Polyhedra

Y8 children will have met tilings on the one hand and solids on the other. The idea of this masterclass is to treat the former as a special case of the latter, namely a polyhedron with zero *angle defect*.

One material is used throughout: Polydron Frameworks, interlocking polygonal tiles. In the workshop we make use only of the regular polygons from this kit. The children access these from the nearest pair of sorters. There is one for even-sided polygons and one for odd-sided ones. Each takes the form of a stack of cylinders of diminishing radius. Other materials are used for particular demonstrations.

There is one large whiteboard (**C**), flanked by two smaller ones (**L** and **R**).

The workshop falls in two parts. In **Part 1** we establish the terms, measures, quantities and rules we need in **Part 2**. There the children, working in pairs, seek the different exemplars allowed by the rules and enter their contributions on a chart.

The 'Euler' work - **Part 1(d)** and **Part 2(3)** - is optional. In **Part 1(d)** we use Euler's formula to show why the total angle deficit for a polyhedron is 4π, a good algebra exercise for senior students.

The important thing is that every formula must be instanced.

Part 1

a) Through a sequence of demonstrations and questions we distinguish first *convex* and *non-convex* polygons, then *irregular* and *regular*. We find the interior angle sum in two different ways. (This work requires board **L**.) Thus we find the individual interior angles for the regular polygons and make a table we can refer to in **Part 2**. (This takes up board **R**.)

b) We fit polygons round a vertex in a particular sequence. We sum the interior angles. We define *angle defect*, measure it in selected cases, study its significance, and relate the angle deficit at a particular vertex to the total.

c) We establish our nomenclature for forms where every vertex is the same.

d) We state Euler's polyhedron formula and check a few cases against it.

Part 2

1) The children are challenged to build models with identical vertices, to name them and enter their code on a giant chart. (This takes up the whole of board **C**.)

2) They check their model against the angle defect rule: does it give the number of vertices correctly? (They can use board **L** for their calculations.)

3) They check their model against the Euler formula: does it give the number of vertices correctly? (Again, they can use board **L**.)

Part 1

a)

figure A1

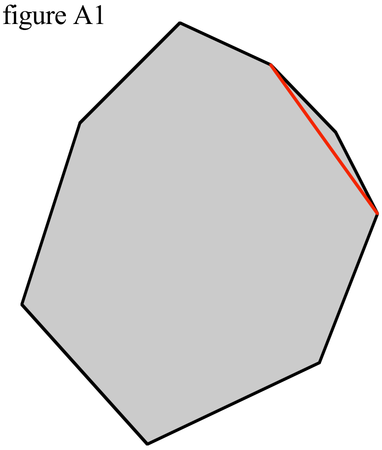


figure B1

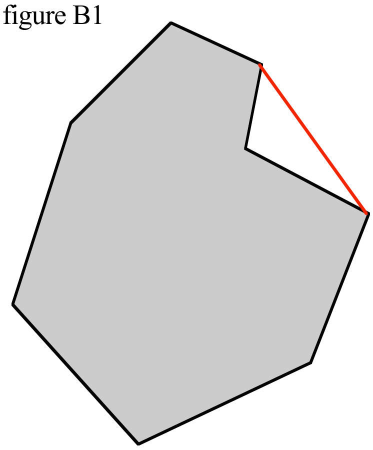


figure A2

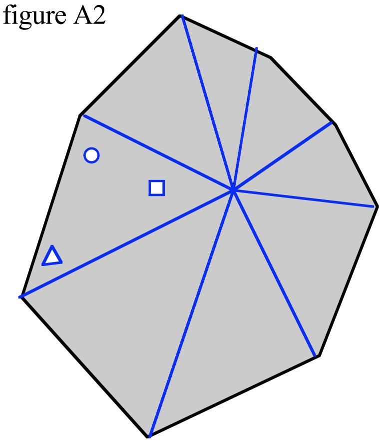


figure B2

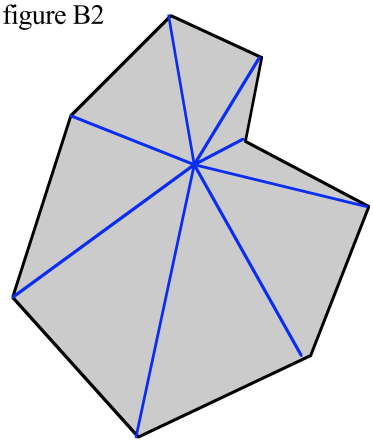


figure A3

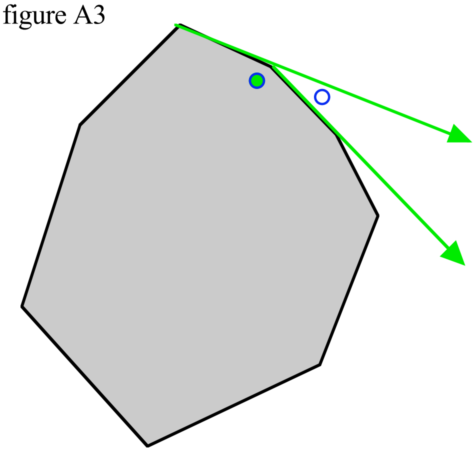
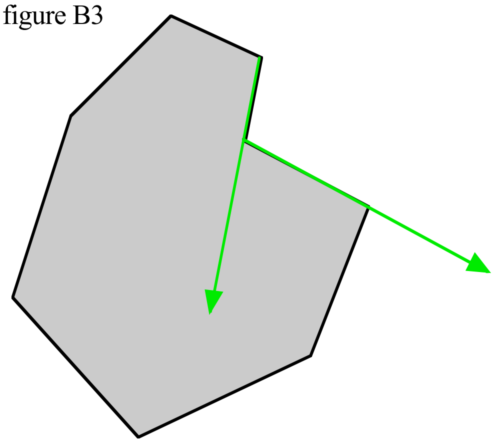


figure B3



Work on board **L**.

[figures A1, B1]:

What is the difference between figures **A** and **B**?
Every diagonal of **A** lies inside the figure.
A is *convex*, **B** *non*-convex.
(Moving to 3 dimensions, exhibit a convex and non-convex *polyhedron*. Show that you can stand the non-convex polyhedron on the edge of a table so that part of it lies beneath the table but you cannot do so with the convex example.)

Two ways to work out the sum of all the angles inside a polygon (the *interior*) angles:

First method [figures A2, B2]:

The number of corners (*vertices*) and sides is the same. Call it n.

Sum of the angles in one triangle? 180°.
How many triangles? n.
That total? $180n^\circ$.
But we must subtract the sum of all the angles round our inner point, 360°.
Therefore the sum we want = $180n^\circ - 360^\circ$.

Second method [figures A3, B3]:

Notice that the interior angle is 180° less the *exterior* angle.

Arrange bean bags on the floor to show **A3**. Invite a volunteer to begin at a chosen vertex, one arm extended along a side; to walk in that direction, at each vertex swinging round so that the arm points along the new side.

The child returns to the starting point. Ask the group: "What angle has his/her arm turned through?" (Answer: 360°.)

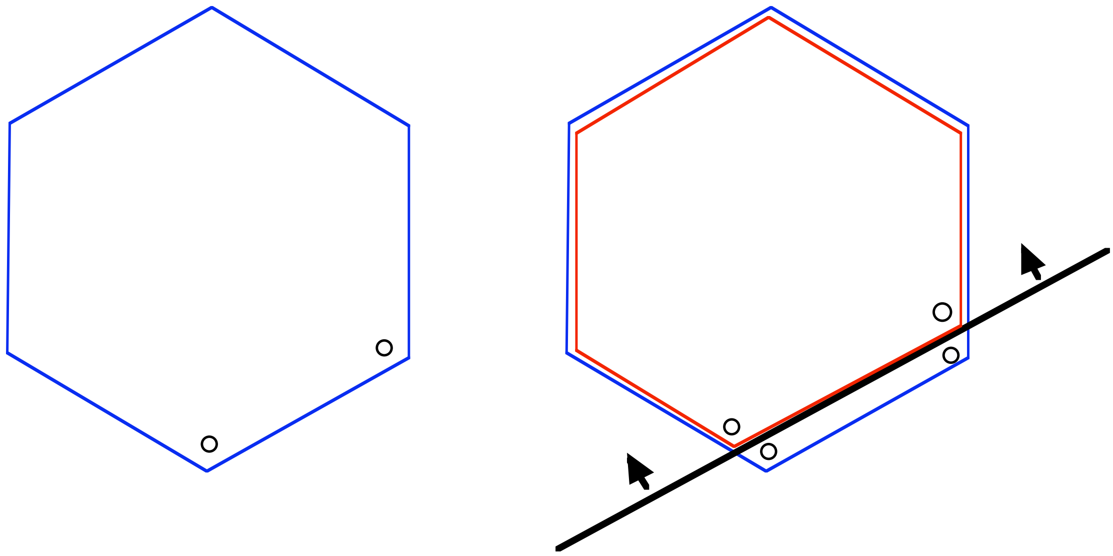
Out of interest, kick one bean bag inwards so that the figure is non-convex and repeat the experiment. Ask the group what they notice. (At that vertex the arm swings in the opposite sense.)

To find the total we want, we take n lots of 180° and subtract the sum of all the exterior angles.
We now know that sum is 360°.
So the final total is $180n^\circ - 360^\circ$, just as before.

Can we make a polygon where all the sides have the same length but not all angles are the same?

Demonstrate with Briomec how you can change the shape of a jointed regular pentagon and square – though not an equilateral triangle (or indeed any other kind of triangle: the triangle is a rigid shape).

Can we make a polygon where all the angles are the same but not all sides have the same length?



Draw a regular polygon. Set a ruler parallel to one edge and, keeping it parallel to itself, drag it across the figure. Redraw the polygon. Point out that the marked angles have stayed the same but three of the sides have changed their length: that in the line of the ruler has become longer; the two joined to it, shorter.

A convex polygon with all sides the same length and all angles the same is called *regular*, (others, *ir*regular).

Because the angles are all the same, we can find the size of one just by dividing our total by n:
The interior angle of a regular n-gon
$$=(180n-360)^\circ \div n = \left(\frac{180n}{n}-\frac{360}{n}\right)^\circ = \left(180-\frac{360}{n}\right)^\circ.$$
We recognise the '360°' as the angle our volunteer turned in the trip round the polygon. Turning the same amount each time gives $\frac{360}{n}^\circ$, the exterior angle. And we know we only have to subtract this from 180° to find the interior angle. Later we shall need to know these for $n = 3,4,5,6,8,10,12$ so we'll make a table.

Display the following on board **R**:

n	interior angle
3	60°
4	90°
5	108°
6	120°
8	135°
10	144°
12	150°

b)

Each pair of children take 4 squares.

Ask how many different corners (*vertices*) they can make. (None possible with 2, one possible with 3, one possible with 4.)
The difference between a corner with 3 and one with 4? (4 just make a flat surface.)

Can they continue their construction with corners of the same kind? (Yes in the first case, to make a finite object, a three-dimensional shape with many faces, a *polyhedron*; yes in the second case to make an infinite object, a tiling or *tessellation*.)

Each pair of children now take 6 equilateral triangles.

Ask them to experiment as before.
Ask them to comment on the relation between how sharp the corner is and the angle sum made by the pieces which meet there. (The bigger the sum, the less sharp the corner; the limiting case being $360°$, when the figure doesn't close up at all.)

Suggest the 'sharpness' measure d, the *angle defect*, how much the angle sum falls short of $360°$.

Ask them now to comment on the relation between the number of vertices a solid has and the angle defect at each. (The greater the number, the smaller the defect.)

Complete this table on board **L** for the models already assembled:

angle defect, d	number of vertices, v	total angle defect, dv
90°	8	720°
180°	4	720°
120°	6	720°
60°	12	720°

Point out the following.

Our volunteer made a tour of a polygon. Each turn s/he made, each exterior angle, was an angle defect, the difference from a straight angle, 180°. The total was 360°. In three dimensions, we make a tour of a polyhedron and complete a total of 720°.
In the section **(d)** we show where this number comes from.

c)

Point out to the children that the models they've made so far have only used one kind of regular polygon and that these are called *regular* or *Platonic* solids.

Now each pair take 8 equilateral triangles and 6 squares.

Ask them to make one solid, keeping the rule that every vertex should be the same.

The result is a *semi-regular* or *Archimedean* solid.

If the vertices are all the same, we can name the figure in a simple way: make a tour of a vertex (anticlockwise by convention) and write down n for each polygon we meet. Thus we code the cube $4.4.4$ or 4^3; the square tiling 4.4.4.4 or 4^4.

Ask the children how they would code what they've just made. (3.4.3.4.)

Can they predict how many vertices it has by using what they've found out about 'angle defect'? ($\frac{720}{360-(90+60+90+60)}=12$.) They can then confirm by counting.

d)

Euler provides yet another way to check.

Present the Euler formula $v+f=e+2$.

To obtain v we therefore only need to know f and e. f is easier to count than e. But e is related to f in the following way. If there are p faces with l sides, q with m, and so on, all we have to do is total the products: $pl+qm+...$ and, because two faces share an edge, halve the result: $e=\frac{pl+qm+...}{2}$.

Now $f=p+q+...$.

So we have $v=e+2-f=\frac{pl+qm+...}{2}+2-(p+q+...)$,

i.e. $v=\frac{p(l-2)+q(m-2)+...+4}{2}$.

We can use this formula to show why the total angle defect for a polyhedron is $720°$ (4π).

From section **(a)** we know that the interior angle sum of an n-gon is $180n^{\circ} - 360^{\circ}$, i.e. $(n-2)\pi$.

So the total, S, of all the interior angle sums for the polyhedron is $\left[p(l-2)+q(m-2)+...\right]\pi$.

We now combine our 'v' and 'S' formulas:

$2\pi v = \left[p(l-2)+q(m-2)+...+4\right]\pi = S + 4\pi$.

So $S = 2\pi(v-2)$.

The total angle deficit, D, is the sum of the '2π's for all v vertices less S, i.e. $D = 2\pi v - S = 2\pi v - 2\pi(v-2) = 4\pi$.

Part 2

Now the children are free to explore the possibilities for themselves.

They enter their models on this chart (board **C**). They should use the numbers down the side for the smallest polygon, the numbers along the top for the next smallest. In anticipation of the number of forms to be recorded, the boxes are sized vertically in inverse order of polygon size. (The boxes below those marked with a horizontal line will not be needed but there is no need to show this.) Provide pens in blue (say) for polyhedra, red (say) for tilings. The children will produce forms where not all the vertices are identical. In those cases, point out that the result can be seen as a hybrid of two standard forms. The children can distinguish these with a comma.

n	3	4	5	6	8	10	12
3							
4							
5							
6							
8							
10							
12							

The models themselves constitute an exhibition. Each shape is available in 4 colours but encourage the children to colour-code the faces; otherwise the structure will be obscured.

Paul Stephenson, 20.9.09