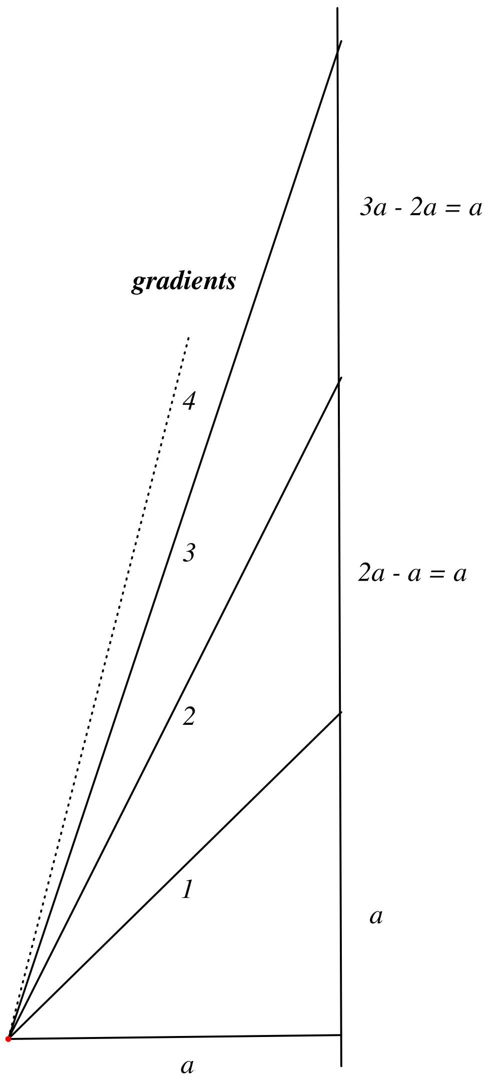
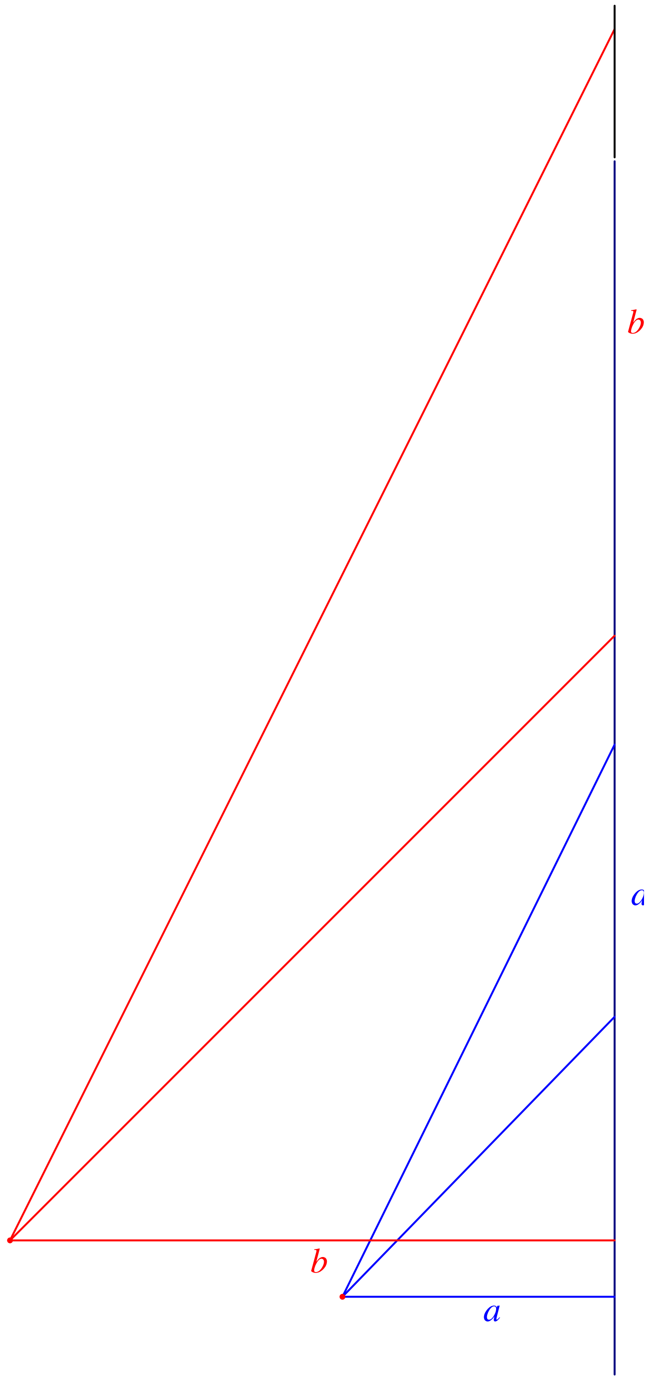
Number tower geometry

The number tower tells us that the number of potential primes is 2/6, so 1/3. We shall see how the searchlights eliminates candidates and so reduces this fraction.

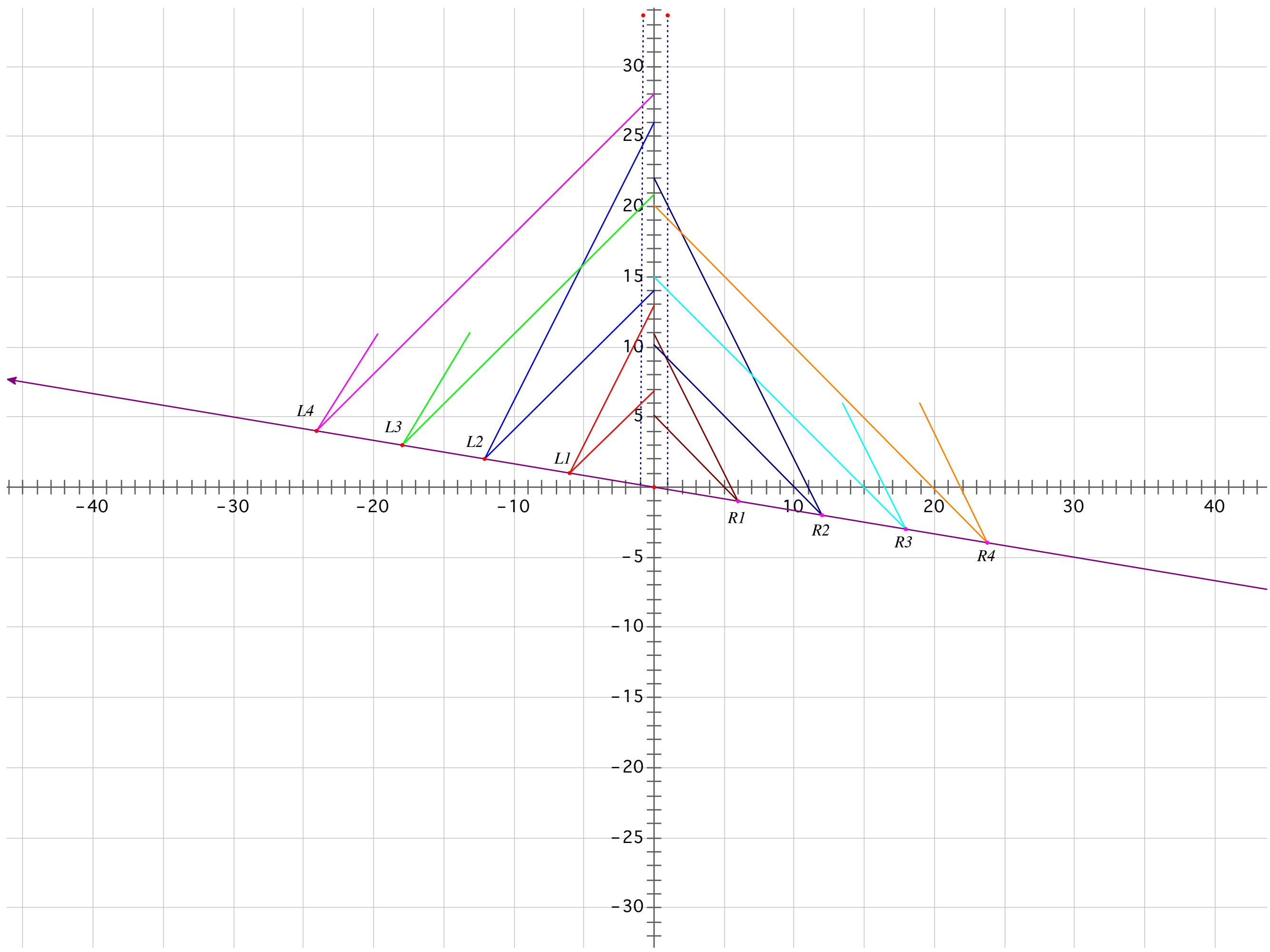
We need not consider the '0' searchlight, which picks out primes for the first time, but must attend to those to left and right.

Choose a single searchlight. The beams cut the vertical lines at equal intervals (see the figure on the left). Thus, though beams from different searchlights will pass through the same number, a single searchlight eliminates potential primes in an arithmetic sequence. The figure on the right shows that these intervals are proportional to how far out to the side a searchlight is.





The figure below exhibits a number of significant features.



We see that the gaps between the intercepts of any two consecutive beams from the left on the line *x* = -1, and those between the intercepts of any two consecutive beams from the right on the line *x* = +1, follow the same arithmetic progression:

L1	L2	L3	L4	…
R1	R2	R3	R4	…
5 x 6	11 x 6	17 x 6	23 x 6	…

If we take intercepts on the line $x = 0$, we have the simpler sequence 1, 2, 3, … scaled by 6.

The difference between left and right emerges when you compare the gaps between intercepts of the *same* beam from *different* searchlights. The positive vertical displacement of searchlights on the left spreads the intercepts out, widens the gap; the negative vertical displacements of those on the right bunches the intercepts together, shrinks the gap. As a result, the gaps between the intercepts of any chosen beam on the left on the line x = -1 are 7 x 6 = 42; the corresponding gaps on the right for the line *x* = +1 are 35.

What this means is that, if we take as vertical interval the lowest common multiple of 35 and 42 = 210, for every such interval we pass as we ascend the number line, 7 + 5 = 12 new searchlights come into play.
But, since the new searchlights are further and further out, their effectiveness diminishes. This effectiveness, the density of intercepts they make on the lines $x = -1$ and $x = +1$, goes in inverse proportion to the terms we noted: 6, 2 x 6, 3 x 6, … .
Thus, were all other things equal, (which they're not), the overall density up to a particular number would be proportional to a partial sum of the harmonic sequence: $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+....$ As the number of terms n approaches infinity, this approaches the natural log of n + the Euler-Mascheroni constant (around 0.58).

What proportion of the beams from a given searchlight is responsible for eliminating primes?

At least three factors need to be considered:

One beam passes through a prime on $x = -1$ and a prime on $x = +1$.

At least 2 beams pass through each candidate. This number increases as we progress up the number line because the potential number of prime factors a number can have increases.

The effect of a composite beam will be duplicated by that of a prime beam from a searchlight further in.

But how can one quantify these effects?

We get some idea how complex the problem is if we trace the beams through particular numbers back to their searchlights:

With each number N is associated a set $S_N = \{(n_1, s_1), (n_2, s_2), (n_3, s_3), ...\}$, where the beam n, a divisor of N, $5 \le n < N$, comes from the searchlight s, the searchlight located at ($6s$, $-s$).

The number N of the form ($6n - 1$) is located at (-1, $\frac{N+1}{6}$).

The number M of the form ($6n + 1$) = $N + 2$ is located at (+1, $\frac{N+1}{6}$).

The gradient of a beam from the left is $\frac{n-1}{6}$.

The gradient of a beam from the right is $\frac{-n-1}{6}$.

We form this equation for a beam from the left: $y = \left(\frac{n-1}{6}\right)x - ns$.

We form this equation for a beam from the right: $y = \left(\frac{-n-1}{6}\right)x + ns$.

Substituting for *N*, we have these four relations:

1. For a beam from the left meeting a number on the left: $N = -n(6s+1)$.
2. For a beam from the left meeting a number on the right: $N = n(1-6s)-2$.
3. For a beam from the right meeting a number on the left: $N = n(6s+1)$.
4. For a beam from the right meeting a number on the right: $N = n(6s-1)-2$.

We note that beams from the right have the same form as numbers on the left, 6*n* – 1; those from the left, the same form as numbers on the right, 6*n* + 1.

Here are some sets:

$S_{979} = \{(11,15)\}$.
$S_{1,001} = \{(7,-24),(11,15),(13,-13)\}$.
$S_{1,015} = \{(5,34),(7,-24)\}$.

Note that, because beams pass through grid points, any two composite numbers have a beam in common.

Note that searchlights which are higher, further out, contribute lower divisors.

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This is as far as I've got at 19.7.14. If you don't see a way forward using the number tower, try the Stechkin-Matiyasevich sieve by going to plus.maths.org and searching for 'Catching primes'.