Can a triangle be rep-6?

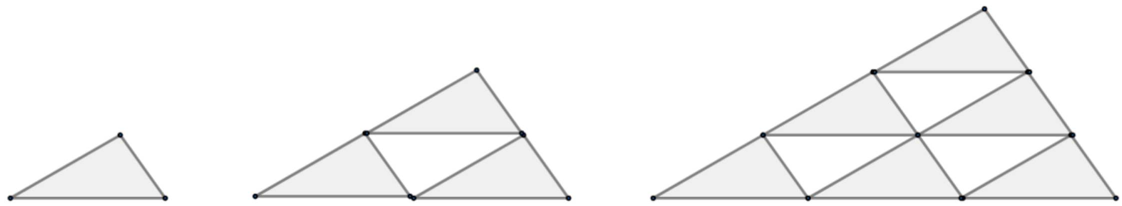
In this little investigation I've made up my own terms and my own notation. I hope students will realise from this that they have carte blanche to do the same – always provided they define them. The work is suitable for KS 3/4 pupils comfortable with exponents and the concept of similarity. Answers to the questions follow the piece.

A rep-tile of order *n* is a figure which can be divided into *n* congruent copies similar to itself, when it is described as 'rep-n'. For example, a metric paper sheet can be rep-2, -4, -8, etc.

In the case of the triangle, we may distinguish three sequences of shapes with their attendant *n* values. We shall give each member of a sequence a symbol. The symbol expresses how many copies of the unit triangle that particular triangle contains. The suffix i or *j*, a positive integer, is the ratio of an edge length to the corresponding edge length in the unit triangle. We shall use the same symbol for the shape and the number. For example, in the first sequence, the general term is $a_j = j^2$, so $a_3 = 9$, the third figure in the sequence contains 9 unit triangles, and 9 is a possible *n* value.

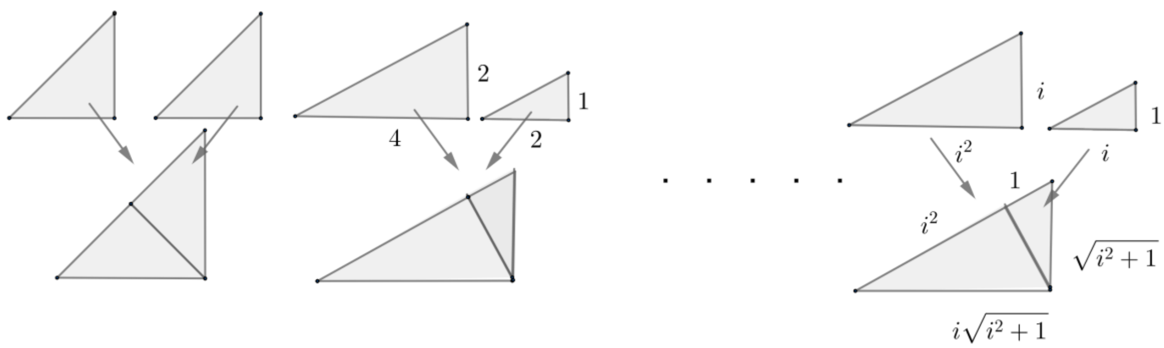
1. We begin with the *primitive* sequences. Sequence *A*, which includes all triangles, comprises all the square numbers, so the general term in the sequence $a_j = j^2$, as stated above.

a_1 $\qquad$ a_2 $\qquad$ a_3

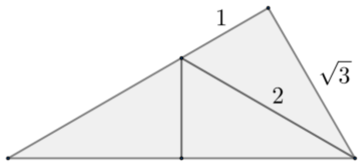


We derive Sequence *B* by taking *right* triangles in Sequence *A* with legs of lengths *i* and i^2 and appending right triangles with legs of lengths 1 and *i* so that the right angles abut. This gives us the general term $b_i = i^2 + 1$. The shape changes with the *i* value.

b_1 $\qquad$ b_2 $\qquad$ b_i

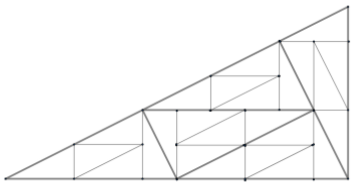


Sequence *C* contains just one member, the $30° - 60° - 90°$ triangle, for which $c = 3$.



2. In *iterated* sequences we continue to subdivide the triangles in the same way. The *A* sequence is unaffected but the general term in the *B* sequence becomes $b_{ip} = (i^2+1)^p$. That in the *C* series becomes $c_q = 3^q$. (*p* and *q* are non-negative integers, so they can take the value 0, required below.)

Here is b_{22}:



Draw c_2. Compare a_3 above.

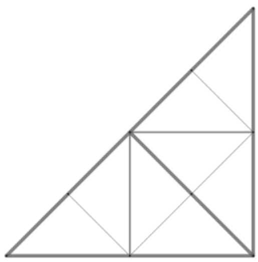
3. In *compound iterated* sequences we further subdivide *B* and *C* as in sequence *A*. This gives us sequences with the extra factor j^2 in the general term:

$b_{ijp} = (i^2+1)^p j^2$,

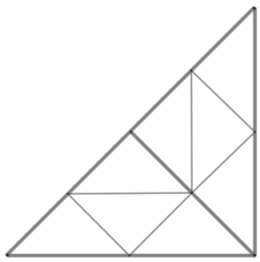
$c_{jq} = 3^q j^2$.

Here are two sequence *B* cases with the same *n* value, 8. As with a_3 and c_2 the unit triangles are differently arranged within the outline.

b_{113}



b_{121}



The completely general term $g_{ijpq} = (i^2+1)^p 3^q j^2$.

If we require that *p* or *q* must be zero – *why?*, we can use this formula to find *n* values in all three sequences, *A, B* and *C,* and of all three types, primitive, iterated and compound iterated. Here is a table for *n* values up to 10.

Extend the table by including 12, 16, 17, 18, 20, 25, 27.

Note that, when $p = 0$, the value of *i* is irrelevant, so there is no entry in that column.

We have two ways of making 4, 8, 9. *Consulting the extended table, what is the next n value we can make in two ways?*

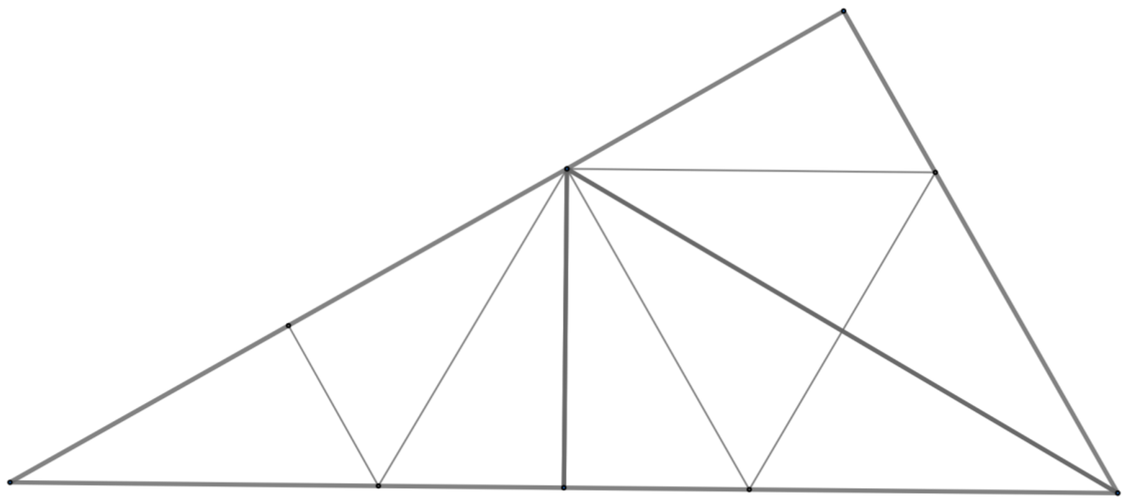
What is the most systematic way of completing the table?

Type	*Symbol*	*i*	*j*	*p*	*q*		g_{ijpq}
1	a_1		1	0	0		1
1	b_1	1	1	1	0		2
1	c_1		1	0	1		3
1	a_2		2	0	0		4
2	b_{12}	1	1	2	0		4
1	b_2	2	1	1	0		5
3	b_{121}	1	2	1	0		8
2	b_{13}	1	1	3	0		8
2	c_2		1	0	2		9
1	a_3		3	0	0		9
1	b_3	3	1	1	0		10

In these notes we have shown certain *n* values are possible. We have not shown which values – if any - are *im*possible. "Can a triangle be rep-6?" I don't think so but I don't know. I don't know because I don't know the mathematics which would tell me.

Can a triangle be rep-6? Solutions

Here is b_2:



"... why must *p* or *q* be 0?"
The exponents *p* and *q* fit different, incompatible, sequences, *B* and *C*. (The 'or' is *inclusive*: *p* or *q or both* must be zero.)

Type	*Symbol*	*i*	*j*	*p*	*q*		g_{ijpq}
3	c_{21}		2	0	1		12
1	a_4		4	0	0		16
3	b_{114}	1	1	4	0		16
1	b_4	4	1	1	0		17
3	b_{131}	1	3	1	0		18
2	b_{22}	2	2	1	0		20
3	b_{212}	2	1	2	0		25
1	a_5		5	0	0		25
2	c_3		1	0	3		27
3	c_{31}		3	0	1		27

The next *n* value we can make in two ways is 16. This is a_4 and b_{114}.

"What is the most systematic way of completing this table?"
I leave that question open. But here is one alternative layout:

In this table, where zeroes are omitted, we've kept all values constant except one.
As a result, only the numbers in one column exceed 1. This catches all the cases in the previous tables bar b_{212}, which gives us a second *n* value of 25.

i	j	p	q		a_j	b_i	b_{ip}	b_{ijp}	c_q	c_{jq}
	1				1					
	2				4					
	3				9					
	4				16					
	5				25					
1	1	1				2				
2	1	1				5				
3	1	1				10				
4	1	1				17				
1	1	2					4			
1	1	3					8			
1	1	4					16			
1	2	1						8		
1	3	1						18		
	1		1						3	
	1		2						9	
	1		3						27	
	2		1							12
	3		1							27

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