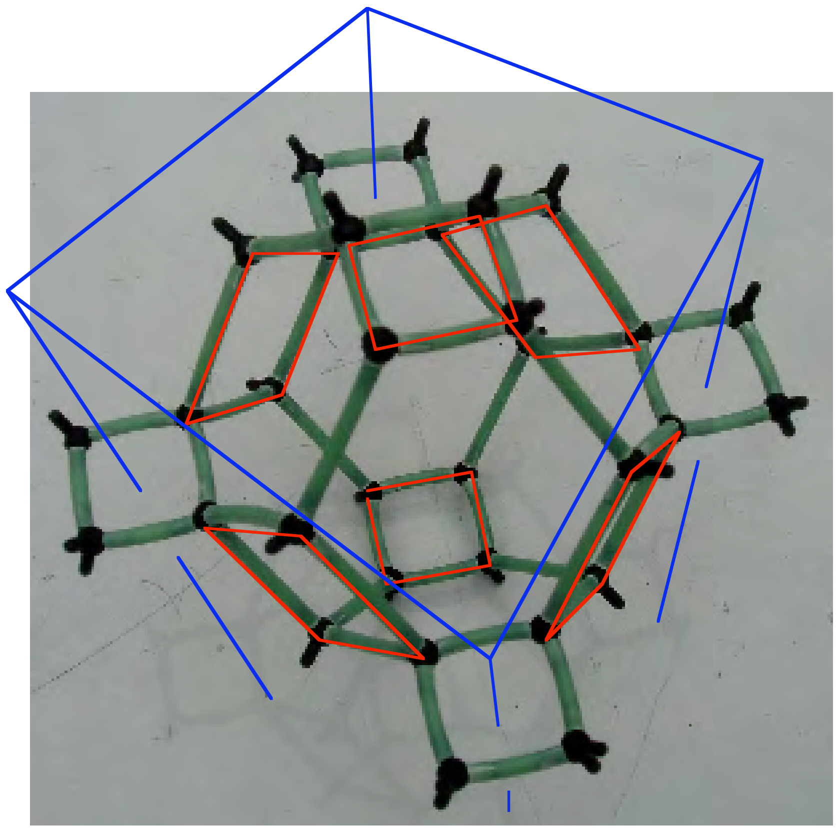
If the wire lattice is held horizontally so that there is a slight imbalance of force on the frame due to gravity, the central 'squares' – actually squares with bowed sides as you see - will all set themselves in the same orientation (horizontally).

By symmetry we can also annul the wire lattice itself, as the forces on each edge are equal and opposite.

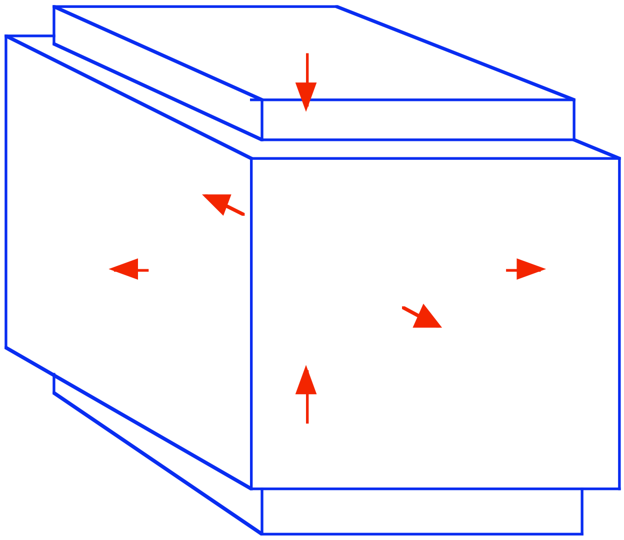
This is what we've now got. (We'll continue to put shapes in quotes to show that they are in fact versions distorted by the soap film conditions we specified.) Notice how a Plateau 'trapezium' in one cell (shown in blue) pairs with one across the cell edge (shown in a fainter blue) to make a 'hexagon'. Notice how a Plateau 'right-angled isosceles triangle' (shown in red) pairs with one across the cell edge (shown in a fainter red) to make a 'square'. But notice also that this ‘square’ is a different size from the one (also shown in red) which appeared at the centre of the Plateau cube.



The resulting polyhedron has 8 equal ‘hexagons’ but 4 ‘squares’ of one size, 2 of another. Kelvin imagines this set within the square prism (shown in blue), a ‘square’ (shown in red) lying in each face. In terms of the cubic lattice we can work out that the blue prism has a height of 2 units, a width of $\sqrt{2}$ units.



He then considers a large cubic section of the lattice formed by such individual prisms. He imagines wire to be introduced wherever a soap film cuts the boundary. This enables him to consider the net force on an entire wall of this cage. The asymmetry caused by the different ‘square’ sizes will cause the top and bottom walls to move inwards and the side-facing walls to move outwards so that the cage – whose wire boundary we imagine constantly to be changing in accordance - becomes a square prism. The small ‘squares’ will grow; the large ones, shrink. Equilibrium will be reached when all ‘squares’ are the same size. Our blue square prism will become a cube:

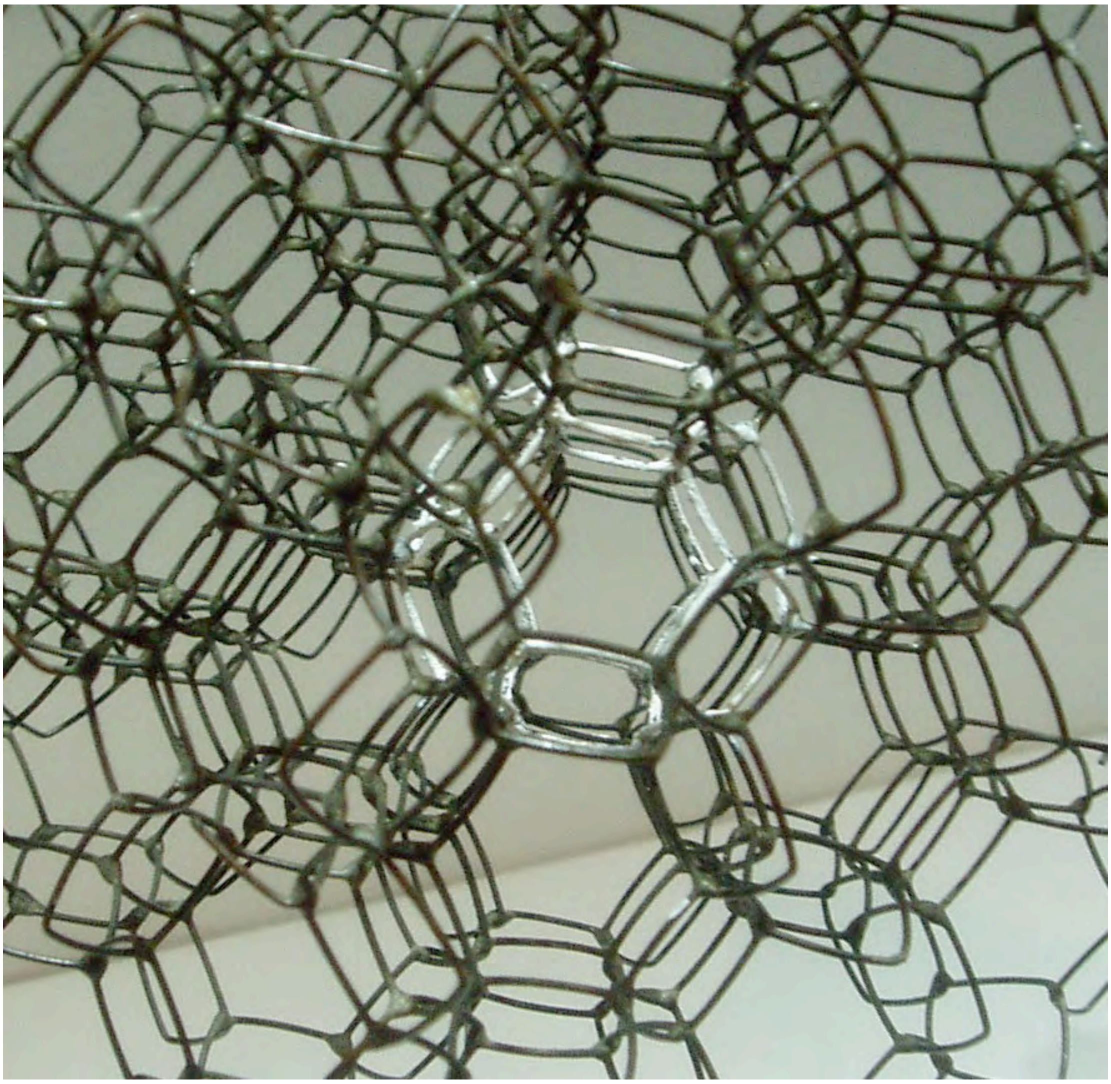


Let the cube have edge e. The volume remains the same so we have:

$e^3 = 2.2^{\frac{1}{2}}.2^{\frac{1}{2}}$, whence $e = 2^{\frac{2}{3}}$.

We can name the 'polyhedron' which results: it is the truncated octahedron [**8**].

Here is Kelvin's own model – known affectionately to his students as 'the bedspring model'. He's picked out one cell in white [**9**]:

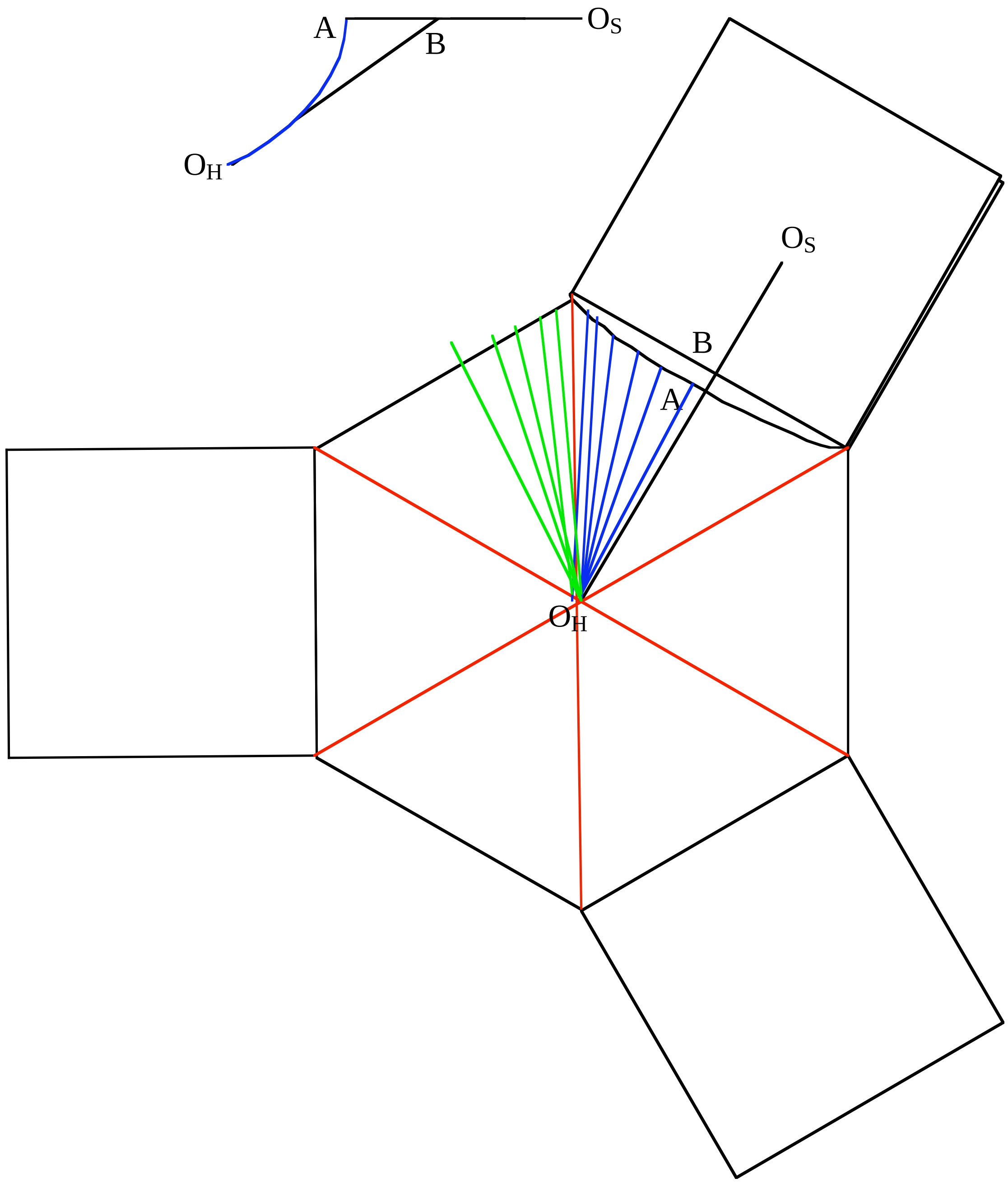


How does Kelvin's 'truncated octahedron' differ from the truncated octahedron (no quotes)? We'll see how it meets our 'foam conditions':

1. Look up the dihedral angles in Cundy & Rollett [**10**] (p.104). That between hexagons is not 120° but our 'tetrahedral' angle, about $109\frac{1}{2}^{\circ}$; that between a square and a hexagon is closer but still wrong: about $125\frac{1}{4}^{\circ}$.

2. The result of changing the interior angles to the tetrahedral angle is that the square sides bow out and the hexagon sides bow in.

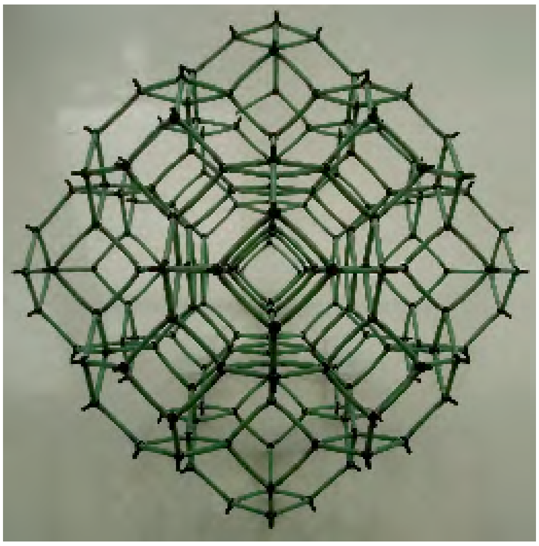
To appreciate the effect of these distortions we must add a third foam condition. If there is a pressure difference across a wall of soap film it will curve in proportion. This curvature is measured by taking the mean in two perpendicular planes. The curvature is the reciprocal of the *radius* of curvature. If these radii are r_1, r_2 in the respective planes, our mean curvature is therefore $\frac{1}{2}\left(\frac{1}{r_1}+\frac{1}{r_2}\right)=\frac{r_1+r_2}{2r_1r_2}$. In a foam in equilibrium there will be no pressure difference, therefore this expression will everywhere take the value zero. It may be that the individual curvatures are each zero and you have a perfectly flat piece of film – observe, for example, the wall between two bubbles which have coalesced; but it may also be that the curvatures are equal but of opposite sign so that you have a saddle. It turns out that the 'square' faces of Kelvin's solid are flat but the 'hexagonal' faces are shaped as follows, each point being a saddle point.

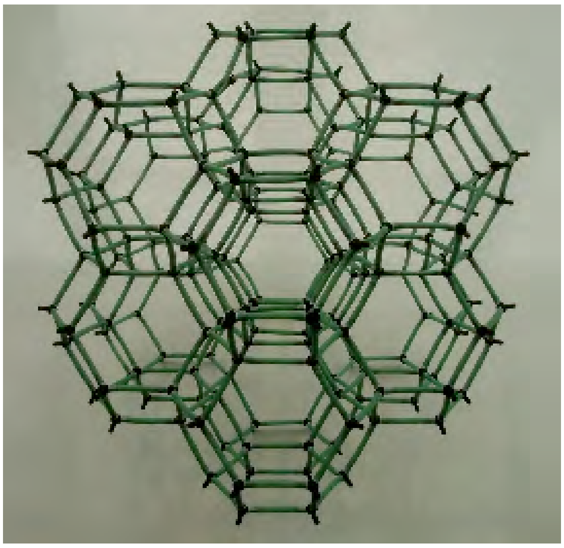


In the figure we're looking on to a particular such face. The vertices and the diagonals (all shown in red) lie in the same plane. Each 'hexagon' is bordered by alternate 'hexagons' and 'squares'. We've just shown the 'squares'. (In the foam 2 'hexagons' and 1 'square' meet in each edge.) Were all the faces regular polygons, a section through a hexagon centre and a square centre perpendicular to an edge would look like $O_H BO_S$ in the inset figure top left. However, the edge of a 'square' face is the arc of a circle (such that the tangents at a corner of the square meet at the tetrahedral angle). We must therefore curve the line $O_H B$ into $O_H A$ (shown in blue). The result is that alternate sextants of the original hexagon assume a trumpet shape. The beautiful thing is that, matching this set (shown in blue), is an identical set of depressions (shown in green).

experiment 4: *our own bedspring model*

We don't need Kelvin's dexterity with wire and solder now. We have 'Orbit' [**11**]. This kit has connecting pieces with pegs sticking out 'tetrahedrally' and plastic straws to join them. Here is part of a Kelvin foam modelled with Orbit. On the left we look down 'square' tunnels; on the right, 'hexagonal' ones:





history 7: *Matzke again, Weaire and Phelan*

Kelvin's solution remained unchallenged until 1993, when two Irish physicists, Weaire and Phelan, showed that, if they were allowed to use two cell shapes instead of one, they could design an aggregate which partitioned space more efficiently [**12**]. You can now explain the title of this piece. The nicknames are not mine but those of John Horton Conway [**13**].

But we won't leave Kelvin quite yet:

"Blow a soap bubble and observe it. You may study it all your life and draw one lesson after another in physics about it."

What do we find when we look into an actual foam? Put a little soapy water in a large glass vessel and blow into it with a drinking straw. (A little glycerol will help stabilise the foam.) You will realise immediately how hard it is to make quantitative observations (**experiment 6**). However, the botanist E.B.Matzke, whom we've already met, did so and by the simplest means [**14**]. He used a dissecting microscope fitted with a camera lucida. With this he could pick out and draw one bubble from a mass of 1,800. The camera lucida is a simple drawing aid – I've had one since a child – consisting of a semisilvered mirror set at 45° which superimposes the scene and your drawing surface. Matzke took great care over the composition and preparation of his soap mixture, resulting in very stable foams. He also formed the bubbles individually with graduated syringes. He conducted one set of experiments with bubbles of $\frac{1}{10}\,cm^3$ and another with those of half that volume, the idea being to check that scale played no part in determining the cell shapes.

His typical bubble had 13 faces: 1 of 4 sides, 10 of 5 sides, 2 of 6 sides.

Kelvin’s theoretical cell is of course different. It has 14 faces: 6 of 4 sides, 8 of 6 sides.

However, if you’ve performed the Plateau experiment, you’ll know that the small central ‘square’ can set itself in any of 3 orientations. Spontaneous jumps into new configurations achieving local energy minima in a Kelvin foam may result in a ‘trapezium’ meeting a ‘trapezium’ to form a ‘hexagon’ at one point, but a ‘triangle’ meeting a ‘trapezium’ to make a ‘pentagon’ at another.

From this point on we’ll drop the convention of distinguishing the shapes of soap walls and cells from their flat, straight-edged cousins by putting them in quotes. I hope the case intended will be clear from the context.

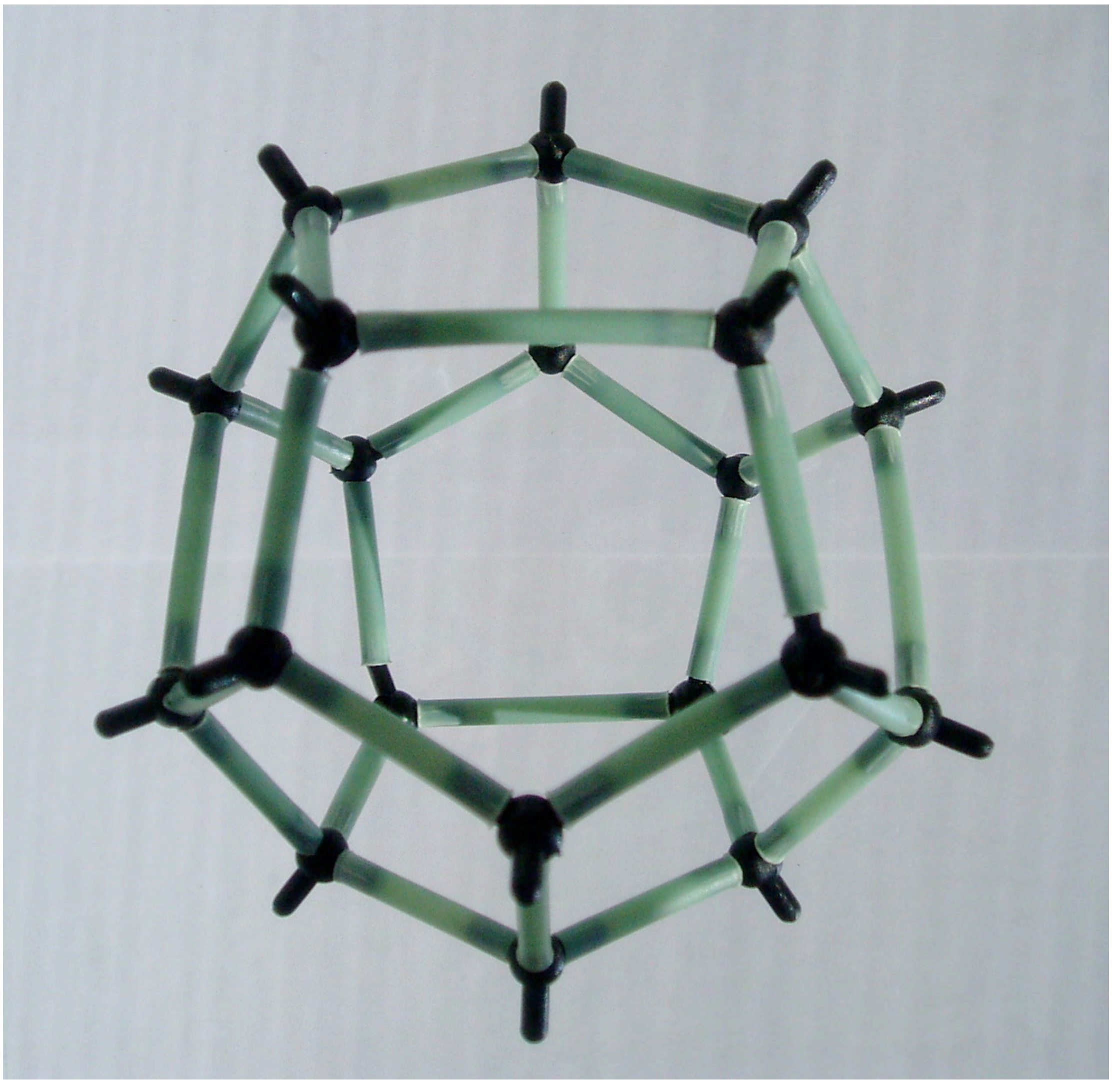
Weaire and Phelan didn’t blow bubbles. They studied the crystalline structure of certain metal alloys [**15**]. A compound has a fixed composition but an alloy is a mixture: you can vary the proportion of different components. It therefore provides a laboratory in which different packings can be observed. For the most part there are steady changes in packing structure as you vary the chemistry. But there are anomalies, warning that Weaire and Phelan may not have spoken the last word on Kelvin’s problem. With the future of your own students in mind, that’s just what you want.

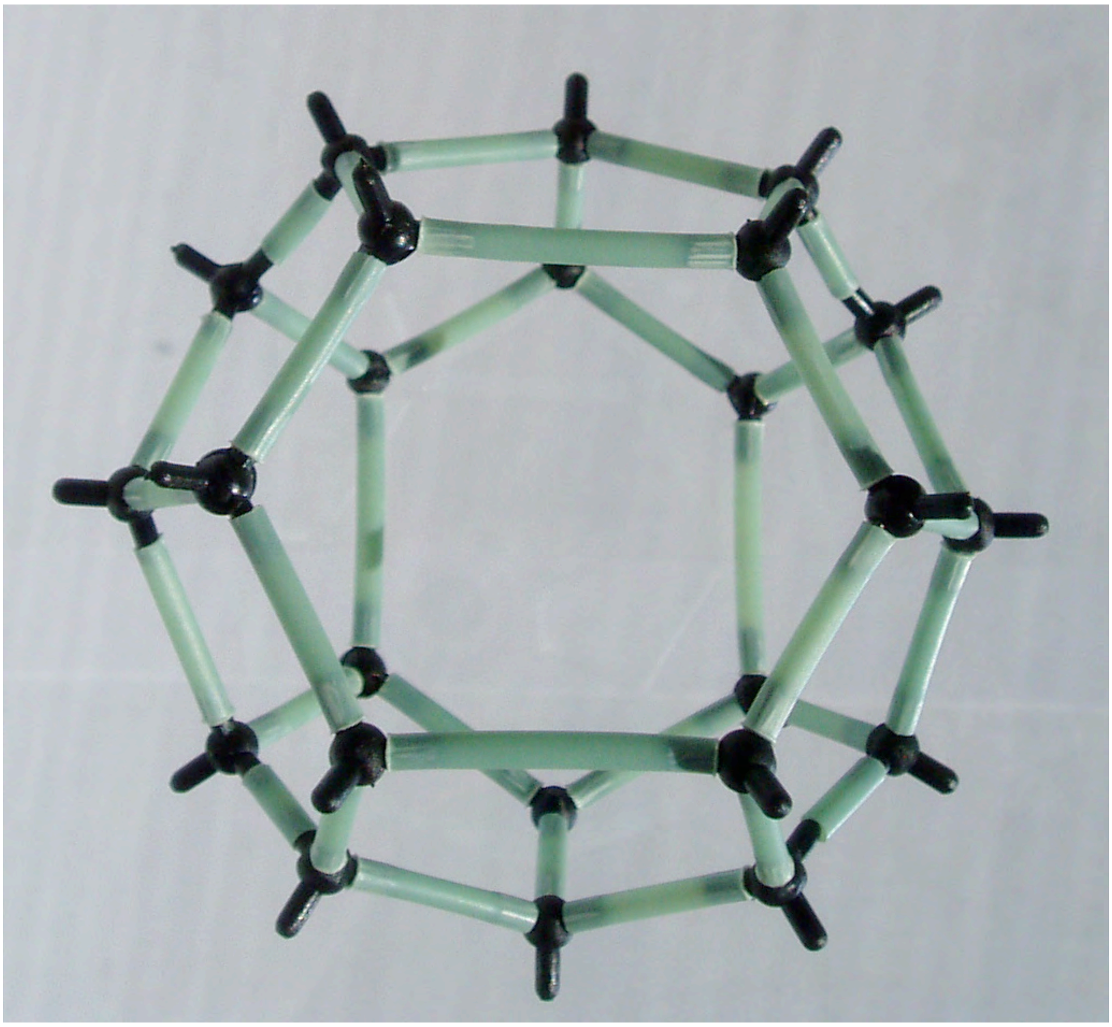
experiment 5: *our Irish bubble*

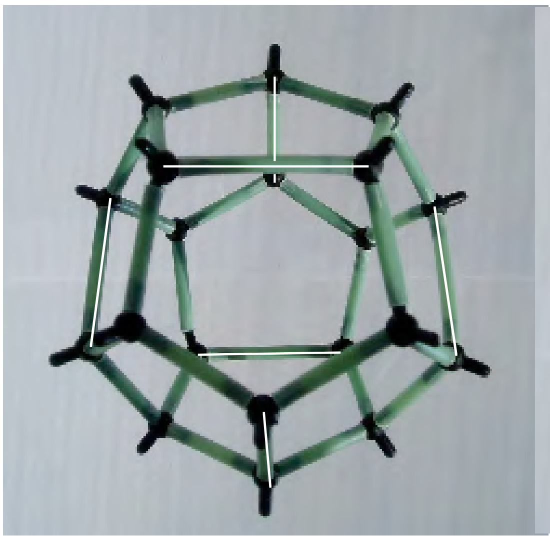
To build the Irish bubble you need to know:

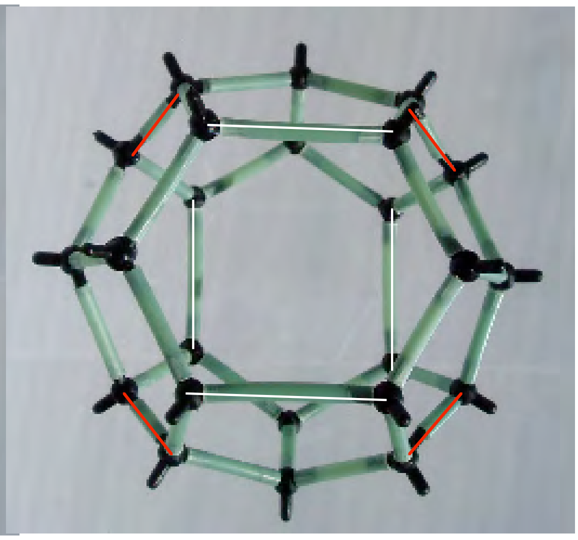
(a) The shape of each of the two cells.

Download nets for flat-faced approximations from Guy Inchbald’s site [**16**]. Here respectively are the Weaire-Phelan dodecahedron and tetrakaidecahedron modelled with Orbit:









Make measurements on Guy Inchbald's nets. You'll find the edges are of 3 different lengths: the dodecahedron uses 2 of them; the tetrakaidecahedron, all 3. The length of the straw supplied is around 29.5 mm. This (+5.0 mm for the connectors) must be your longest edge (shown white above). Though you can't aim for anything like this accuracy, you'll need to

cut the remaining 2 to: 21.8 mm (left unmarked above) and 10.3 mm (shown red. The pegs don't allow the pieces to bend in the immediate vicinity of a vertex. In the case of the shortest piece, the result is that it springs out. In the case of the complete translation unit, shown below, we've cheated by making this piece 50% longer than it should be.)
Applying simple proportion to work out the lengths is a useful exercise in itself, but more is involved:
- working out *from a net you're not going to fold* how the faces fit together (though if in doubt you can always experiment with paper);
- finding the planes of mirror symmetry and the axes of rotation symmetry for each solid:

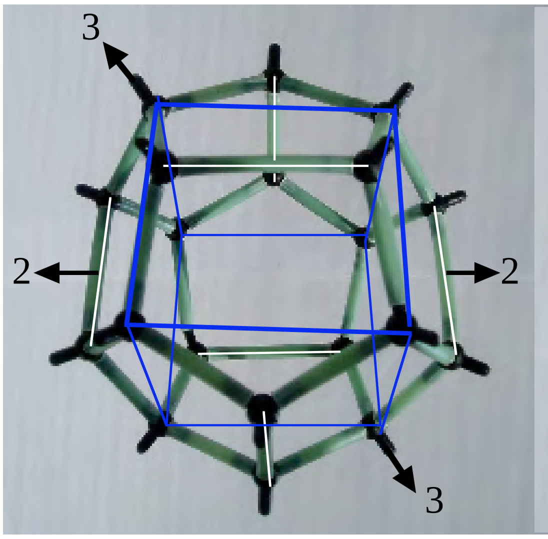
(i) The dodecahedron.

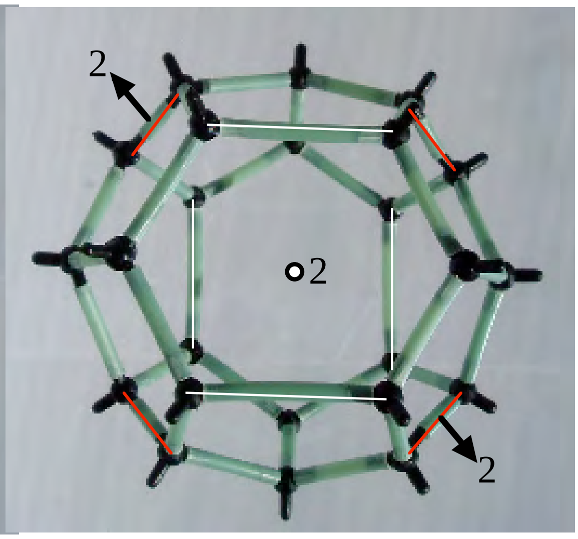
The faces of the dodecahedron are congruent pentagons. Each pentagon has one long side. An axis of symmetry runs at right angles to this. The symmetries of this irregular form and the regular pentagonal dodecahedron are readily compared by considering both as cubes with a 'house roof' on each face. In the regular form we're free to choose the 8 vertices forming the cube. Here, the long edges are the roof ridges so there's only one such cube. The space diagonals of this are the axes of 3-fold rotation symmetry of the dodecahedron. A 2-axis runs through the mid-points of opposite house roofs (and through no other edges). The long edges define 3 orthogonal planes of symmetry and (as with the regular form) lie in the faces of a cube.

(ii) The tetrakaidecahedron

The two parallel, hexagonal faces of the tetrakaidecaheron, through whose two symmetry axes run its two planes of symmetry, are set at right angles. To each long side is attached a face of the type characterising the dodecahedron. To the remaining four sides (of the common, middle length) are attached pentagons in which a long(est) side is replaced by a short(est) side, retaining the symmetry axis of the former type. These are arranged symmetrically so that the whole 'crown' belonging to one hexagon is congruent with that belonging to the other. There are three axes of 2-fold rotation symmetry: one passes through the centres of opposite hexagons; the other through the mid-points of opposite shortest edges.

Here we show representative rotation symmetry axes for the pair, and the dodecahedron's inscribed cube:



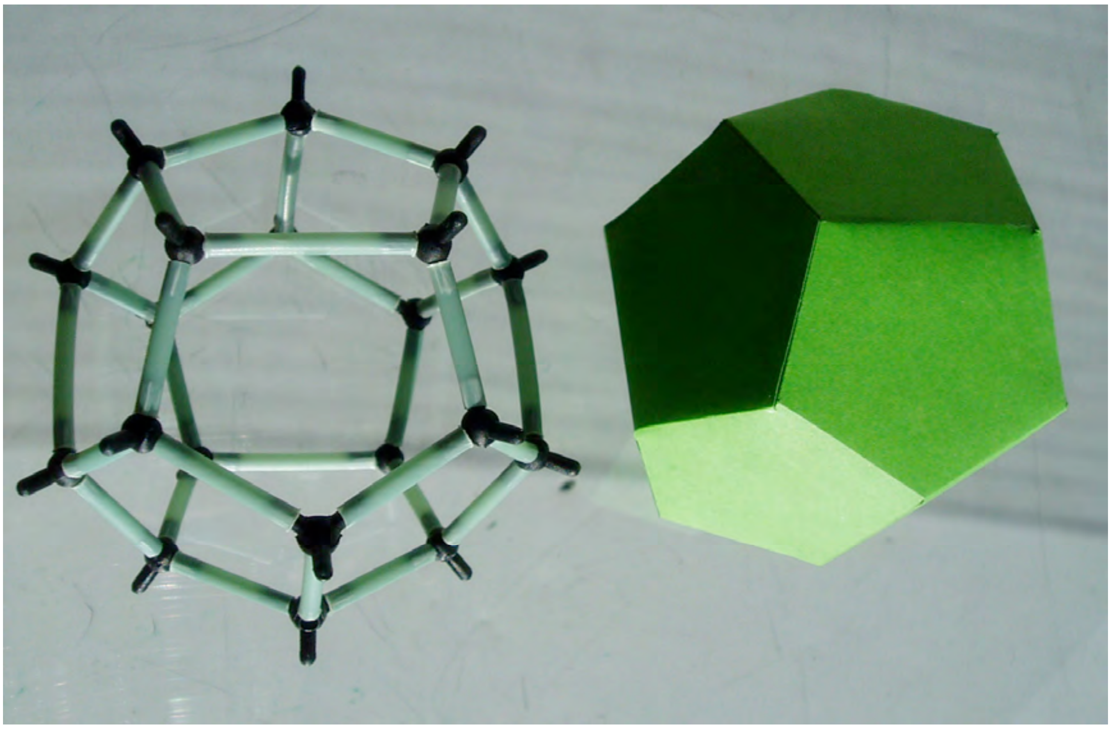


(b) How the cells fit together in the aggregate.

Ken Brakke shows you [**17**].
Brakke's name appears in [**15**] in connection with his program 'Surface Evolver'. I e-mailed Professor Brakke to ask him how it dealt with the partition of space [**18**].

You'll have to decide whether just to grow the aggregate or, more wastefully, mass-produce cells and take out common pieces where they join.

You may of course prefer not to build a skeleton structure at all but abandon Orbit for stiff paper and use Guy Inchbald's nets as intended [**19**]. Here are Orbit and paper dodecahedra for comparison:



The paper model has one advantage: your class or maths club can mass-produce cells – and do so cheaply - so that the way the aggregates of 8 fill space becomes apparent.

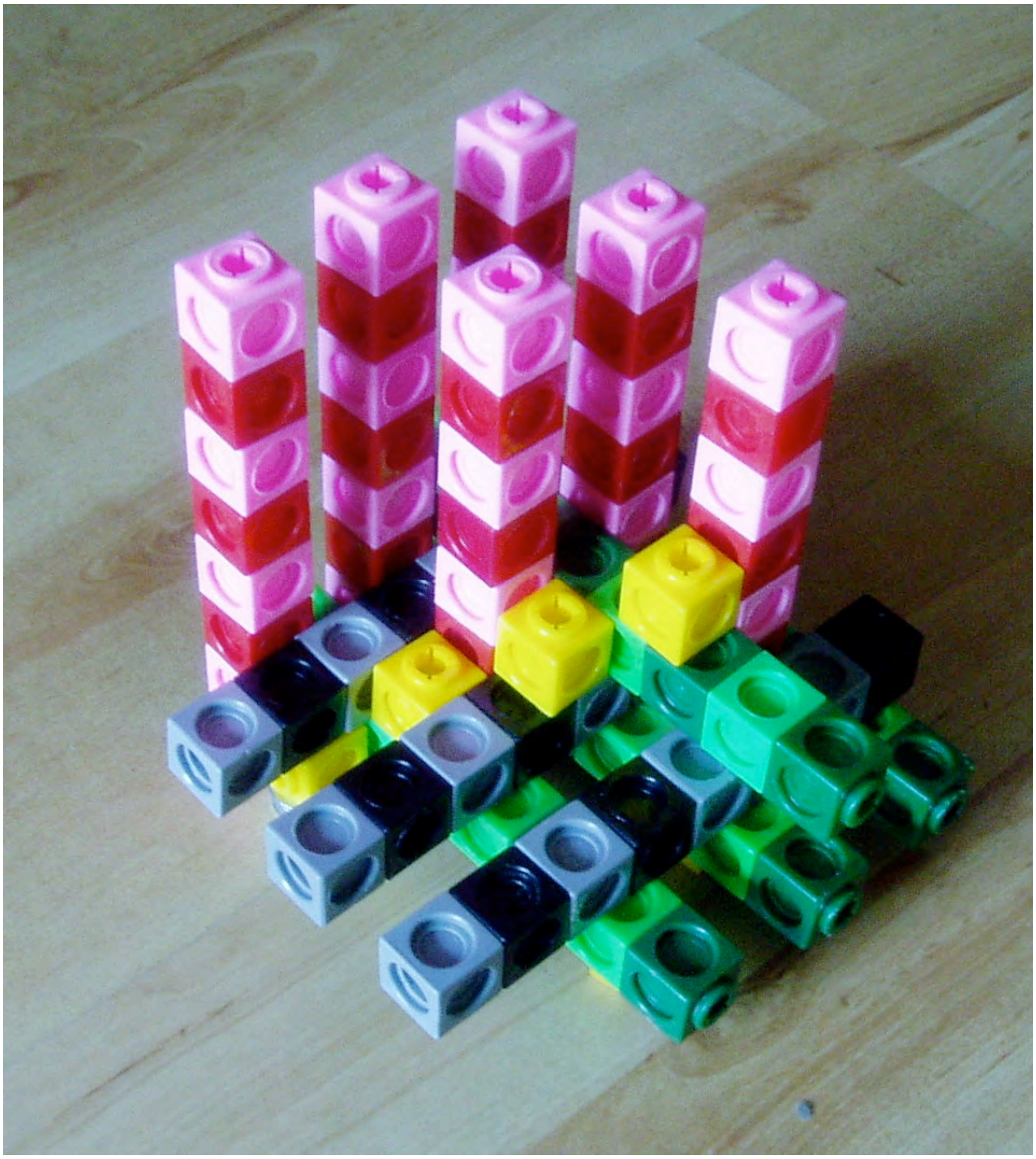
You may also, however, decide to use both materials.

There are only two kinds of vertex: in one, three edges of medium length meet one of the shortest; in the other, three edges of medium length meet one of the longest. In other words, the nearest neighbours of a given vertex form a regular tetrahedron which is either squashed along one altitude or stretched along one.

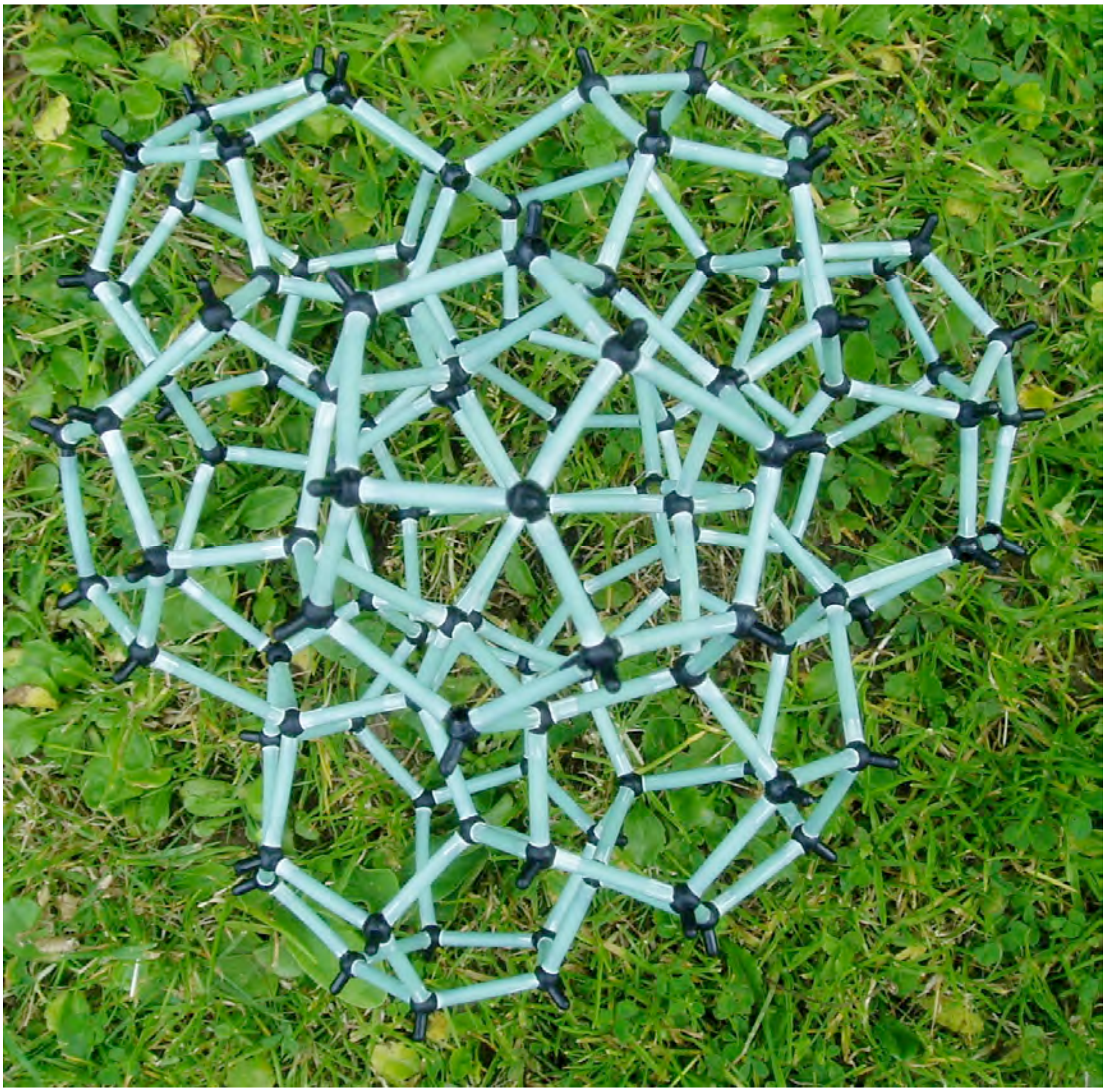
Here is the aggregate, the cells (with the approval of [**13**]) represented as cubes:

The 3 colour-pairs (black-grey, red-pink, dark green-light green) show the tetrakaidecahedron pairs, which are arranged back-to-back. In the cube edges picked out in white lie the long edges of the dodecahedron which fits there. (See **(a)**). The second dodecahedron fits behind. *You have to realise, however, that the equal cube dimensions are not to be found in the individual tetrakaidecahedra: it is the symmetry of the whole complex which allows this representation.*

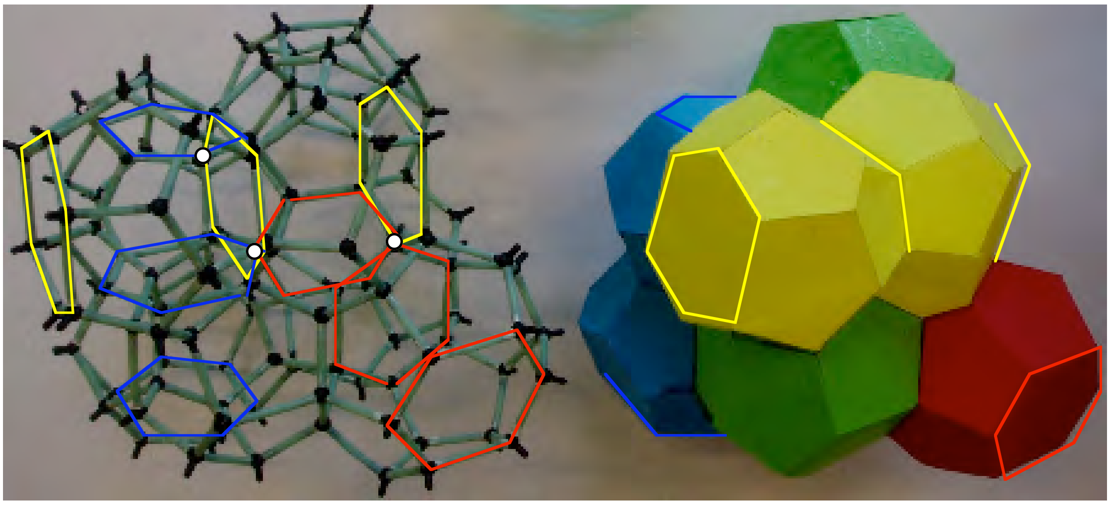
To model the whole array, all you have to do is to make many long sticks of cubes of 3 kinds: black-grey alternating, red-pink alternating, dark green-light green alternating, and pack them orthogonally, just like the individual pairs shown in the picture. These are the tetrakaidecahedra. You then imagine the dodecahedra filling the spaces (again, see [**13**]). This picture shows how the cubes representing the dodecahedra (yellow) join corner-to-corner – *though the dodecahedra themselves are separated by the shortest edge of a tetrakaidecahedron. The cubic unit cell for the whole 3-D lattice has a long diagonal equal to twice the sum of this shortest length plus the space diagonal of the dodecahedron's* *inscribed cube. It has an edge equal to the thickness of a back-to-back pair of tetrakaidecahedra.*



Here is a complete translation unit, modelled with Orbit. You are looking down an axis of 3-fold rotation symmetry, as in the (Multilink) cube model:



Here it is again, shown alongside the corresponding card model. Hexagonal faces are picked out. The white dots mark vertices on the long axes of the hexagons, where one joins another, (belonging necessarily to another tetrakaidecahedral stack). Contingent hexagons lie in perpendicular planes so that they're strung out along a line like Christmas streamers:



calculation 7: *the Scottish v. the Irish solutions*

The time has come to compare the Kelvin and Weaire-Phelan models. We do so against the statistically ideal cell dictated by the 'foam' conditions, following Stevens' treatment [**20**].

1) We suppose this cell has v vertices, e edges and f identical faces.

3 faces, each of interior angle $\approx 109.47^{\circ}$, meet in a vertex. The angle defect there is therefore $\approx 360^{\circ} - 3(109.47)^{\circ}$.

Since the total angle defect is $720°$, $v \approx \dfrac{720}{360 - 3(109.47)} \approx 22.79$.

Let the number of edges – and therefore also vertices - per face be e_f. We can write the angle sum of our polygonal face in two ways and set them equal:

$180(e_f - 2) \approx 109.47 e_f$, whence $e_f \approx \dfrac{360}{180 - 109.47} \approx 5.104$.

We have e itself by summing over the f faces, remembering that 2 faces share an edge:

$$e = \frac{1}{2} e_f f \approx 2.552 f .$$

We now substitute in Euler's polyhedron formula and solve for f:

$$v + f = e + 2 .$$

$$22.79 + f \approx 2.552 f + 2.$$

$$f \approx \frac{22.79 - 2}{2.552 - 1} \approx 13.39.$$

Note the reappearance of our near neighbour number.

From above, $e \approx 2.552 f \approx 2.552 \times 13.39 \approx 34.17$.

(Though we have applied Descartes' angle defect formula and the Euler polyhedron formula at separate stages in this derivation, the first can be derived from the second – as Euler himself did [**21**].)

2) The Kelvin solid has 6 square and 8 hexagonal faces, thus $f = 6 + 8 = 14$.

We find e just as we did above:

$$e = \frac{(6\times 4)+(8\times 6)}{2} = 36 .$$

Since 3 edges share a vertex but 2 vertices share an edge, $v = \dfrac{2e}{3} = 24$.

(We can check that number by substituting our f and e values in Euler's formula.)

3) For the Weaire-Phelan aggregate we must form the weighted mean of the two cell types:

$$f = \frac{(1\times 12)+(3\times 14)}{1+3} = 13.5 .$$

$$e = \frac{1\times\left[(12\times 5)\right] + 3\times\left[(12\times 5)+(2\times 6)\right]}{2(1+3)} = 34.5 .$$

$v = e + 2 - f = 34.5 + 2 - 13.5 = 23$.

Note that for the purpose of this analysis we have treated the two cell types as if they are distributed at random through the lattice. Their aggregation in the periodic arrangement prescribed by Weaire and Phelan has not entered into our calculation.

4) We now bring those results together:

	v	f	e	$e_f = \frac{2e}{f}$
The statistical ideal	22.8	13.4	34.2	5.10
The Kelvin model	24.0	14.0	36.0	5.14
The Weaire-Phelan model	23.0	13.5	34.5	5.11

We see that in a Weaire-Phelan foam there are rather fewer vertices, faces and edges per cell than in a Kelvin foam and that these figures fall closer to the statistical ideal.
Looking in particular at f, we note good agreement not only with our theoretical 'near neighbour number', derived first in **Ch. 2** and independently above, but also with Bernal's experimental finding (reported in **Link 2-3**).

All our averaging out conceals the discrete observation apparent from the models: the majority of the faces in the Kelvin model are hexagons, the rest squares; the *vast* majority in the Weaire-Phelan model are *pentagons* and there are *no* squares.

experiment 6: *a real foam*

Now's the time to return to your home-made foam in the glass vessel. You have to look a few layers in, not to be misled by edge effects, but two things should strike you. The first, as expected, is the adherence to the 'foam' conditions (the 'wall' condition is the more obvious to the eye). The second is the preponderance of pentagons. Clearly then real foams are not very Kelvinian. But how Wairephelanian are they?

Notes

[1] The *greatest* in terms of influence on the physics of the next century is indisputably a younger Scottish contemporary, Clerk Maxwell, but no one published as much and so widely as Kelvin. As a physicist he averaged 2 papers a month for half a century. As an *engineer* he filed 67 patents.

[**2**] Visit http://en.wikipedia.org/wiki/William_Thomson,_1st_Baron_Kelvin .

[3] The Michelson-Morley experiment suggested that, to the contrary, whatever the aether might be, it stuck to the earth. In a lecture Kelvin acknowledged that the result cast a "dark cloud" over contemporary physics. Einstein famously removed an absolute frame of reference altogether and this anomaly with it.

[4] 'On the Division of Space with Minimum Partitional Area', Philosophical Magazine, vol. 24, no. 151.

The year Kelvin published this paper was also the year he abandoned the aether theory altogether, as attested by this entry in one of his famous green notebooks regarding the Michelson-Morley experiment: "The result of the hypothesis of a stationary ether is shown to be incorrect, and the necessary conclusion follows that the hypothesis is erroneous."

For a historical survey of the aether concept and its demise go to http://en.wikipedia.org/wiki/Luminiferous_aether .

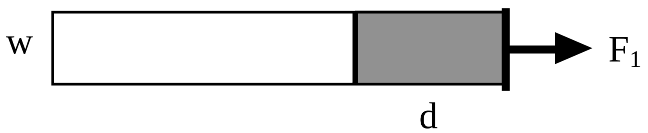
[5] calculation 8: *justifying the '2-D' foam conditions*

We need to know two things:
(a) why soap films meet at equal angles,
(b) why they do so in 3s.

(a) We work in terms of bands of soap. (If we fit spacers between a pair of parallel perspex plates, dip the model into soap solution then withdraw it, such ribbons will connect the spacers.)

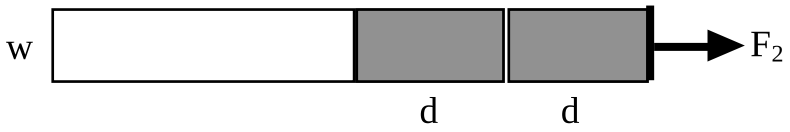
When we stretch a rubber band, the force we need increases as the band gets longer. This is not so with a band of soap.

Here is a band of soap of width w cm.
In the first diagram we pull with a force F_1 through a distance d cm.



We have added an area dw cm^2. Each cm^2 added increases the potential energy of the band. Call this increase E. We have achieved this by doing work: moving the force F_1 through the distance d. Work done = $F_1 d$ = potential energy gained = E.

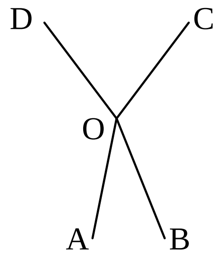
In the second diagram we pull with a force F_2 through a further distance d.
The increase in energy is just the same as before so we can write $F_2 d = E$.



But $F_1 d$ also equals E, so $F_1 = F_2$. In other words the force stays the same.

Since each band is subject to the same force, symmetry dictates that, for the edge in which they join to remain in equilibrium, the angles between adjacent bands must be equal. Thus, where 3 meet, they do so at angles of $2\pi/3$.

(b) We start from the observation that soap films minimise their surface area. Since the bands under consideration are of constant width, their total length will be a minimum. We can therefore work in terms of a cross-section parallel to the imagined perspex plates. Consider 4 bands from the edge O to spacers A, B, C, D at unit distances from it:



If 4 lines meet in a point, at least 2 angles must be $< 2\pi/3$. Take one of these, $\angle AOB = \phi = 2\theta$.

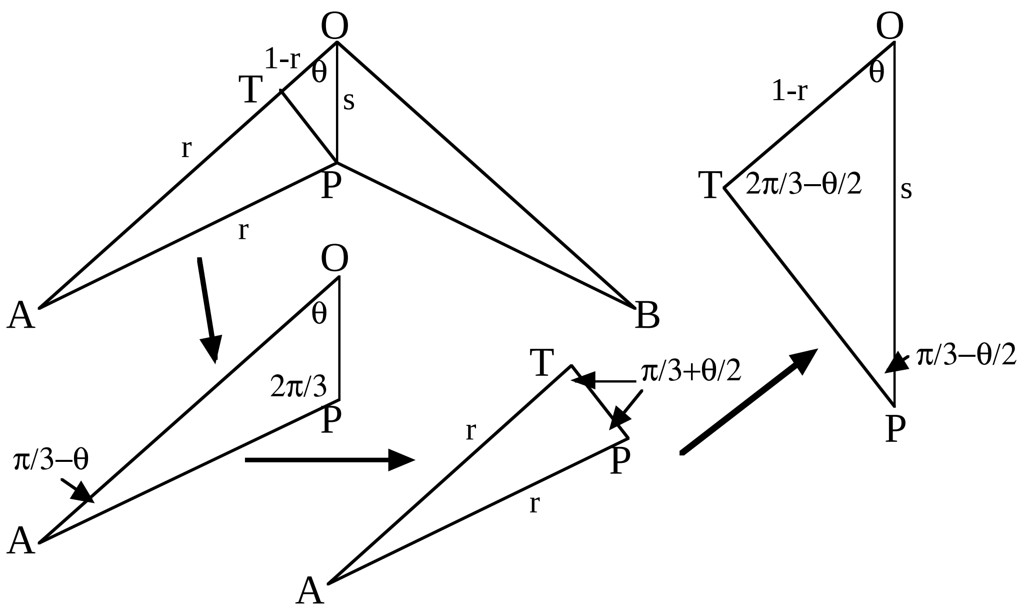
In the top left figure below we compare the sums of the lengths $AO + BO = 2$ and $AP + BP + PO = 2r + s$, where $\angle APB = \angle BPO = \angle OPA = 2\pi/3$.

Let $d = 2 - (2r + s) = 2 - 2r - s$.

If $d > 0$ we know that the soap film will abandon the configuration AO, OB and adopt the configuration AP, BP, PO, reducing the number of lines meeting in O to 3.

We can measure off r on AO, BO so that, rearranging the above expression, we have $d = 2(1 - r) - s$.

Here is the angle-chasing we must do in order to work out the angles in triangle POT. (Notice that triangle APT is isosceles.)



We now use the Sine Rule in triangle POT :

$$\frac{s}{\sin\left(2\pi/3-\theta/2\right)}=\frac{1-r}{\sin\left(\pi/3-\theta/2\right)}\Rightarrow 1-r=s\frac{\sin\left(\pi/3-\theta/2\right)}{\sin\left(2\pi/3-\theta/2\right)}.$$

Hence $d=2(1-r)-s=2s\dfrac{\sin\left(\pi/3-\theta/2\right)}{\sin\left(2\pi/3-\theta/2\right)}-s$, which simplifies to:

$$s\frac{\cos\theta/2-\sqrt{3}\sin\theta/2}{\cos\theta/2+1/\sqrt{3}\sin\theta/2}.$$

If $\sqrt{3}\sin\theta/2<\cos\theta/2$, i.e. $\tan\theta/2<1/\sqrt{3},\theta/2<\arctan 1/\sqrt{3},\theta/2<\pi/6,\phi=2\theta<2\pi/3$, then $d>0$.

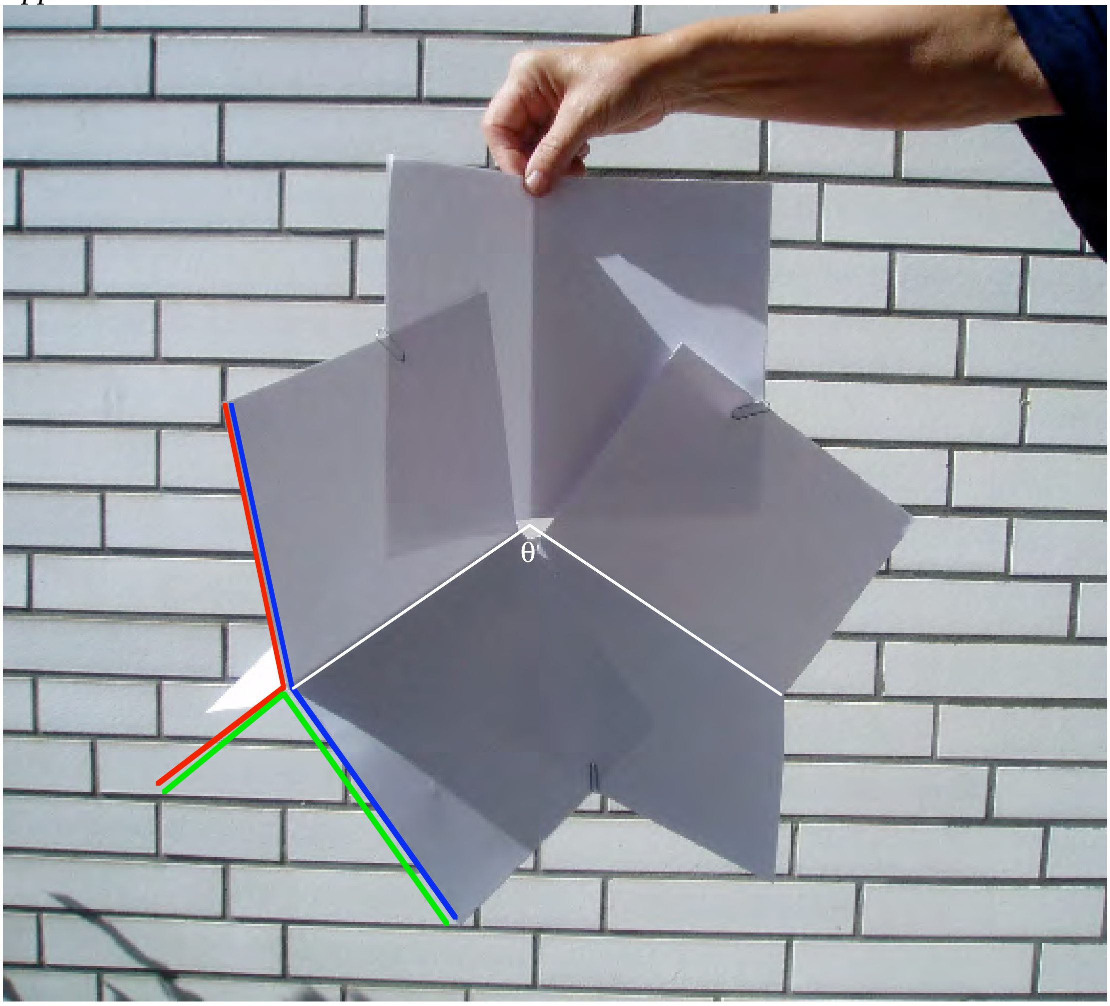
But we know that ϕ *is* $<2\pi/3$, so d *is* >0 and the soap band will adopt the configuration AB,BP,PO, reducing the number of bands meeting in O to 3.

[6] experiment 7: *justifying the '3-D' foam conditions*

We assume that what goes for the uniform bands of film in **[5]** goes for flat extended films: 3 meet in an edge at $2\pi/3$.

We can do an experiment to understand the symmetry this imposes where edges themselves meet. (You may, if you prefer, perform this as a *thought* experiment.)

Apparatus:

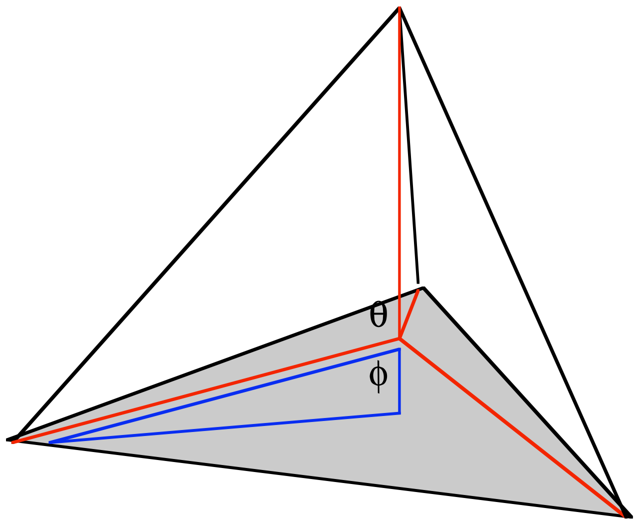


4 120° trigonal prisms made by folding 3 sheets of A4 card width-ways and sticking them together like the red, blue and green sheets shown. 6 paper clips. Each prism represents 3 walls of soap meeting in an edge.

Test: Fit 2 prisms together so that a pair of walls merge – the corresponding sheets slide over each other. You can vary the angle θ and fix the prisms together with a paper clip. But in order to fit a 3rd prism to these two you find you have to adjust θ to a special value. When you've found this, fix the 3rd prism in place with paperclips, one for each pair of overlapping sheets. You now find that the 3 prisms in place allow a 4th to be added without any further adjustment. Secure each of the 3 overlapping pairs of sheets with paper clips.

Observation: The final model has the greatest possible symmetry.
There are 6 walls. Each defines a plane of symmetry.
There are 4 edges. Each defines an axis of rotation symmetry of order 3. Each lies at the intersection of 3 planes of symmetry.
Every pair of edges makes an equal angle – about $109\frac{1}{2}°$. We shall calculate the exact value of θ.

calculation 9: *the tetrahedral angle*



The red lines divide the tetrahedron into 4 congruent pieces.
Each must therefore have $1/4$ the volume of the whole.

Since this is proportional to height, the blue height must be $1/4$ of the total height.

The blue height therefore stands to the red length as $1:3$.
This gives us the cosine of angle ϕ

and θ is the supplement of this = $\arccos\left(-1/3\right)$.

experiment 8: *the tetrahedral angle from a sheet of A4*

The first pair of figures below shows a simple way of producing this angle.

Take a sheet of A4. Fold P on to P'.
Show that the red angle is θ by inserting the folded sheet into the above apparatus.

calculation 10: *justifying the paper-folding construction*

We now find out why this works.

The last figure on the first line shows the folded sheet and the ghost of the original.
Look at the sequence of 4 figures on the second line.

First figure:
If the left-hand angle is ϕ, so is the right-hand angle by the symmetry of the fold.

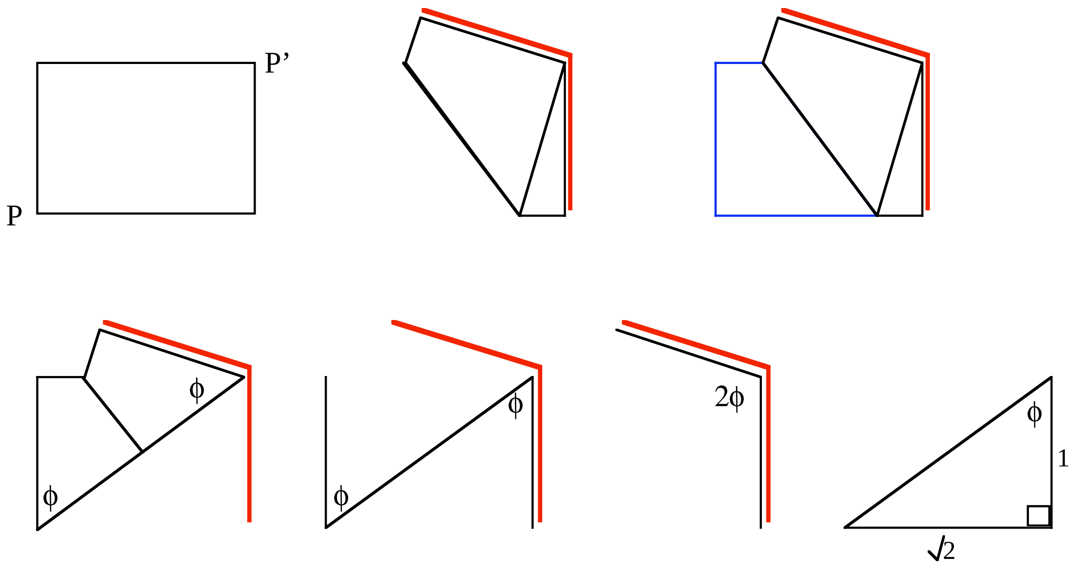
Second figure:
If the left-hand angle is ϕ, so is the right-hand angle (alternate angle between parallel lines).

Third figure:
Therefore $\theta = 2\phi$.

Fourth figure:
But from the proportions of a sheet of metric paper $\phi = \arctan\left(\sqrt{2}\right)$

$$\Rightarrow \theta = 2\phi = 2\arctan\left(\sqrt{2}\right) = \arctan\left(\frac{2\sqrt{2}}{1-\left(\sqrt{2}\right)^2}\right) = \arctan\left(-2\sqrt{2}\right) \text{ or}$$

$$\arccos\left(-\sqrt{\frac{1}{1+\left(2\sqrt{2}\right)^2}}\right) = \arccos\left(-\frac{1}{3}\right).$$



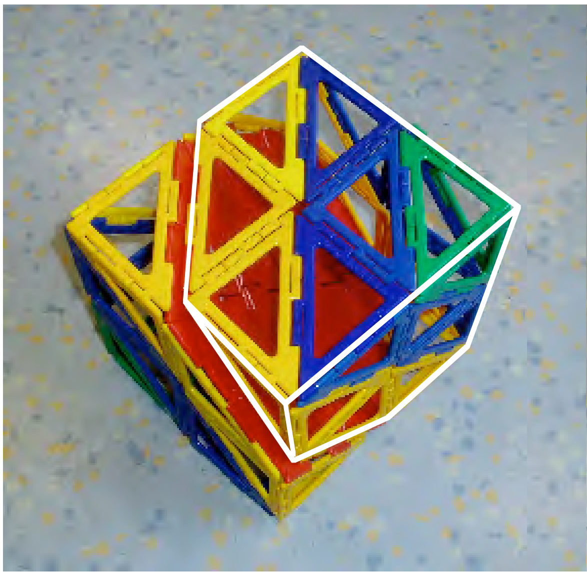
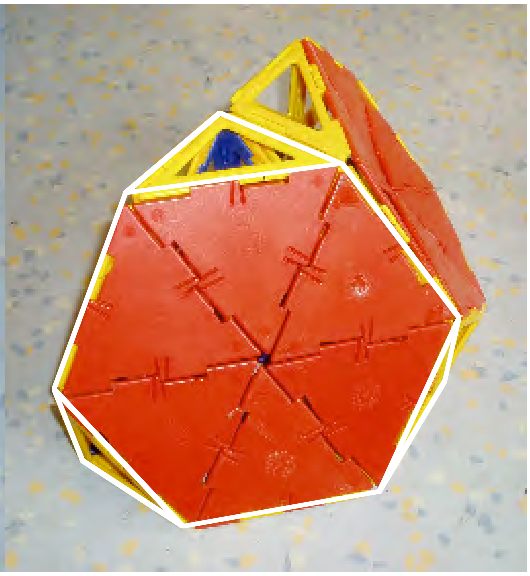
[7] The model here uses Zome Tool. Go to http://www.zometool.com .

[8] calculation 11: *isoperimetric quotients for the truncated octahedron and the rhombic dodecahedron*

Comparing the I.Q.s for the truncated octahedron itself (i.e. the one with planar faces) and the rhombic dodecahedron makes a good exercise – though we can guess from Kelvin's argument which one is going to come out on top.
The rival solids dissect into convenient shapes.

(a) The truncated octahedron

If you halve a cube with a slice through the midpoint of a space diagonal (also an axis of 3-fold rotation symmetry) and perpendicular to it, you can assemble 8 such pieces into a truncated octahedron. (You see from this dissection incidentally why the shape packs.)

Working in terms of unit cubes, we have a volume therefore of 8 x $\frac{1}{2}$ = 4 units.

The area of each triangle is $\frac{1}{8}$ of a square face = $\frac{1}{8}$ of a unit.
We have 6 x 4 = 24 of these, giving a total area of 24 x $\frac{1}{8}$ = 3 units.
Each hexagon is made up of 6 equilateral triangles, each with an edge of $\frac{1}{\sqrt{2}}$ units.

The area of each is therefore $\frac{1}{2}$ x $\left(\frac{1}{\sqrt{2}}\right)^2$ x $\sin(\pi/3)$ = $\frac{\sqrt{3}}{8}$ units.

There are 8 hexagons, giving an area of 8 x 6 x $\frac{\sqrt{3}}{8}$ = $6\sqrt{3}$ units.

So the total area = $3+6\sqrt{3}=3\left(1+2\sqrt{3}\right)$ units.

Our I.Q. $=36\pi\frac{volume^2}{area^3}=36\pi\frac{4^2}{\left[3\left(1+2\sqrt{3}\right)\right]^3}$.

The part after the '36π' = 6.66 x 10^{-3} (3 S.F.).

(b) The rhombic dodecahedron

As we have already seen, if you split a cube into 6 pyramids with apices at its centre and stick one on each face of a like cube, you have a rhombic dodecahedron:

Working in terms of unit cubes again, we have a volume this time of $1 + 6 \text{ x } \frac{1}{6} = 2$ units.

The area is 6 x 4 = 24 x that of a pyramid face.

Since each such triangle has an altitude half a face diagonal of the cube, it has an area of $\frac{1}{2}$ x 1 x $\frac{1}{\sqrt{2}}$.

The total area is therefore $24 \text{ x } (\frac{1}{2} \text{ x } 1 \text{ x } \frac{1}{\sqrt{2}}) = 6\sqrt{2}$ units.

We feed the volume and area values into our I.Q. expression and find that, this time, the part after the '36π' = $6.55 \text{ x } 10^{-3}$ (3 S.F.).

So **(a)** wins, as expected.

[9] Photograph taken in the recently-enlarged Kelvin gallery in the Hunterian Museum, Glasgow ('Kelvin Revolutionary Scientist'). Go to http://www.hunterian.gla.ac.uk/whatson/whatsOnitem.php?item=1 .

[10] Cundy, H.M. & Rollett, A.P. (1961), *Mathematical Models*, 3rd ed. 1981, Tarquin Publications, Norfolk.

[11] Go to www.cochranes.co.uk/show_category.asp?id=43 .
Choose the 'Orbit Basic Structures Class Set' (ref. 0046).

[12] See Thomas, R. (2008), 'Swimming in Mathematics', http://plus.maths.org/latestnews/sep-dec08/watercube/ .

[13] Burgiel, H., Conway, J.H., Goodman-Strauss, C. (2008), *The Symmetries of Things*, A.K.Peters Ltd., p. 332, p. 333 fig. 23.5, p. 351.

(In fact both Kelvin *and* Weaire came from Belfast.)

[14] Matzke, E.B. (1945), 'The Three-Dimensional Shapes of Bubbles in Foams', Proceedings of the National Academy of Sciences, vol. 31, no. 9, pp. 281-289.

[15] The following paper will be of equal interest to chemists, metallurgists, materials scientists and mathematicians since the authors explain all their terms as they go along: Kusner, R. & Sullivan, J.M., 'Comparing the Weaire-Phelan Equal-Volume Foam to Kelvin's Foam'. You can download it as:
http://torus.math.uiuc.edu/jms/Papers/foams/forma.pdf
simply by clicking on the relevant entry on the first Google page.

[16] Go to http://www.steelpillow.com/polyhedra/wp/wp.htm . Go to the bottom of the page and click on 'dodecahedron' and 'tetrakaidecahedron' respectively.

[17] Go to http://susqu.edu/brakke/kelvin/Kelvin.html .

[18] Here is my question and his answer:

"… Am I right in thinking your 'Surface Evolver' software is a computer equivalent to the analogue solution of such problems by soap films? …"

"… Yes, my Surface Evolver does computer simulations of soap films. On the most basic level, all you have to understand is that the surface is made up of triangles, and the total area is just the sum of the areas of all the triangles. The surface tension is a force trying to shrink the area, and Evolver just calculates how that force wants to move the triangle vertices to shrink area, with the pressure in the bubbles providing an opposing force. If you can explain to your students why soap bubbles are round, then you can explain what my Evolver is doing. …"

[19] To glue the tabs use a petroleum-based adhesive like Evostik. Let the glue get tacky so that, when you bring the surfaces together, you get an instant bond and previously glued tabs don't pop open as you proceed.

[20] Stevens, P.S. (1974), *Patterns in Nature*, Little, Brown and Company, notes to ch. 7, p. 231.

[21] calculation 12: *from Euler's formula to Descartes'*

Let a polyhedron have p faces with l sides, q faces with m sides, and so on. $f = p + q + ...$.

$$e = \frac{pl + qm + ...}{2} \ .$$

$$v = e + 2 - f = \frac{pl + qm + ...}{2} + 2 - (p + q + ...)$$

$$= \frac{p(l-2) + q(m-2) + ... + 4}{2}.$$

The interior angle sum of an $n-$gon is $(n-2)\pi$, so the total,
S, of all the interior angle sums for the polyhedron is
$\left[p(l-2) + q(m-2) + ...\right]\pi$.

We now combine our 'v' and 'S' formulas:
$2\pi v = \left[p(l-2) + q(m-2) + ... + 4\right]\pi = S + 4\pi.$

So $S = 2\pi(v-2)$.

The total angle defect, D, is the sum of the '2π's for all v vertices less S,
i.e. $D = 2\pi v - S = 2\pi v - 2\pi(v-2) = 4\pi$.

Conclusion

We have looked at three different problems.

In **Chapter 1** we investigated how far apart we could set n points on a sphere.
In **Link 1-2** we took the case $n = 12$ and worked out the near neighbour number (about 13.4), which we thought might help us with the problem set in **Chapter 2**: how tightly you can pack equal spheres. It didn't. But the number cropped up in **Link 2-3**, where we studied what happens when you crush the spheres together. This prepared us for **Chapter 3**, where we sought the partition of space into cells which had the greatest volume for their surface area.

The consensus is that the problem set in **Chapter 2**, the 'Kepler' problem, is solved.
One cannot say the same of the problem set in **Chapter 3**, the 'Kelvin' problem. Weaire and Phelan changed the rules by allowing a second cell shape and forming an aggregate of the two. Who is to say that a future mathematician may not find a superior solution by aggregating 3 or more?

Acknowledgements

Some of the theoretical ideas behind this course of activities, in particular what I'm calling here the near neighbour number, are anticipated in [**1**]. But a decade on, my interest in the topic was rekindled by Tony Robin, who made me aware of the 'Irish bubble' and reminded me of a standard measure, the isoperimetric quotient. I am also indebted to Paul Gailiunas for suggesting an alternative dissection in my 'magic number' derivation and drawing my attention to the Stevens reference (**Ch. 3, note 20**).

Notes

[1] Stephenson, P. (1999), 'Solitary Solarians and Densely Packed Spheres', The Mathematical Gazette, **83**, 498, pp. 426-432.