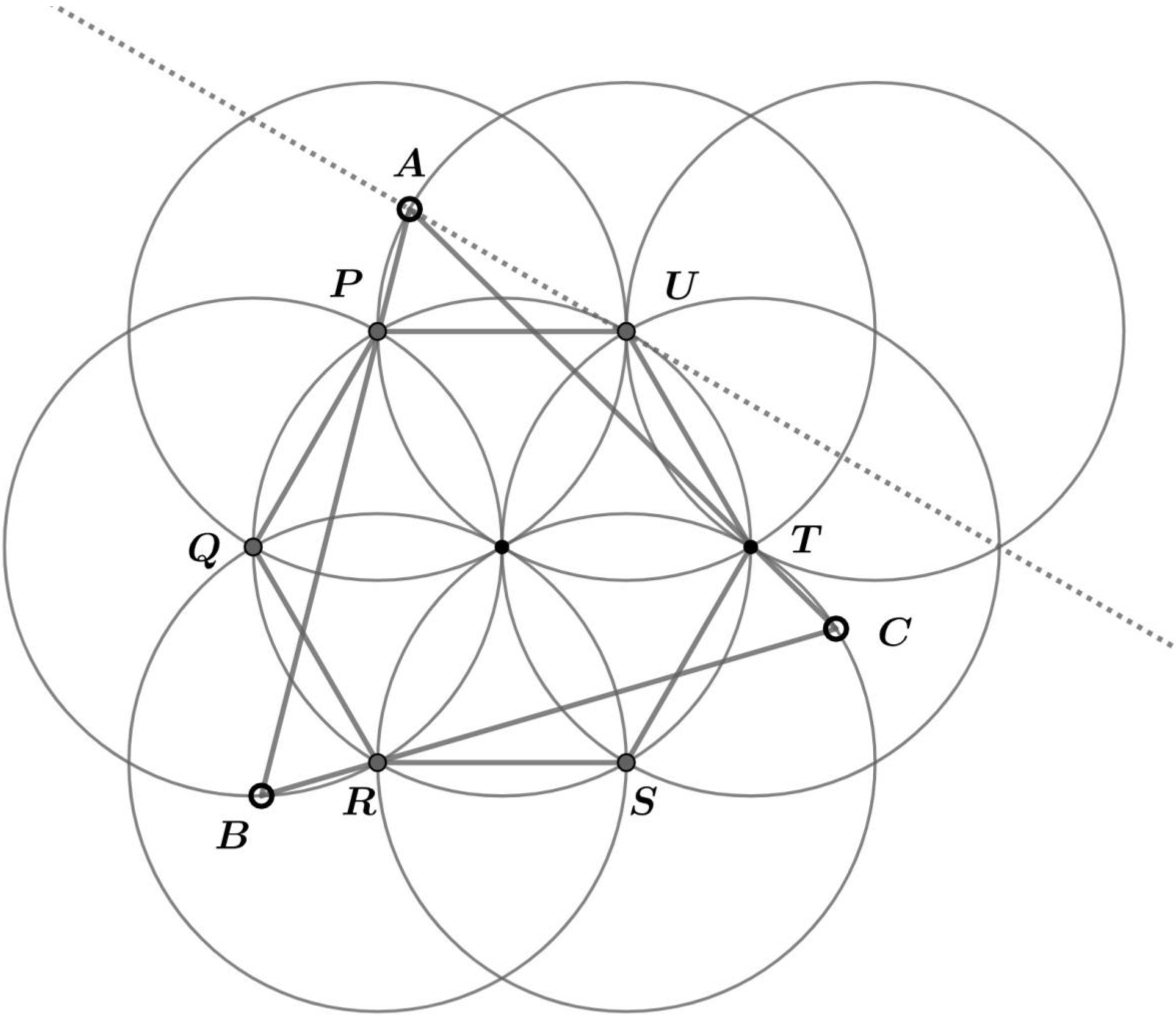
Constructing regular n-gons with the same area as regular $2n$-gons

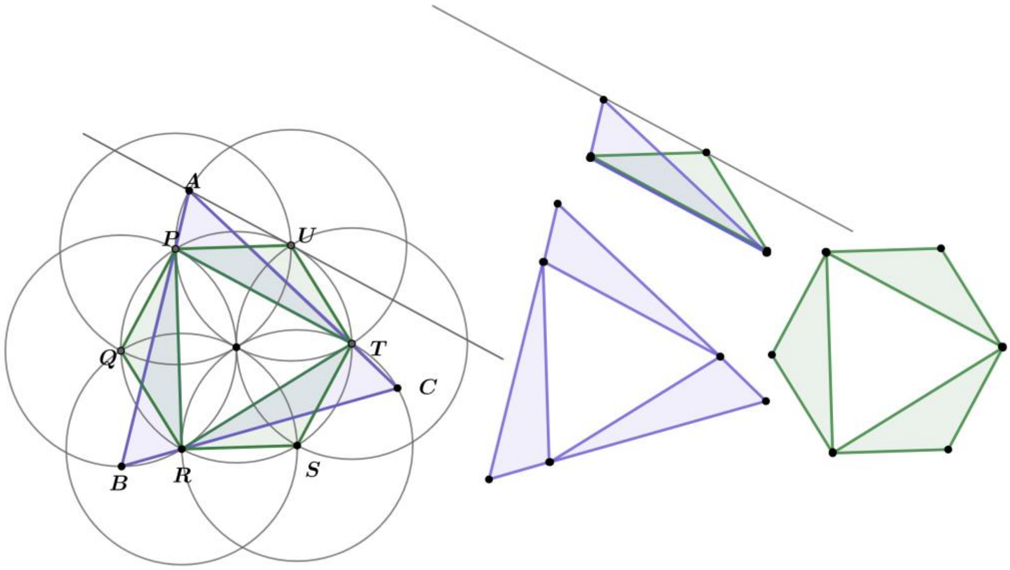
(A) Triangle = hexagon

The labelled figure is drawn on a covering of equal circles, where six circles pass through the centre of each one.

Show that triangle *ABC* and hexagon *PQRSTU* have the same area.



The following proof proceeds in four steps. **(1)** is used in **(4)**, **(2)** in **(3)**. They spell out this 'proof without words':



(1) The key observation

Triangle *PRT* is common to triangle *ABC* and hexagon *PQRSTU*.

(2) Establishing that two particular triangles have the same area

By construction *AU* is parallel to *PT*.

Triangles *PAT* and *PUT* therefore have the same height and,
since they have the same base,
the same area.

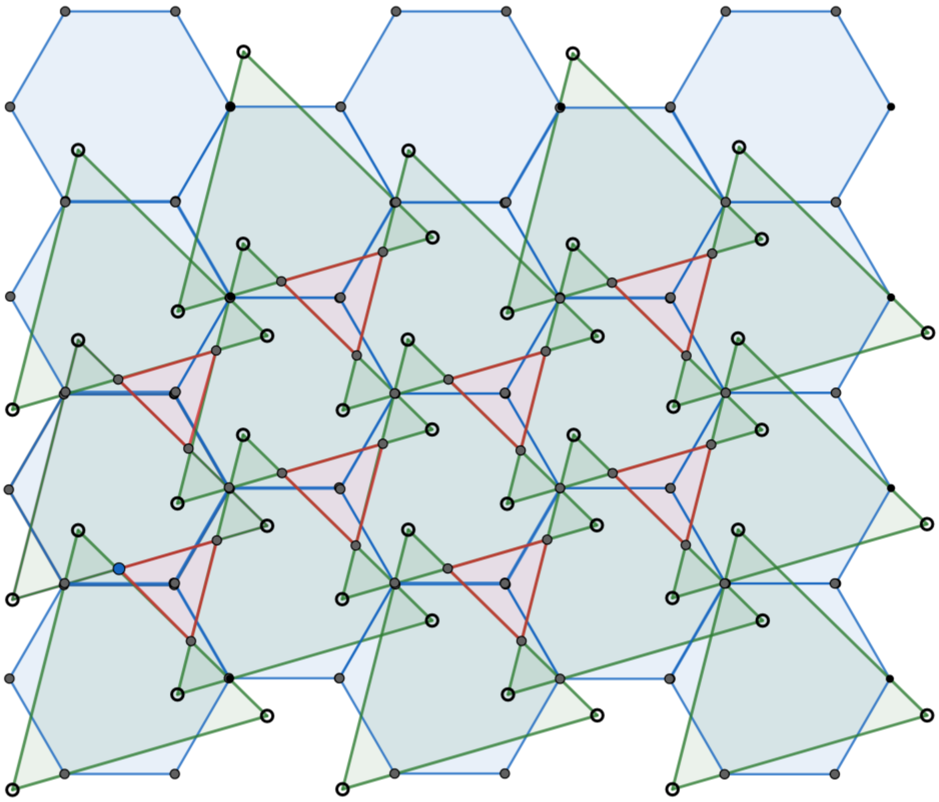
(3) Using the symmetry of the figure to generalise (2)

From the symmetry of the circle arrangement on which the rectilinear figure is constructed, we infer that triangle *ABC* is equilateral, hexagon *PQRSTU* regular.
We can therefore identify one set of three congruent triangles, of which *PAT* is one,
and another set of which *PUT* is one.

(4) Proceeding outwards from the common triangle identified in (1)

Since three copies of *PAT* complete triangle *ABC*,
and three copies of *PUT* complete hexagon *PQRSTU*,
and *PAT* and *PUT* have the same area,
triangle *ABC* and hexagon *PQRSTU* must have the same area.

A nice design results from superimposing grids of hexagons and mutually-rotated triangles. Since a triangle has the same area as a hexagon, a red triangle has the same area as 3 dark green ones.



Other cases

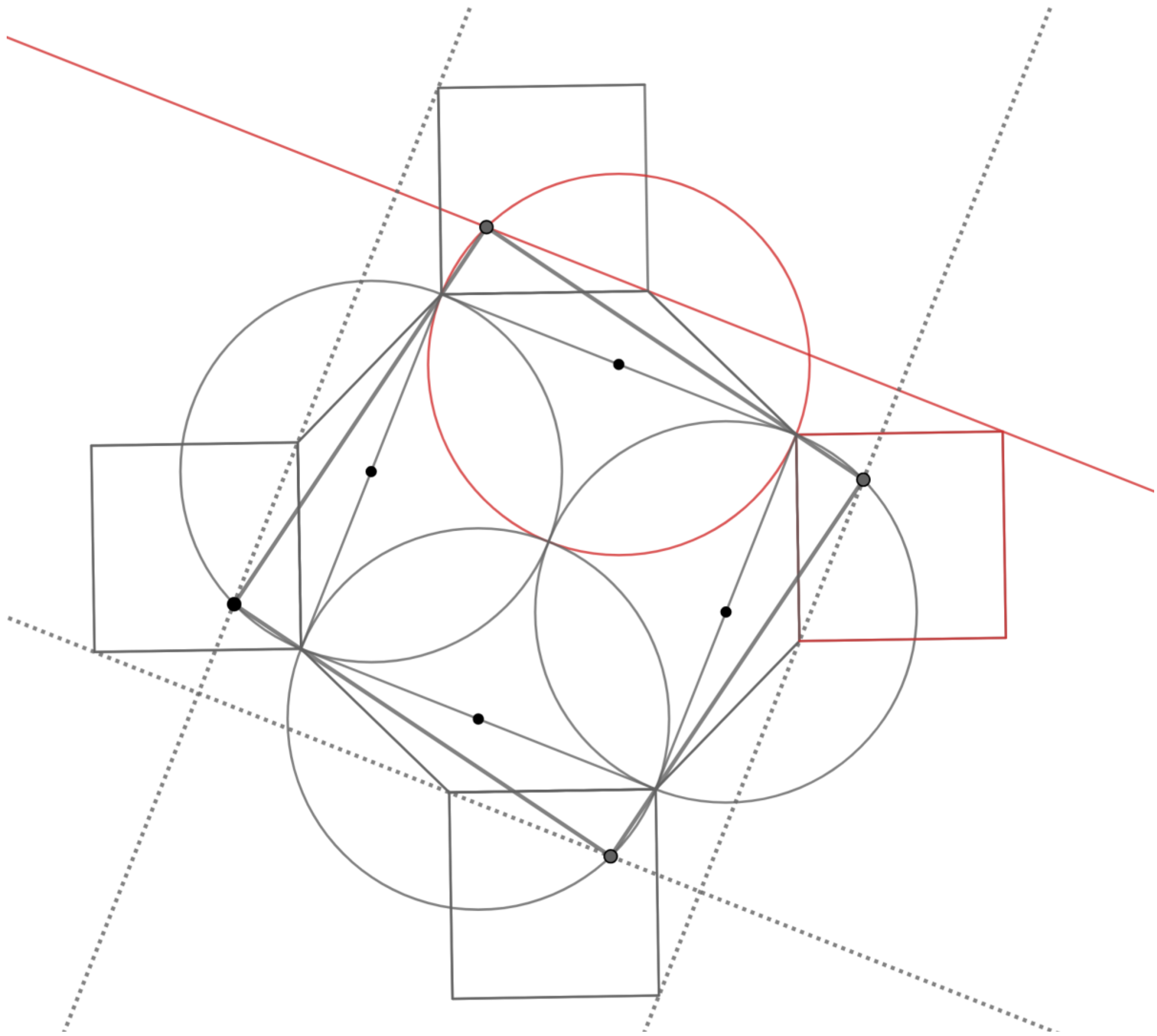
Two constructions are needed: the parallel corresponding to AU, and the angle of the new n-gon, corresponding to angle PAT. In case **(A)** the circles did all this work for us. In other cases more contrivance may be needed. However, the constructions are particularly simple in the following two cases. But notice that, where it looks as if we've added an exterior square or an exterior triangle, what in fact we've added is one of the n isosceles triangles into which a regular n-gon can be dissected. Check this.

(B) Square = octagon

This figure is part of the semiregular tiling 4.8.8.

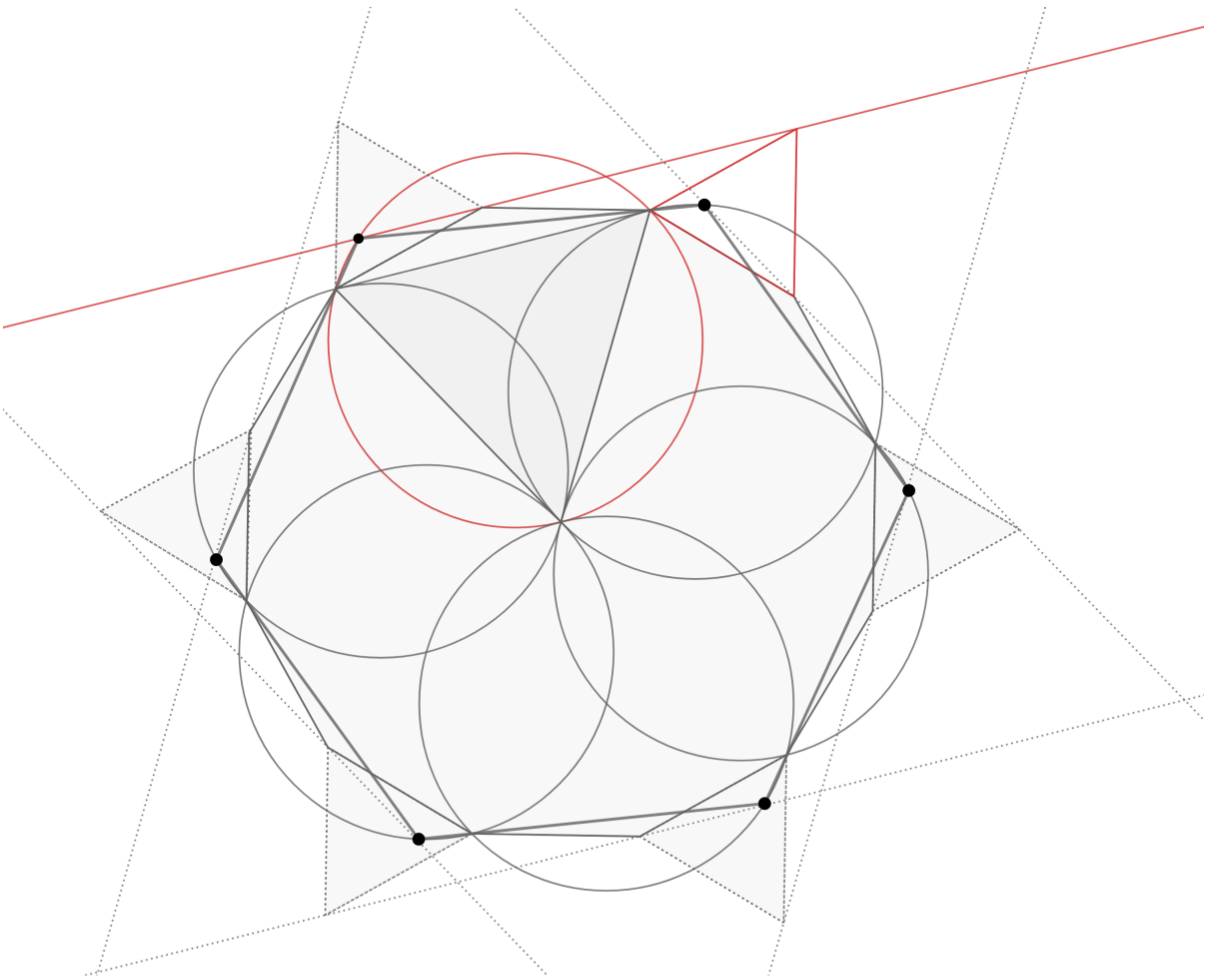
There are two constructions which check each other. The line through the vertex of the external square and the vertex of the octagon, and the line from the last new square vertex through the next octagon vertex both meet on a circle in the next new square vertex.

Why does the external square construction work? The circles produce right angles by the converse of Thales' theorem.



(C) Hexagon = dodecagon

This figure is part of the semiregular tiling 3.12.12. Check that the external triangle construction works. The circles contain quadrilaterals with a 60° angle at the central vertex and therefore the supplementary angle of 120° where we need it.



P.S. 15.7.23