The Dynamic Geometry of Sightseeing: Part 1

In [Robin, 2013] Tony Robin invites us to investigate the different orders in which landmarks appear when viewed from different positions. He shows how lines of sight divide a map into regions in each of which the landmarks appear in a different order. When you cross a line, you switch (*transpose*) the 2 symbols lettering the points on it.

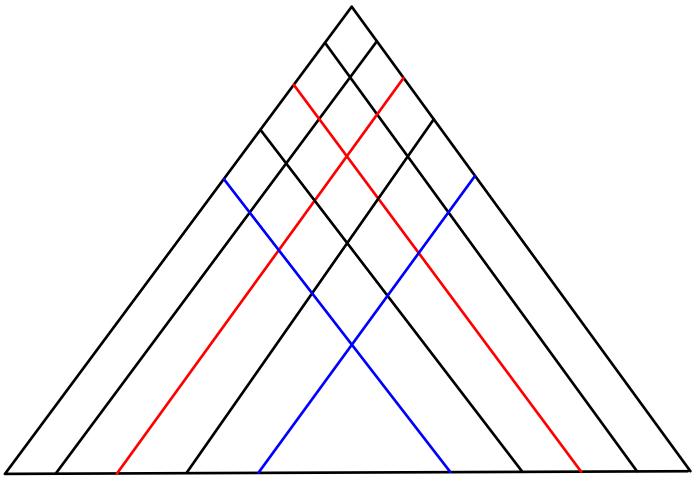
Introduction

For your KS5 students, counting the regions is an exercise in combinations.
They will only need to use these explicit formulae:

${}^{n}C_{2}$ (*n*-choose-two: how many choices of two things can be made from *n*?):
$\frac{n(n-1)}{2!}$

${}^{n}C_{4}$ (*n*-choose-four: how many choices of four things can be made from *n*?):
$\frac{n(n-1)(n-2)(n-3)}{4!}$

… and know where to find them on Pascal's Triangle:



I shall not attempt to consider cases, flagged up by Tony, where vertices lie on parallel lines, nor those where more than 2 are collinear, nor those where more than 2 lines intersect at points other than the polygon vertices.

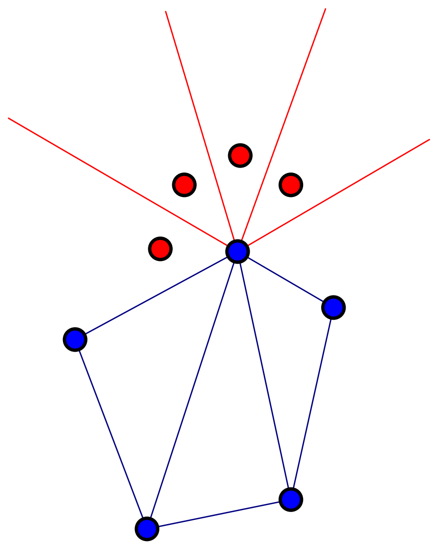
I shall have much to say about landmarks defining convex polygons but precious little about other arrangements. *Perhaps your students can find more. If not, they only need wait for the next issue of Mathematics in School, where, with Tony's help, we show how, by going back to the start and using an iterative procedure which takes no account of the precise arrangement of the landmarks, you can tackle the general case.*

For KS 2/3 children the topic would seem to offer (… meaning I haven't tried it) scope for 'people' maths. All that's needed is a playground, some volunteer 'landmarks' and a child with a camera(/mobile/i-Pad/…). Of course, the experiments will only have point if (on the basis of sketches, mental images, …) the children first make predictions. 'Fun' maths is fun in proportion to the amount of mathematical thinking it calls for.

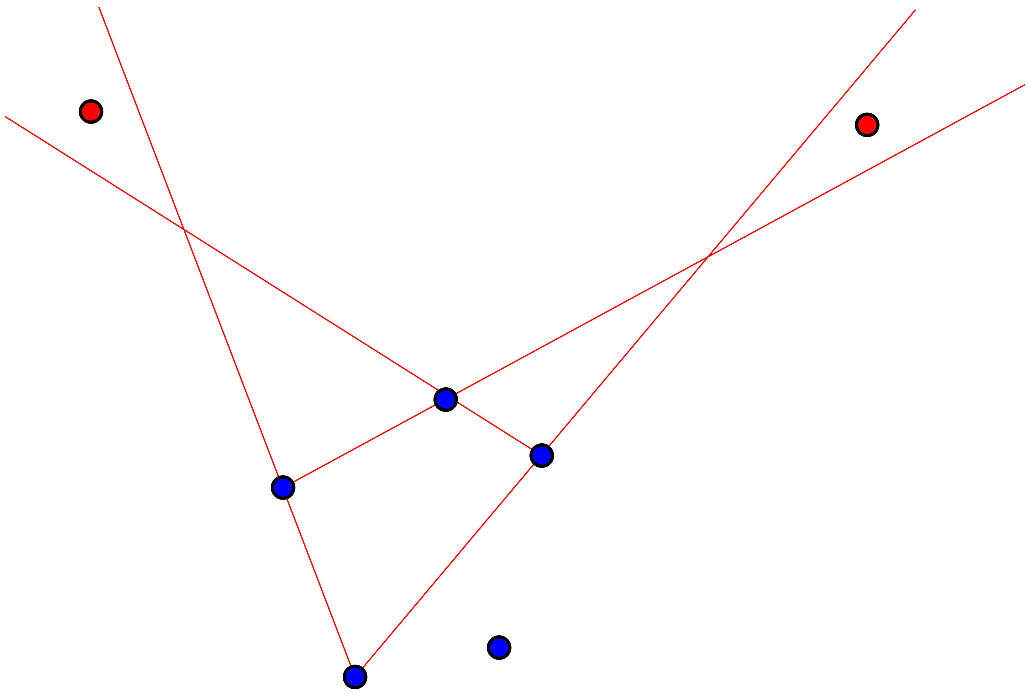
Counting the regions produced by convex polygons

$F_c(n)$

In Tony's first example the landmarks, shown here in blue, lie at the vertices of a convex n-gon. Within this we cannot see all the landmarks at once, so we must move outside. From each vertex $(n - 1)$ lines emerge, creating n 'neighbouring' regions, (those which touch the polygon at one or two vertices; other regions we call 'remote'.) Adjacent vertices share a region so the total number of neighbouring regions is $n(n - 1)$.



Each set of 4 vertices constitutes a quadrilateral. Each of the 2 pairs of opposite sides converges to create a remote point of intersection, and therefore a new region, shown in red.



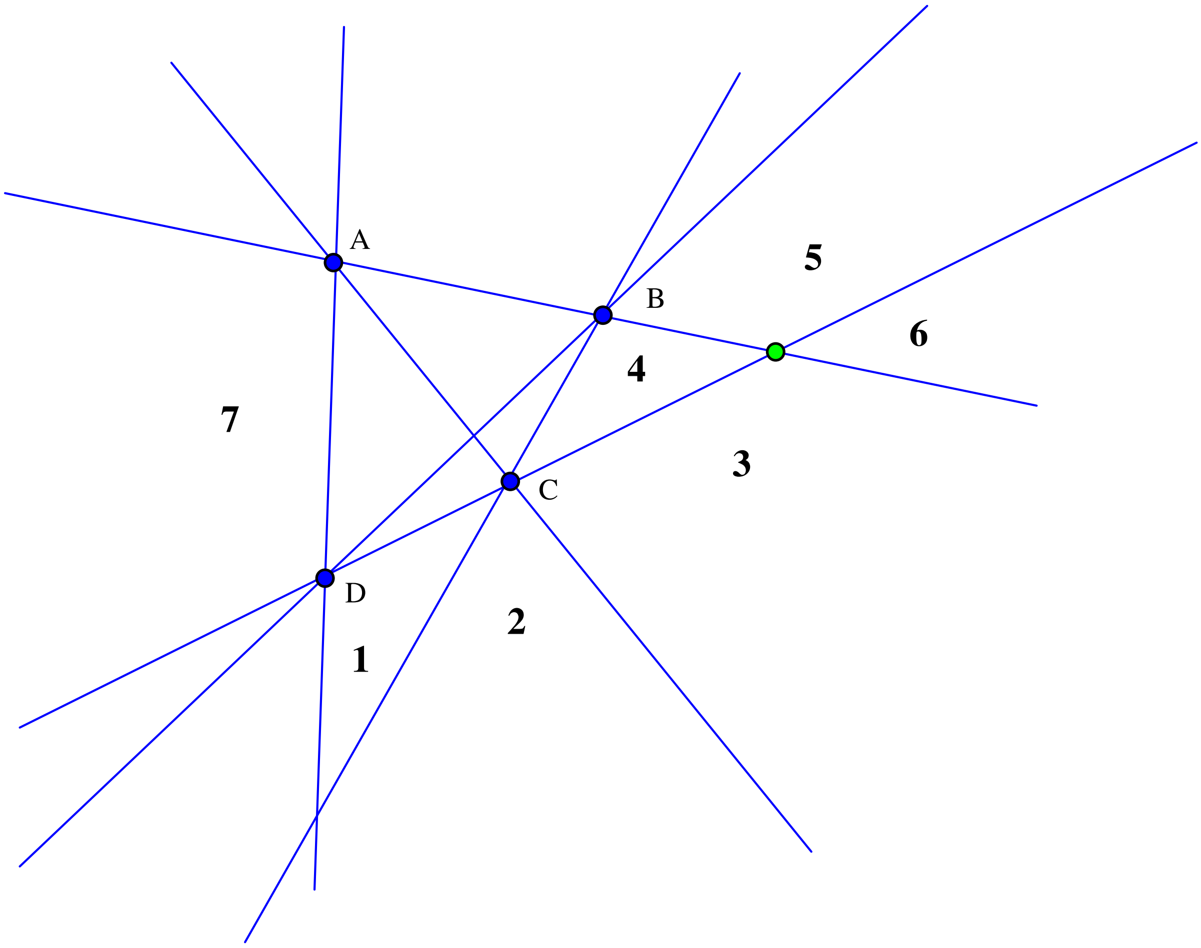
This gives $2\,^nC_4$ more regions, and the following grand total. (The *F* stands for *f*ixed camera, the suffix *c* for *c*onvex polygon.)

$$F_c(n) \ = n(n-1) + 2\,^nC_4 = 2\left[\,^nC_2 + \,^nC_4\right] \text{ regions,} \qquad \textbf{(1)}$$

and therefore orders in which the landmarks are seen.

Moving around on such maps

It's instructive to watch how Tony's algebra works as you move around.



The vertex ***C*** shares neighbouring regions ***1*** to ***4***. Moving round it anticlockwise, we have:

DABC DACB DCAB CDAB

The effect of the 3 transpositions, taking you straight across the vertex, is to cycle the sequence one place. Because we shall always have n symbols and $(n - 1)$ transpositions, the result generalises to other n-gons.

The green crossing-point shares 3 neighbouring regions, **3**, **4** & **5**, and 1 remote one, **6**. The region **7** on the far side of the polygon shares the same pair of lines. Moving from **7** to **4** to **6**, we have:

ABCD CDAB DCBA

The first operation cycles the sequence 2 places. The second transposes 2 pairs. The combined operation reverses the sequence. But I'm afraid the only general observation we can make is this: if you draw parallel lines through the vertices (east-west, say), choose one, and move far enough along it east and west respectively, the landmarks will appear in the reverse order. The effect would be the same if you printed the figure on acetate and read it from the back.

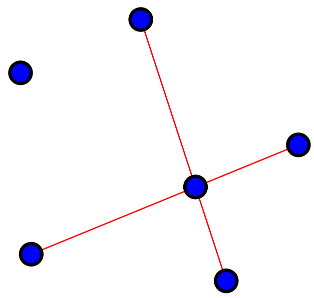
Counting the regions when the camera can make a 360^0 'panning' shot

$I_c(n)$

If the camera can observe a complete panorama, we must include the $I_c(n)$ regions ('*I*' for '*i*nterior') inside the polygon. Christian Blatter [Blatter, 2012] gives us a neat way to count these using the Euler formula. We don't want the exterior region so our …

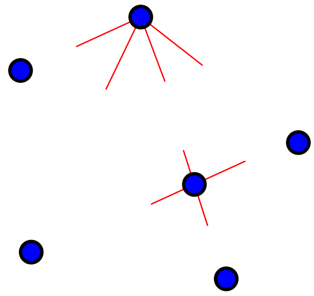
f regions = e edges – v vertices + 1. **(2)**

v comprises the n boundary points and the ${}^nC_4$ interior points - ${}^nC_4$ because each set of 4 boundary points is responsible for a pair of lines which meet in the point:



so $v = n + {}^nC_4$ points in all. **(3)**

$2e$ (the '2' because we count each edge twice, once at each end) comprises $(n – 1)$ sides and diagonals from each of the n boundary points and 4 from each of the interior points,



so $2e = n(n-1) + 4\,{}^nC_4 = 2\left[{}^nC_2 + 2^2C_4\right]$

and e = half this, ${}^nC_2 + 2\,{}^nC_4$. **(4)**

Putting **(3)** and **(4)** in **(2)** gives

$I_c(n)$ $= f =$ $e - v + 1$ $=$ ${}^nC_2 + {}^nC_4 - n + 1$. **(5)**

So, adding **(1)** and **(5)**, the grand total, $P_c(n)$ ('P' for '*p*anning' or '*p*anorama') is given by

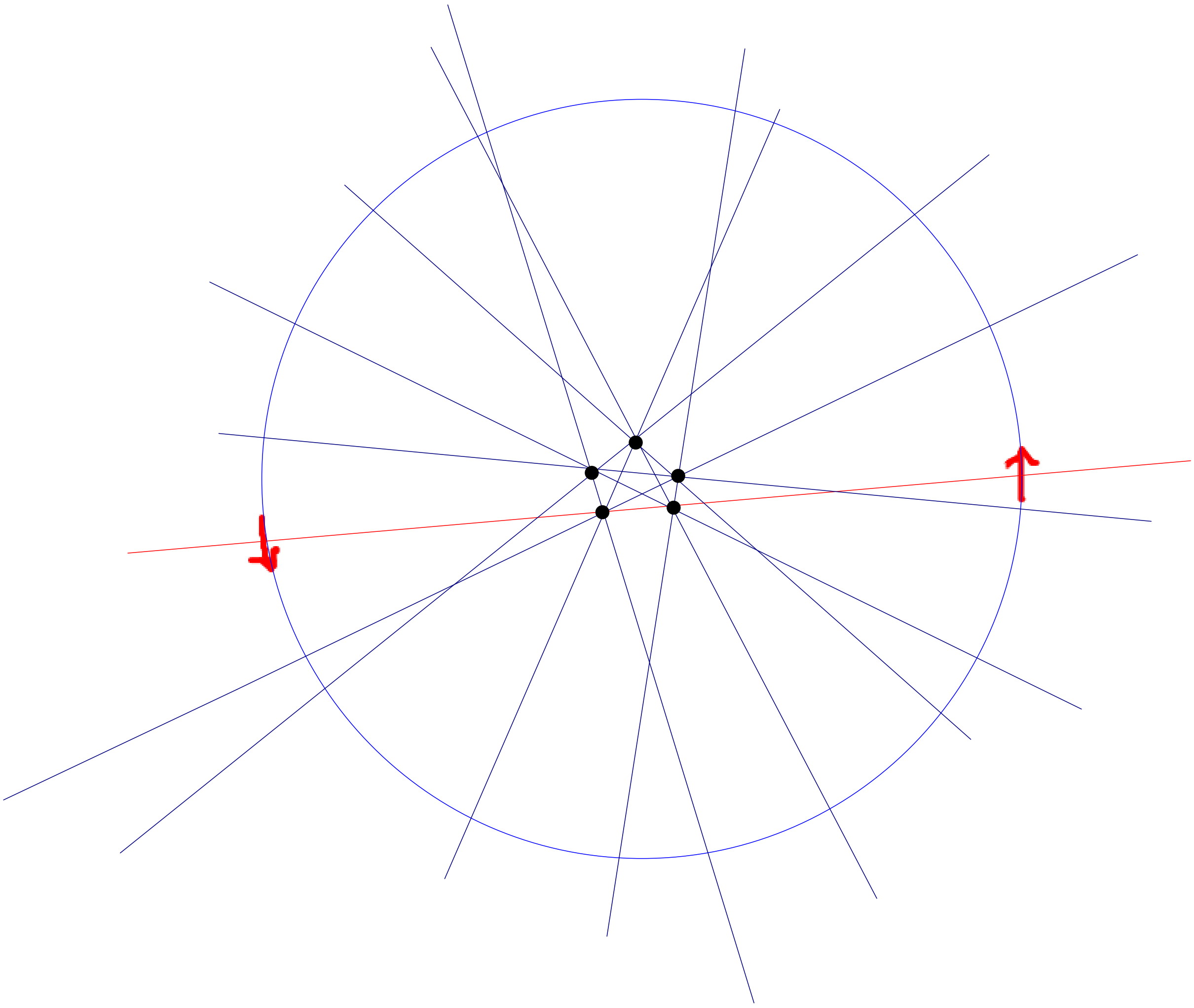
$P_c(n) = F_c(n) + I_c(n) = 3[{}^nC_2 + {}^nC_4] - n + 1$. **(6)**

Notice these relations:

There is 1 interior point for every 2 remote points, and therefore remote regions.
There is 1 polygon edge (side or diagonal) for every neighbouring region.
$F_c(n) = 2I_c(n) + 2(n-1)$.
$P_c(n) = 3I_c(n) + 2(n-1)$.

If the vertices are concyclic, and you add to $I_c(n)$ the n regions between the inscribed polygon and the circle, you have the formula for the number of regions in a circle produced by joining n points lying on it, ${}^nC_2 + {}^nC_4 + 1$, notorious because the sequence runs 1, 2, 4, 8, 16, 31, … .

One last observation. Imagine moving so far out from the polygon that, when you make a complete circuit, every region you pass through is infinite (unenclosed). Each time you cross a line, you enter a new region. But you cross each line twice. There are therefore twice as many infinite regions as lines. Since there is a line through each pair of vertices, there are ${}^nC_2$ lines, therefore $2\,{}^nC_2$ infinite regions. That the number of infinite regions and the number of neighbouring regions is the same, is no coincidence. The lines bordering a neighbouring region either diverge, thus producing an infinite region, or converge, thus producing a remote region, and remote regions are infinite.



Your students may check counts given by the formulae against Tony's map for n = 4 and their own for bigger n.

n	1	2	3	4	5	6	…
$F_c(n)$	0	2	6	14	30	60	…
$I_c(n)$	0	0	1	4	11	25	...
$P_c(n)$	0	2	7	18	41	85	…

Counting the regions produced by polygons, whether convex or not

But the landmarks may not be so obedient as to form a convex polygon, and non-convex polygons are lawless: there may be many ways to join the dots. Thus we come to Tony's second example.

We'll adopt a couple of approaches to the problem, hoping to find a property that will generalise.

We drop the suffix 'c' to show that we're dealing with polygons in general.

1^{st} Approach

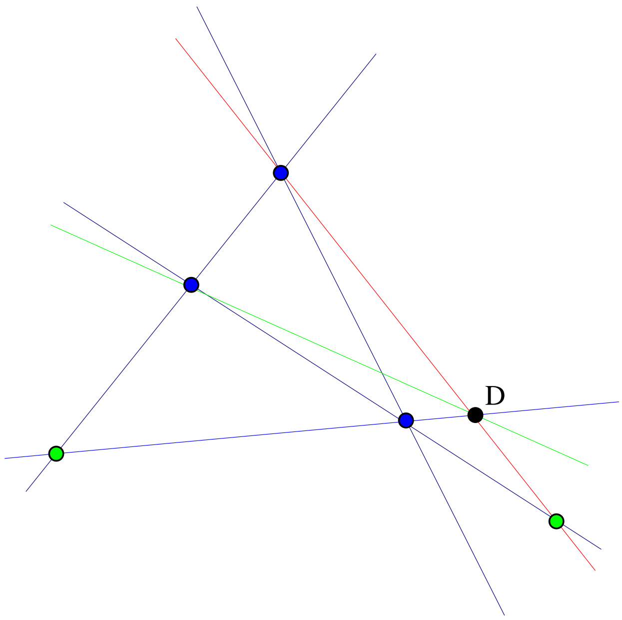
With 3 landmarks there are no remote regions and all 3! = 6 views are represented.
Put a 4^{th} point *D* inside the triangle *ABC* and imagine making a complete circuit of the group. *D* is always framed by the other letters so cannot appear to left or right but must fall between both the first pair and the second:

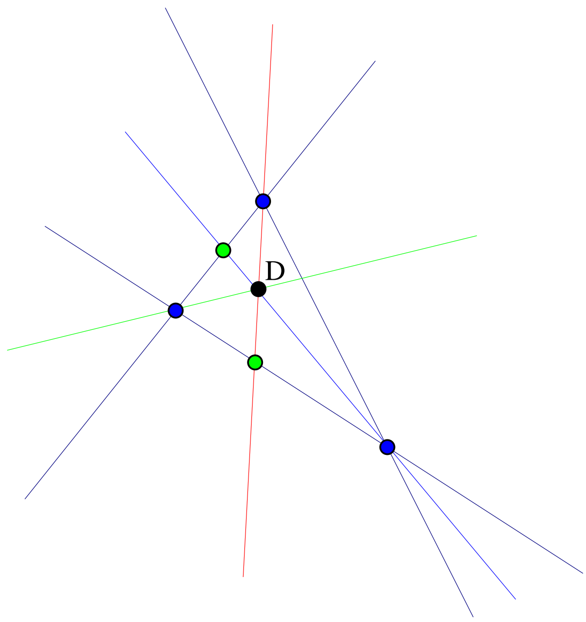
A D B C	*B D C A*	*C D A B*	*C D B A*	*B D A C*	*A D C B*
A B D C	*B C D A*	*C A D B*	*C B D A*	*B A D C*	*A C D B*

This gives us a check on the number arrived at by another method but this is all, since we know from Tony that not all $n!$ orders are represented for $n > 3$ [note **1**].

2^{nd} Approach

Using Geometer's Sketchpad, construct the first, convex figure but then drag one vertex, *D*, across a diagonal, thus producing a non-convex quadrilateral. Colour the lines passing through *D* so that you can see what happens to them. Here we colour the 2 remote crossing points green.





What we notice is that the green points now lie in the sides of the polygon. We know they must appear somewhere because *opposite* sides are defined for all quadrilaterals, whether convex or not: they're the sides which don't share a vertex. So our $2\,{}^{4}C_{n}$ so-called 'remote' points will always contribute to $F(n)$ and $P(n)$.

But what happens to the $n(n-1)$ neighbouring regions? This is the point at which I get stuck. But there's a prior question: *Is* there a unique $F(n)$ and $P(n)$? *I(n)* certainly has no meaning beyond $n = 4$. Your children may, like me, draw a number of sketches without ever being sure that they are not – in whatever way – special cases.

Clearly we must find a different approach altogether. This is what, with Tony's help, we will do in the next issue.

Photogrammetry

These questions are interesting but academic. However, if we ask questions like, "What do these photographs tell me about the landscape?", i.e. "How do points on the photograph map on to points on the ground?" (a 1-to-many mapping since we're going from 2 dimensions to 3) or "What programs can I apply to reconstruct the landscape from my photographs?", we're into the important science of *photogrammetry*. Going back sixty years, the best you could do was take photographs from two positions and view the pair in a stereoscope. As you can imagine, the computer has changed all that. Nevertheless, the geometry behind photogrammetry and traditional surveying is the same [note **2**].

References

Blatter, Christian, www.math.stackexchange.com/questions/241887

Robin, A. 2013 'The Different Orders of Landmarks on a Photograph', *Mathematics in School*, **42**, 5, p. 27.

Notes

1. There is one case where they are, but the landmarks move! Following Galileo, you can see the 4 biggest moons of Jupiter with quite a small telescope, strung out along a line. Since there is no rational relation between their angular speeds round the planet, they can appear in any order.

2. Look for instance at: www.geodetic.com/v-stars/what-is-photogrammetry.aspx .

Keywords: Combination; Transposition

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The Dynamic Geometry of Sight-seeing: Part 2

This short series was prompted by [Robin, 2013], in which the following question was posed: *In how many different orders do n landmarks appear when viewed from all possible positions?* In Part 1 answered the question for cases where the landmarks formed a convex polygon. I distinguished the $F_c(n)$ views possible with a fixed camera and the $P_c(n)$ views where the camera pans through $360°$.

But my argument cannot be applied to the $F(n)$ and $P(n)$ cases respectively where the polygon is not necessarily convex. In this second part I am helped by Tony to show how, by adding vertices one at a time, $P(n)$ may be found.

Introduction

The argument is illustrated for the case where a 5^{th} (green) vertex is added to the existing 4.

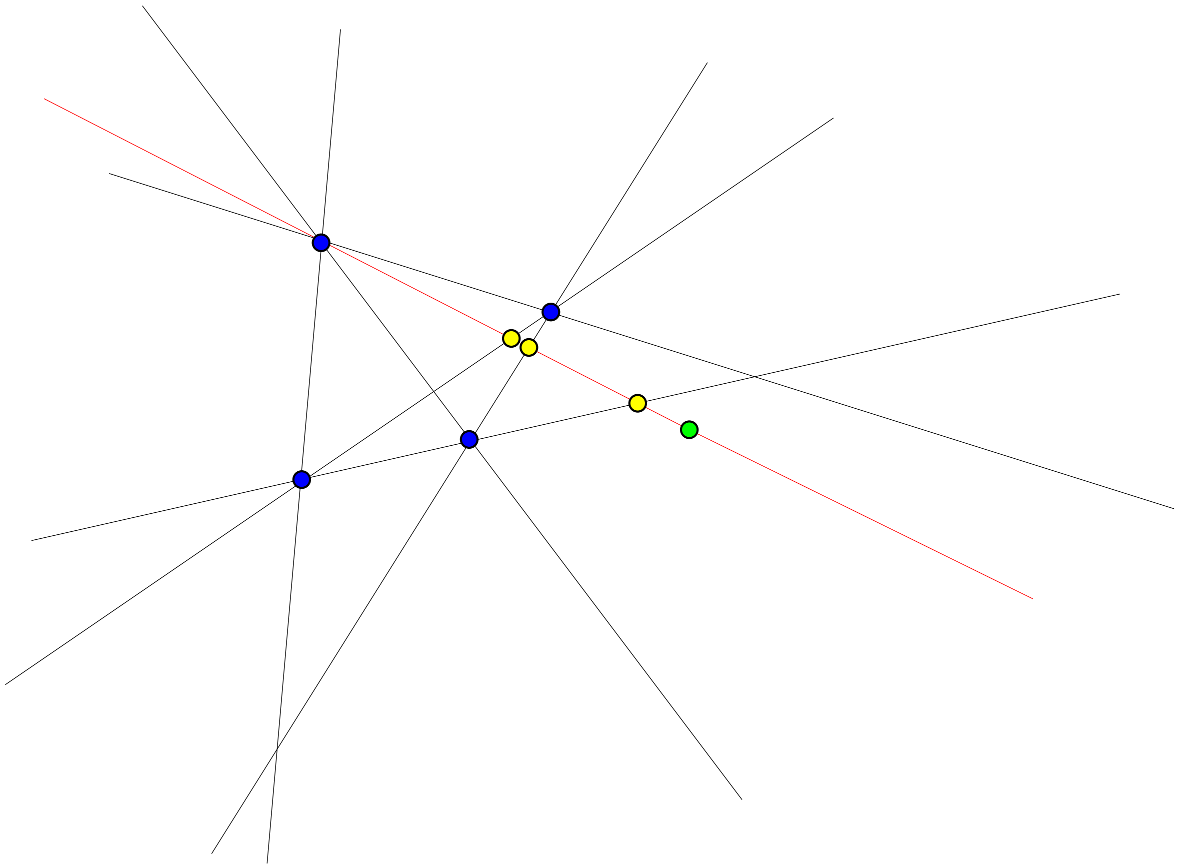
Your children may like to draw the successive diagrams using Geometer's Sketchpad. On the left we show the count for our chosen example; on the right, the count for the general case.

The effect of adding the first line from the new point

Since a line passes through each pair of vertices, there are

${}^4C_2$ lines so in general: ${}^nC_2$ lines.

We now draw the red line from the new point to a chosen vertex.



It crosses the

lines so in general: lines

except that 3 so in general: $(n-1)$

all pass through the same vertex. The number of new crossing points, shown in yellow, is therefore

${}^4C_2-(4-1)={}^4C_2-4+1$ so in general: ${}^nC_2-(n-1)={}^nC_2-n+1$.

For the total number of crossing points we must add the old (blue) point, giving

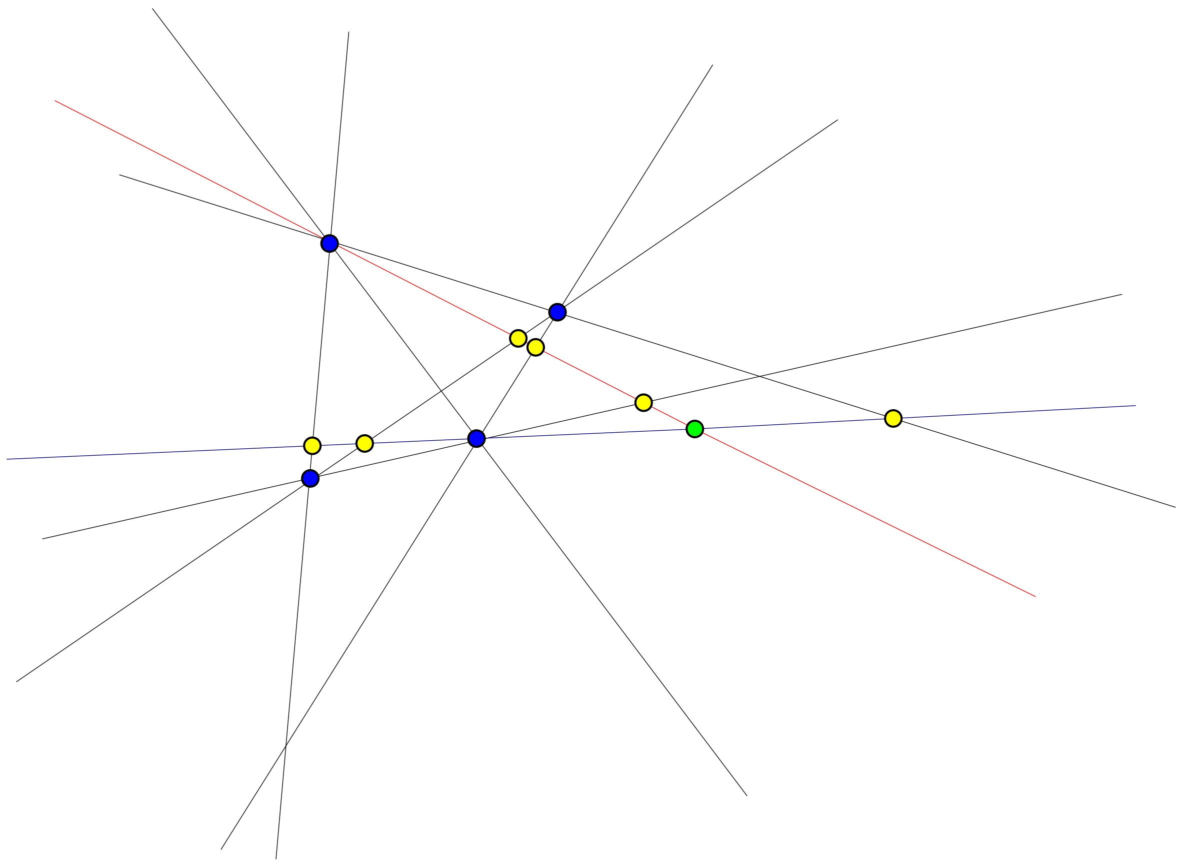
${}^4C_2-4+2$ so in general: ${}^nC_2-n+2$.

These divide the line into

${}^4C_2-4+3$ segments so in general: ${}^nC_2-n+3$ segments.

Each segment creates a new region. So the number of new regions is the same as the number of segments.

The effect of adding the second and subsequent lines



For the line through the second vertex and all following, the green point itself must be counted. This increases by 1 the number of regions added for this and all the remaining

4 – 1 lines so in general: $n - 1$ lines.

The grand total of new regions for which the red point is responsible is therefore

$({}^4C_2 - 4 + 3) + 3\left({}^4C_2 - 4 + 4\right)$ so in general:

$$({}^nC_2 - n + 3) + (n-1)({}^nC_2 - n + 4) = n\,{}^nC_2 - n^2 + 4n - 1.$$

Substituting $\dfrac{n(n-1)}{2}$ for ${}^nC_2$, we have the difference between $P(n)$ and $P(n + 1)$:

$$\frac{n^3 - 3n^2 + 8n - 2}{2}. \qquad \textbf{(7)}$$

Finding the *P*(*n*) values one at a time

Starting with P(1) = 0 regions for $n = 1$ point, we can find each $P(n)$ value by substituting for n in **(7)** and adding that number:

n	$P(n)$	$P(n+1) - P(n)$
1	0	
		2
2	2	
		5
3	7	
		11
4	18	
		23
5	41	
		44
6	85	

Using the method of finite differences to find a formula

Though we have a formula for $P(n + 1) – P(n)$, we don't yet have one for $P(n)$. What we do is take our table of values and keep subtracting pairs as we move to the right. We have to make 5 difference columns in this way before we get down to zero. This tells us that the highest power in the polynomial we seek is 4: $P(n) = an^4 + bn^3 + cn^2 + dn + e$.

n	Δ_0	Δ_1	Δ_2	Δ_3	Δ_4	Δ_5
1	0 $a+b+c+d+e$					
		2 $15a+7b+3c+d$				
2	2 $16a+8b+4c+2d+e$		3 $50a+12b+2c$			
		5 $65a+19b+5c+d$		3 $60a+6b$		
3	7 $81a+27b+9c+3d+e$		6 $110a+18b+2c$		3 $24a$	
		11		6		0

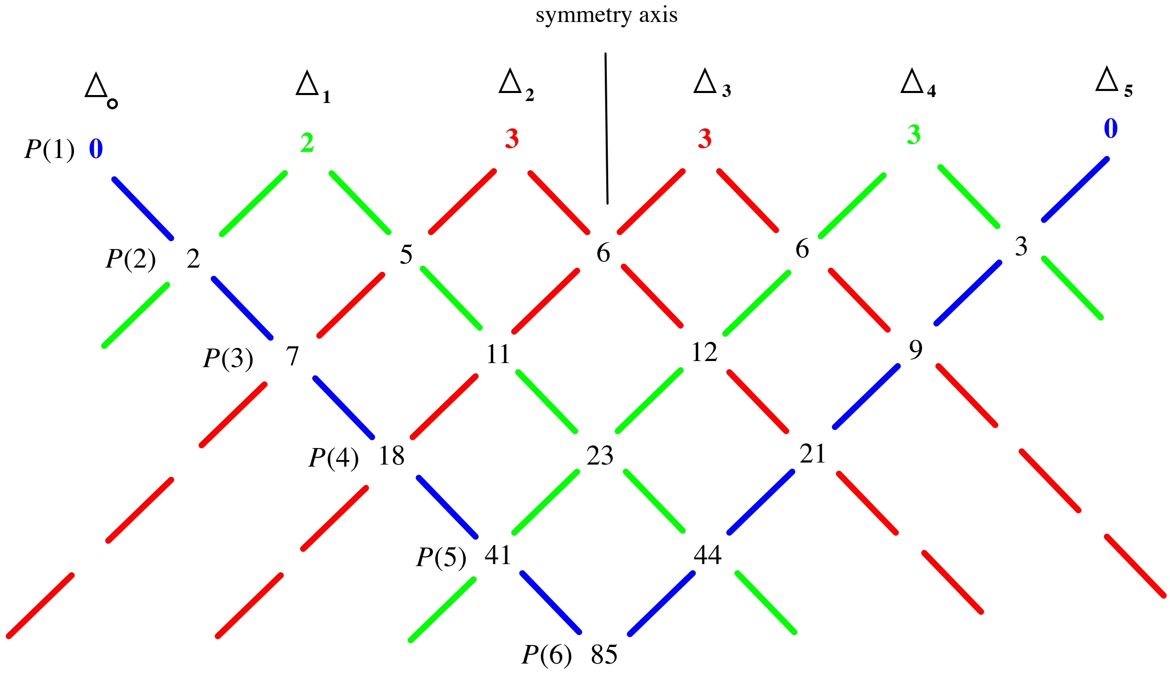
		$175a+37b+7c+d$		$84a+6b$	
4	18 $256a+64b+16c+4d+e$		12 $194a+24b+2c$		3 $24a$
		23 $369a+61b+9c+d$		9 $108a+6b$	
5	41 $625a+125b+25c+5d+e$		21 $302a+30b+2c$		
		44 $671a+91b+9c+d$			
6	85 $1296a+216b+36c+6d+e$				

To each number in our difference table corresponds an expression obtained by the same process of successive subtraction as we move to the right. Our final subtraction gives us an equation in *a* alone. We then move to the left. We have an equation in *a* and *b* but can now substitute for *a*. And so we proceed, till we have the values for *a*, *b*, *c*, *d* and *e*:

$$P(n)=\frac{n^4-6n^3+23n^2-26n+8}{8}. \qquad \textbf{(8)}$$

That's fine, but we can't leave our difference table without advertising a second approach, (which readers will know from the books as 'Newton's forward difference method'), because it provides important and interesting extension work on Pascal's Triangle for A-level students.

The head of each column in a difference table is the apex of a Pascal Triangle weighted by that number. The entries sum the overlapping cells. This is clearer if we reorientate the table:



For example, $85 = 0({}^5C_5)+2({}^5C_4)+3({}^5C_3)+3({}^5C_2)+3({}^5C_1)+0({}^5C_0)$ or, observing symmetry, $(0+0)({}^5C_0)+(2+3)({}^5C_1)+(3+3)({}^5C_2)$.

Generalising:
$$P(n) = 0({}^{n-1}C_5)+2({}^{n-1}C_4)+3({}^{n-1}C_3)+3({}^{n-1}C_2)+3({}^{n-1}C_1)+0({}^{n-1}C_0)$$
$$= \frac{n^4-6n^3+23n^2-26n+8}{8}.$$

Proof by induction

Because we are sure of our starting value, and sure of the number we have to add to obtain each new value of $P(n)$, we are sure that the expression we have derived is correct. However, it is good mathematical form at this point – and a good exercise for your senior students – to construct a proof by induction.

Taking stock

We already had formulae for $F_c(n)$, $I_c(n)$ and $P_c(n)$. We now also have one for $P(n)$. If you substitute for ${}^nC_2$ and ${}^nC_4$ in equation **(6)** from **Part 1**: $3\left[{}^nC_2+{}^nC_4\right]-n+1$, you will find (as you may have already suspected from Tony's example in [Robin, 2013]) that $P_c(n)$ and $P(n)$ are one and the same.

I leave you and your students with this question: *Is there a meaningful formula for F(n)?*

References

Robin, A. 2013 ‘The Different Orders of Landmarks on a Photograph’, *Mathematics in School*, **42**, 5, p.27.

Stephenson, P. ‘The Dynamic Geometry of Sight-seeing: Part 1’

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