Symmetries of the mystic rose

The individual pictures for the students are available for projection as .pdfs, accessed at: www.magicmathworks.org. Go to 'Masterclasses', 'Mystic rose', 'Senior'. The files are '**Image 1**', ' **Image 2**', '**Image 3**', etc. The numbers below correspond to these.

Suggested tasks for each figure and teachers' notes

'Mystic rose' is the name given to a regular polygon together with all its diagonals. Each of the dozen or so pictures brings out a different property of the figure. Some properties jump out at you; others are less obvious. Study the colours and the labels for clues. Equal lengths, equal angles, regular polygons, triangles isosceles and scalene, ... all these are to be found here in abundance. We take advantage of the fact that, with software packages like GeoGebra, there is a regular polygon tool the students can use to tackle these tasks. This provides in effect a virtual, circular geoboard. In every case the students may spot features I haven't noticed, and should be encouraged to pursue their observations.

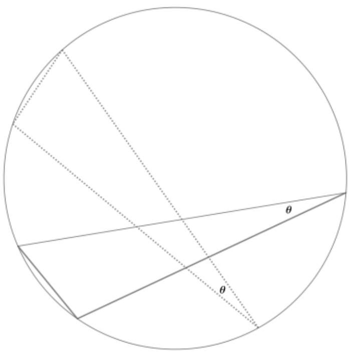
(a) Level: Advanced KS3/KS4

($'180°'$ can be substituted for $'\pi'$ throughout.)

The first task in each case is to describe the figure as if to a friend over the phone so clearly and unambiguously that they could draw it. This requires you to answer the question, sometimes asked explicitly below: *How has the figure been constructed?*

1. The mystic rose is the regular *n*-gon together with all its diagonals.
It has the full symmetry of the regular *n*-gon: *n* symmetry axes and rotation symmetry of order *n.* [See notes **mystic rose 1** following.]

2. *What can you say about all the diagonals from a chosen vertex?*

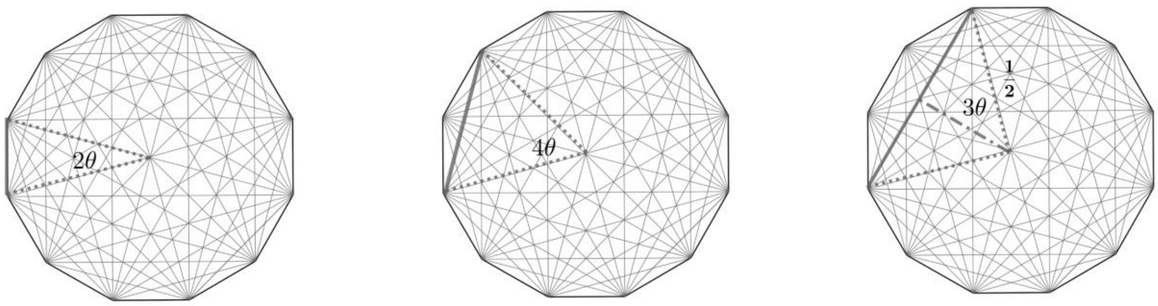


They're the same angle apart. This follows from the following extension of the 'same segment' theorem. In the figure, the triangle has been rotated about the circle centre. The chord maintains its length and the angle subtended remains the same. If a whole series of equal chords are run together - as they are in a regular polygon - the angles they subtend at a point on the circumference - in particular another polygon vertex - are therefore equal.

Because of the rose's rotation symmetry, all angles between sides or diagonals within the rose are multiples of this angle, $\theta = \frac{\pi}{n}$. [See notes **mystic rose 2** following.]

3. If the circumcircle of the polygon has unit diameter, the graph shows that the lengths of those diagonals follow the sequence $\sin\theta, \sin 2\theta, \sin 3\theta, \ldots$.

To see why this is so, build isosceles triangles on the diagonals with their apices at the centre, and do the trigonometry. We see here that the length of the third diagonal is $2 \times \frac{1}{2} \sin 3\theta = \sin 3\theta$.



4. Triangles in the same shade of blue are congruent.

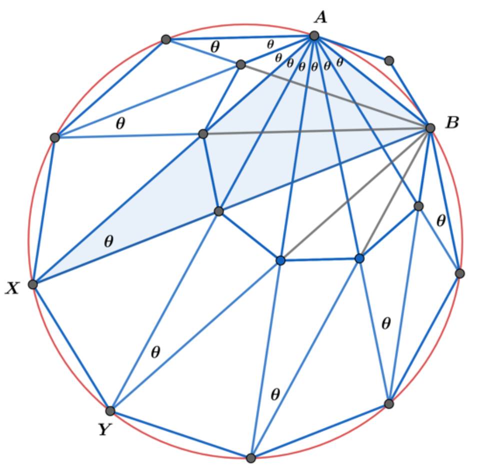
Notice how a triangle has been swung round a vertex of the small polygon to complete a bigger triangle with one a shade paler inside the polygon.

What can you say about that bigger triangle?
We have transferred the angle θ from A to X (see figure below), thus producing a triangle with two equal angles. It is therefore isosceles.

What can you say about all the bigger triangles?
Their angles are the same. Therefore they're similar.

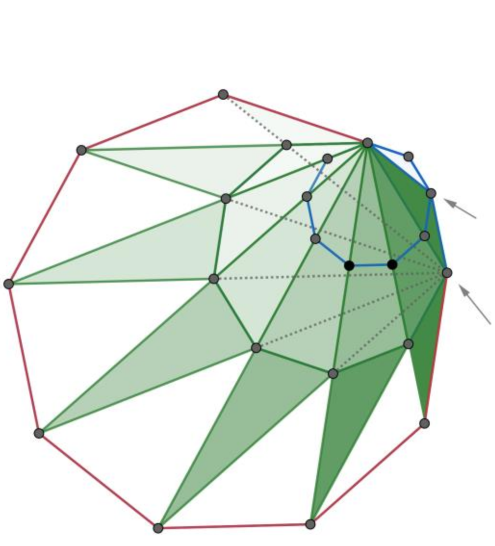
How has the bigger polygon been constructed?
It's based on the first diagonal of the smaller one.

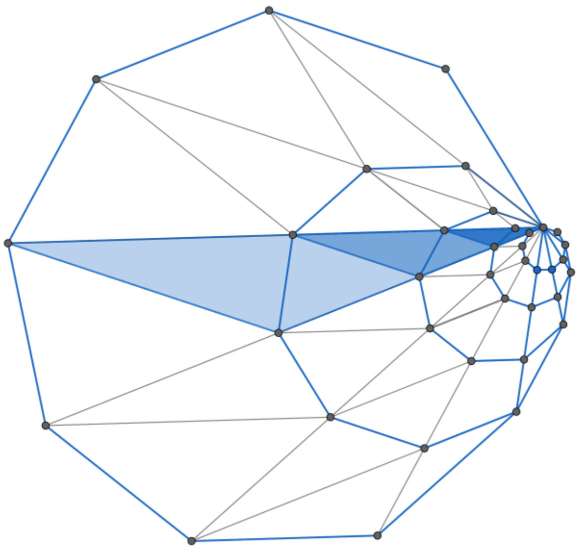
Why are the outer vertices of the bigger triangles vertices of the bigger polygon?



The equal angles on the same base, *AB*, imply that a circle passes through the outer points, which therefore define a cyclic polygon. But we can say more. Line segments like *AX*, *AY* are separated by the same angle and thus form diagonals of a regular polygon - see **2.** Since this angle is the same for the two polygons, the larger one is similar to the original.

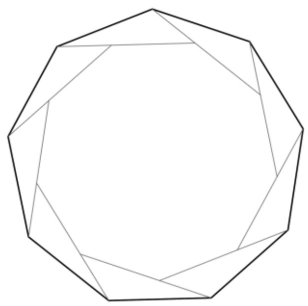
We can iterate the process. The arrows in the left hand figure below show the successive diagonals on which the new polygons are based. The right hand figure below shows how we end up with nested similar triangles like those in blue.





When we come to consider **11.**, we shall find that the exterior vertices of a generation g polygon are interior vertices (diagonal intersections) of a generation $(g + 1)$ polygon.

5.



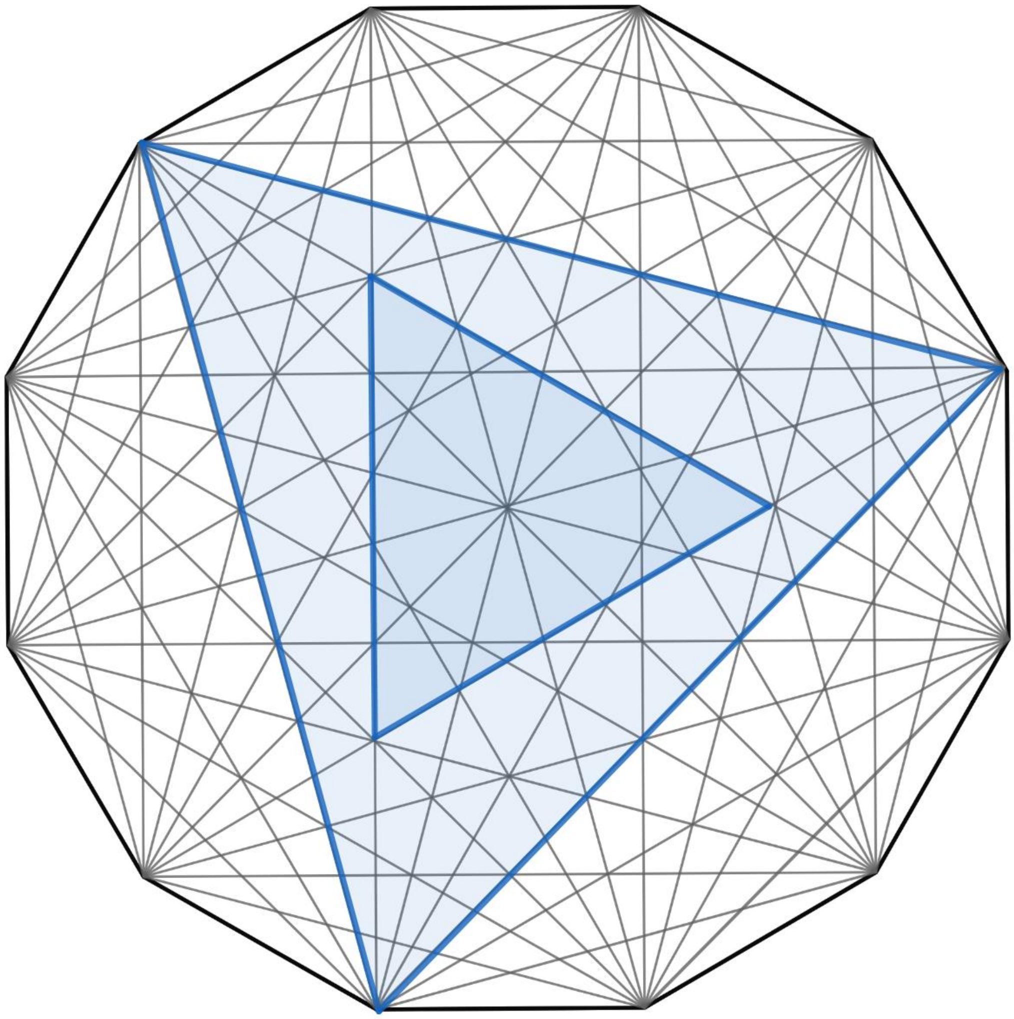
The diagonals picked out by the dashed blue lines are like the blades in the iris diaphragm of a camera, closing up on smaller and smaller hexagons. The difference is that these blades close in discrete angular jumps of θ.

We could have shown a similar diagram with squares or equilateral triangles.

What can you say about the number of sides possible for a regular polygon traced in the diagonals of a regular 12-gon?
By symmetry it must be a factor of 12.

We have shown one hexagon of each size but they are multiplied by symmetry. So there are $\frac{12}{6} = 2$ central regular hexagons, $\frac{12}{4}$ central squares, $\frac{12}{3} = 4$ central triangles sharing vertices with the outer polygon.

But internal regular polygons need not share a centre with the 12-gon. It contains, for example, 172 equilateral triangles, of which only 8 are central, these two multiplied by 4:

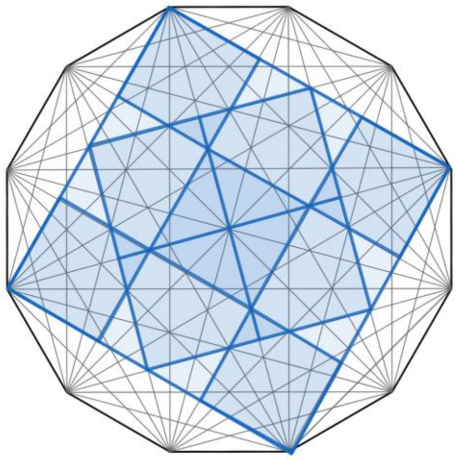


To be sure that number is correct, we need to make the count in different ways.

How many ways can you find?
See notes **mystic rose 5** following.

For an easier example, find the number of squares in the order 12 rose.

There are 33, of 5 different sizes.
All have diagonals lying in
symmetry axes of the rose.



*How many distinct triangle **shapes** can you find?*
In any triangle, the three angles are multiples of $\theta = \frac{\pi}{n}$, so we have $a\theta + b\theta + c\theta = \pi = n\theta$, $a + b + c = n$. So the question becomes: How many ways are there to partition *n* into distinct sets of three numbers?
For *n* = 9, for example, we have:

1 1 7	2 2 5	3 3 3
1 2 6	2 3 4	
1 3 5		
1 4 4		

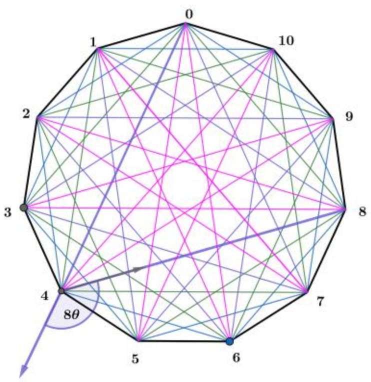
A total of 7. *Try n* = 12. 13

6. All the diagonals of the mystic 11-gon are here. *But what is the colour scheme? (The clue is in the vertex numbers.)*

The colours indicate star polygons.

A star polygon has the full symmetry of the convex regular polygon but its sides follow diagonals according to the following scheme. The difference between the convex polygon {11} and the star polygon {11,4} is that, for {11}, beginning at vertex 0, we just count on 1 to the next vertex; for {11,4}, we count on 4 modulo 11. This brings us from 0 to: 4 8 1 5 9 2 6 10 3 7 and back to 0 again.
The path closes after 4 circuits. We can see this from the angle a vector would turn through along the path - see figure. We have 11 turns of 8θ, where $\theta = \frac{\pi}{11}$, that is, $11 \times 8 \times \frac{\pi}{11} = 8\pi$ = 4 whole angles.]

How big a turn does the vector make at each vertex of {11,k}? 2kθ

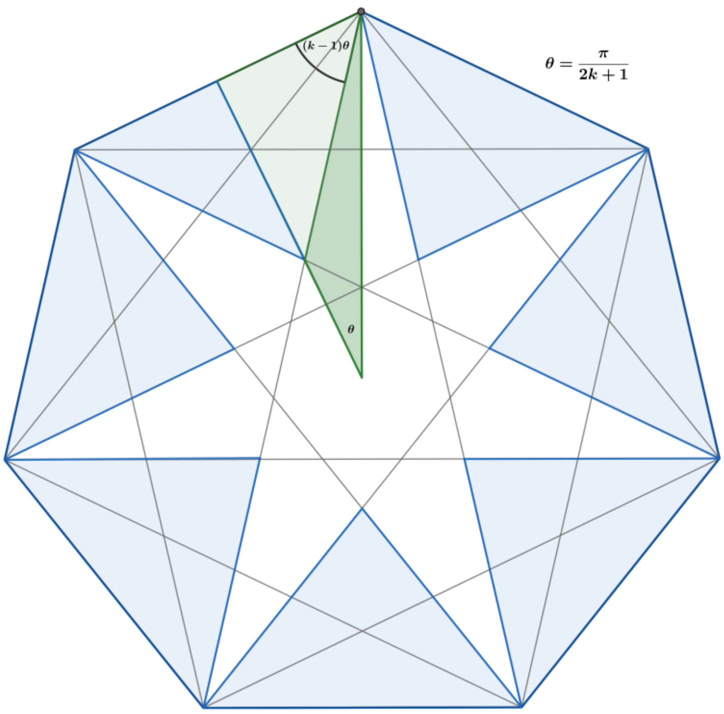


Draw the same diagram for the set of star 9-gons. What problem do you meet with {9,3}?
{9,3} has to be made from three distinct triangles: it is a 'degenerate' case.
For a valid star polygon, n and k must be coprime.

Why does the complete set of star polygons for a regular n-gon correspond exactly to the set of diagonals?
Both represent all possible ways of joining the vertices of the polygon.

How does the perimeter of a star polygon compare with that of its parent?
Because the number of sides is the same for {n,k} and {n}, namely n, we only need compare the length of the diagonal forming a side of {n,k} with a side of {n}. We know from **3.** that the ratio is $\sin k\theta : \sin\theta$.

How does the area of a star polygon compare with that of its parent?

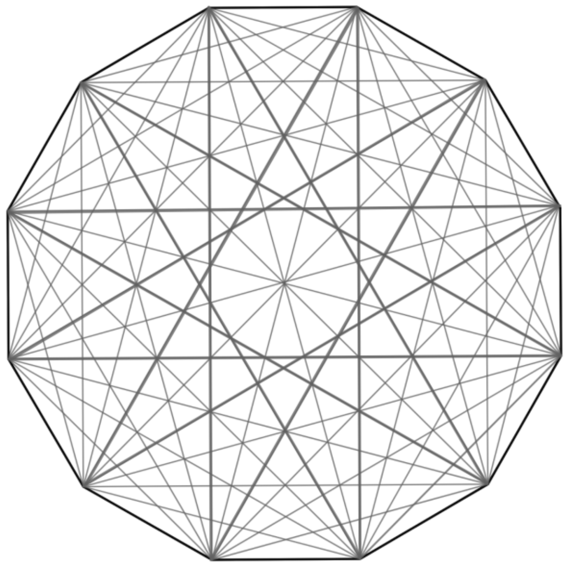


The white region is {7,3}.
The ratio of areas of {7,3} to {7} is that of the dark green triangle to the whole green triangle. Because they have the same height, this is the ratio of their bases. You can confirm that this is

$$\frac{\cos\theta - \sin\theta\tan(k-1)\theta}{\cos\theta},$$

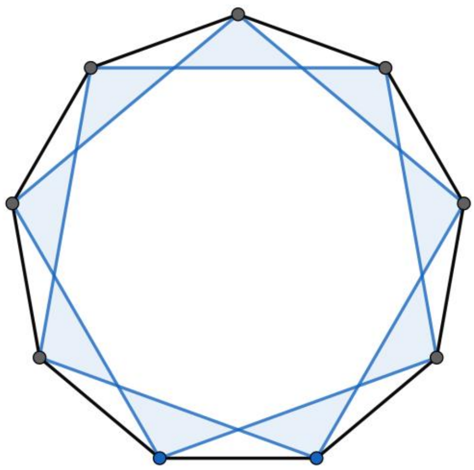
where, in this case, $k = 3$, $\theta = \frac{\pi}{7}$.

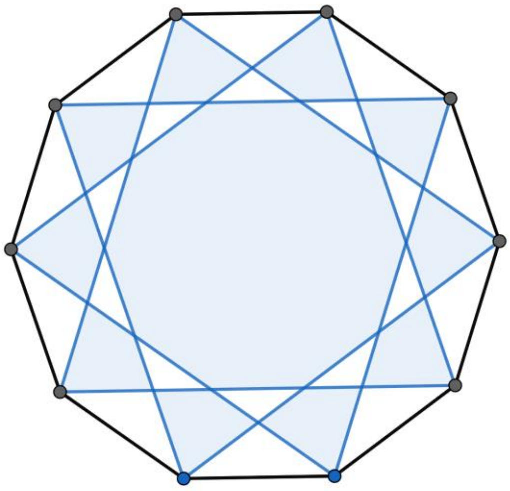
The formula also works for a $2k$-gon. *Try {12,5} below.*



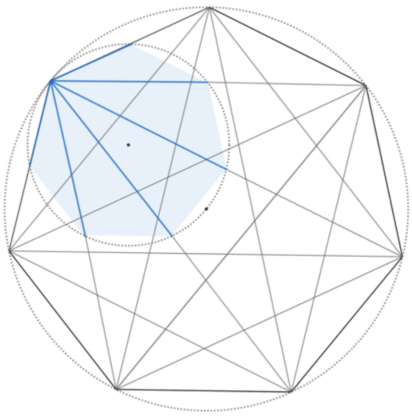
A valid star polygon is a closed curve but not a *simple* closed curve because it crosses itself. *Use the Geogebra 'polygon' tool to draw a star polygon.*

What rule does it appear to follow?
The first edge is simply a line. With the second line, we have a defined angle and the tool colours the side of the figure where that angle is less than a straight angle, left say. It colours the part right of an imaginary line between the point it has reached and the first vertex. When it crosses itself, it switches the colour to its other side, always filling the part between its present position and what it has already drawn. The result is a 'map-colouring' of the figure. (See **13.**) Here are two examples, {9,2} on the left, {10,3} on the right.





7. The diagonals change colour at their midpoints. This is because, for each colour, we have chosen a vertex and drawn the half-edges and half-diagonals leading from it - see figure.



Draw your own one seventh of the rose which completes the whole rose by the 7-fold rotation. Make it a lot less tidy than mine: the elements don't even need to be connected.

How do you ensure that your set of elements is complete?
Counting diagonals anticlockwise, say, from a vertex, you have to make sure that, for example, a third line segment along a second diagonal only occurs once - and so for every line segment - and that every line segment does occur. The only way to check is to rebuild the rose by rotating your seventh.

How has the figure been constructed?
We begin with a regular 7-gon and its centre. The circles are drawn with polygon circumradii as diameters.
Where two circles cut, there is a dot.

What is special about the dots?
They're vertices of regular 7-gons. And they lie at midpoints of diagonals of the outer 7-gon. We can think of the small 7-gons as the big 7-gon enlarged from a vertex by scale factor ½.

The parallel lilac triangles are congruent.

Show this.
The sides of the triangles correspond to the same diagonals in equal regular 7-gons.

Draw a figure of your own, with a different original polygon and a different coloured polygon.

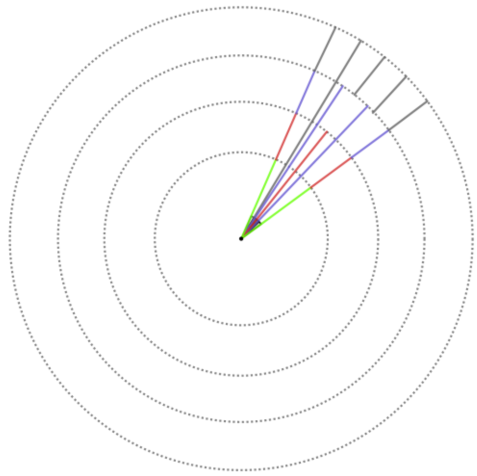
8. The set of polygons share an edge. The numbers of sides are shown. The tramlines reveal pairs of superimposed diagonals which match perfectly.

What is the significance of the angle, $\frac{\pi}{6}$ (30°)?
The tramline pairs make multiples of this angle with the base.

How are the numbers of sides related?
They are multiples of the side number of the smallest polygon.

How can the perfect fit be explained?

Like **11.,** this depends on **2.** From a chosen vertex, the 24-gon has diagonals every $\frac{\pi}{24}$; the 18-gon, every $\frac{\pi}{18}$; the 12-gon, every $\frac{\pi}{12}$; the 6-gon, every $\frac{\pi}{6}$. Writing integers proportional to the highest common divisor of these angles, $\frac{\pi}{72}$, we have respectively, 3, 4, 6, 12, of which 12 is the lowest common multiple. Therefore the diagonals all line up every $12 \times \frac{\pi}{72} = \frac{\pi}{6}$ - see figure.



For the 5 figures, the subdivisions run as follows. All represent the angle on the left, written below each of the 5 sets, divided by 2, 3, 4, ...

$90°$ $45°, 30°, 22\frac{1}{2}°, 18°$
$60°$ $30°, 20°, 15°, 12°$
$45°$ $22\frac{1}{2}°, 15°, 11\frac{1}{4}°$
$36°$ $18°, 12°, 9°$
$30°$ $15°, 10°, 7\frac{1}{2}°$

9. *What do we see?*

10 10-gons, a series of enlargements from one vertex. The blue dots show interior vertices; the red dots do not fall on an intersection. On diagonal d_3 and its symmetrical partner, d_5, all dots mark interior vertices.

10. *What is the colour code on the left?*
Points of the same colour lie at equal distances from the centre.

How has the figure on the right been constructed?
The polygon has been broken at a vertex and straightened out; the diagonals have become semicircles.

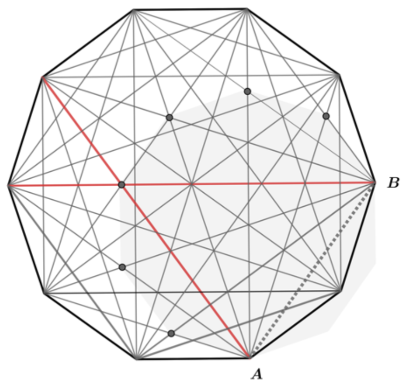
On the right we have lost the rotation symmetry but preserved a single mirror line.

The colour code on the right matches that on the left. Because 7 is an odd number, it turns out that we can work out from a simple formula the number of diagonal intersections there should be and match that total on the two figures (35). [See **mystic rose 10** following, where we also show how to count line segments.]
(For even *n* > 4 there are intersection points through which more than two diagonals pass. These points do not survive a transformation into the straightened rose.)

Draw a similarly transformed figure for the order 5 rose.

11. Let's call the points where polygon sides meet, exterior vertices; the points in which diagonals cut, interior vertices. On the left, one 12-gon shares a diagonal (dotted) with another 12-gon. What we notice is that exterior vertices of the polygon on the right coincide with interior vertices of the polygon on the left. On the right, a 15-gon shares a diagonal with a 5-gon. Again, the exterior vertices of the 5-gon coincide with interior vertices of the 15-gon.

If a regular p-gon, P, shares a diagonal with a regular q-gon, Q, (AB in figure below) and the exterior vertices of Q within P coincide with interior vertices of P, what do we know about p and q?
The key to this is figure **2.** We'll take the left hand figure of **11.** first. Anchoring the two diagonals to each other means that, within *P*, all the diagonals of *Q* from *A*, and all the diagonals of *Q* from *B*, lie over diagonals of *P* - see figure. So, where *Q* diagonals meet in an exterior *Q* vertex, *P* diagonals meet in an interior *P* vertex.



In the right hand figure of **11.**, a *Q* diagonal does not lie over every *P* diagonal, but over every third one. But this still means it still lies over a *P* diagonal, which is all we require.
Thus the requirement is simply that *p* is a multiple of *q*.

What can we say about the lengths of line segments between interior vertices?
What can we say about the relation between the lengths of complete diagonals?
See notes **mystic rose 11** following.

12. *How has the figure been constructed?*
The sides and diagonals have been extended from segments into lines. The circles join intersections at equal distance from the centre.

Copy the figure, extending the lines outwards till there are no more intersections. Confirm that there are more diagonal intersections outside the polygon than inside.

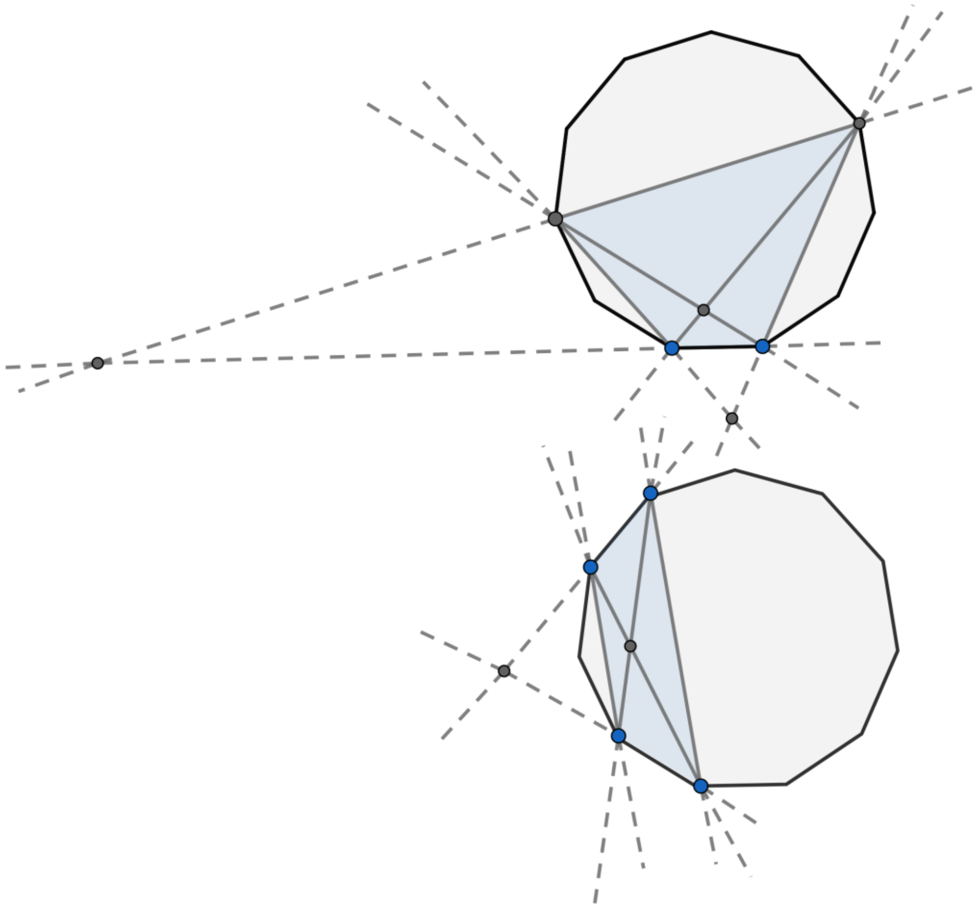
Given that, for odd n, no more than two diagonals pass through any intersection inside or outside the figure, show that this must be so for all odd n > 5.

Any four vertices define a quadrilateral.

Since two lines define a unique intersection, an intersection identified in a particular position with respect to the polygon is repeated exactly *n* times by symmetry. Such an intersection will be produced by one particular quadrilateral. We therefore only need consider the set of quadrilaterals distinct up to symmetry.

For odd *n* these quadrilaterals can only be of the two kinds shown.
(a) Where no sides are parallel, there is 1 interior cut, 2 exterior: a gain of 1.

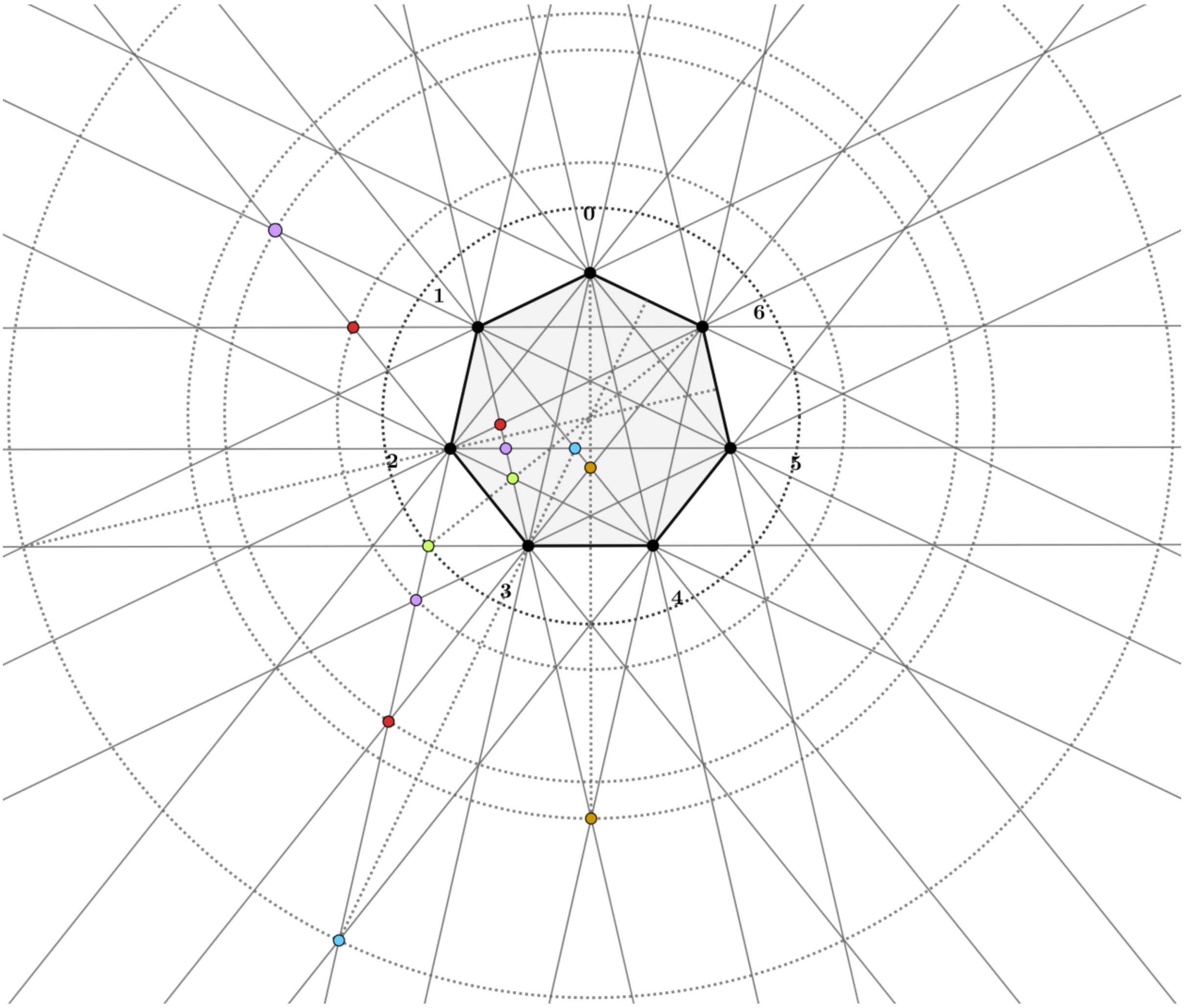
(b) Where there is one pair of parallel sides, there is 1 interior cut, 1 exterior: no gain, no loss.



Since both kinds of quadrilateral are present in any odd *n*-gon with more than 5 sides, there is always a net gain if *n* is odd and > 5.

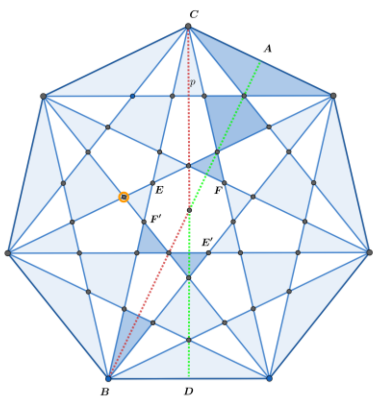
We can work out the numbers. See **mystic rose 12** following.

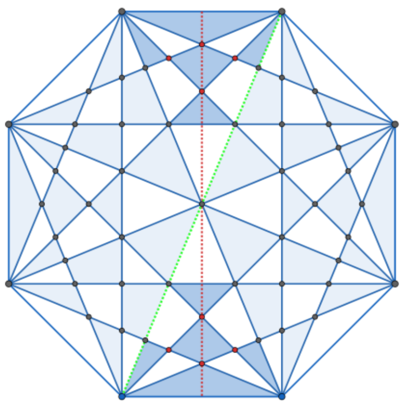
It's interesting to see how this particular case works out. Here is the 7-gon. The vertices are numbered so that you can label diagonals and intersections if needed. The dotted lines are symmetry axes. Points showing intersections due to the sides and diagonals of the same quadrilateral have the same colour. *Study how interior points map to exterior ones.*
The red and lilac dots, which are due to quadrilaterals of type (a), are responsible for the excess $2 \times 7 = 14$ points. *Notice how the red and lilac dots, which lie off symmetry axes, swap circles.*



If we take the right hand figure in **10.**, where the diagonals become semicircles, the equivalent figure for **12.** would complete the circles to produce a mirror image below the line. But we see from what we've just found that this would give an incorrect count for intersections outside the polygon.

13. These figures have been so coloured that regions sharing an edge have different colours. We only need two colours. *What guarantees this?*





Here are two approaches.

A) Every node - like that circled in the left hand figure - has an even number of half-lines coming out of it. I can make a tour of the node blue-white-blue-white so I have used just two colours. If I cross the node, I emerge in a region of the same colour: like colours are ‘vertically opposite’. What we have to show is that we shall never have to allocate whites and blues so that they end up in the wrong positions.

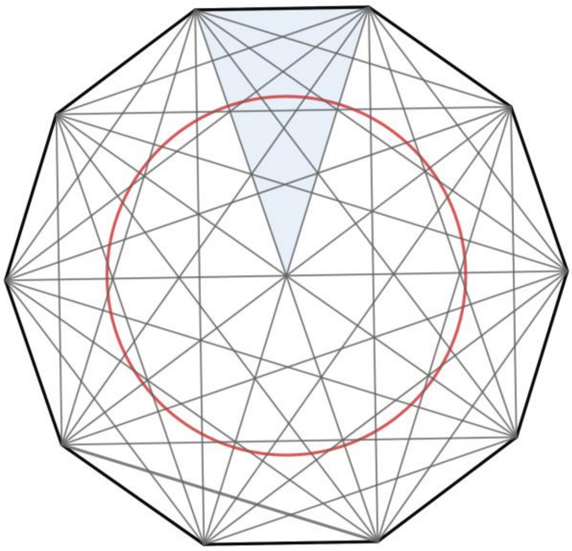
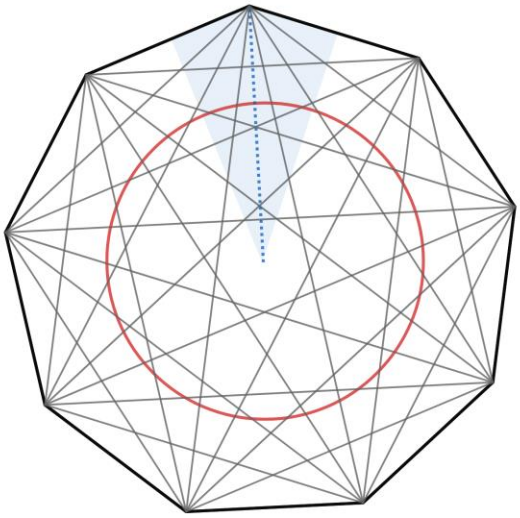
On the left we can move down the symmetry axis *AB* following the rule that we keep the same colour when we cross a node and swap colours when we cross an edge. We then reflect what we’ve coloured in the symmetry axis *CD*. We’ve missed the triangle *p* but, by rotating *E’F’B* to *EFC*, we see that we can give that the colour blue and that this colouring is consistent with the rest. *Confirm that this scheme is possible for some other odd value of n.*

On the right, in a similar way, we can consistently colour the top and bottom 1/8s of the polygon. If we reflect what we’ve completed in the green symmetry axis, we find we can simply swap blue for white and obtain a consistent colouring. This scheme will be possible for all even-sided regular polygons. *Try* $n = 10$.

B) This second approach depends only on the rotation symmetry of the rose and we do not need to distinguish roses of odd from roses of even order. But it depends on the same properties of an ‘even’ node. We proceed as follows.

1. We move from the centre and outwards, drawing concentric circles. Every time we meet an intersection point, we draw a black dotted circle. When those are complete, we insert red circles between them.
2. We start at an outer region and colour it blue.
3. We move in along a radius. If we cross an intersection point, we do not change colour; if we cross an edge, we do. We thus end up with a radius along which each region has a colour.
4. From every one of those regions we follow a red circle round the polygon. We shall not meet an edge, since those lie on dotted black circles, but we shall cross edges, upon which we must change colour.

We must first establish a property common to odd and even roses.

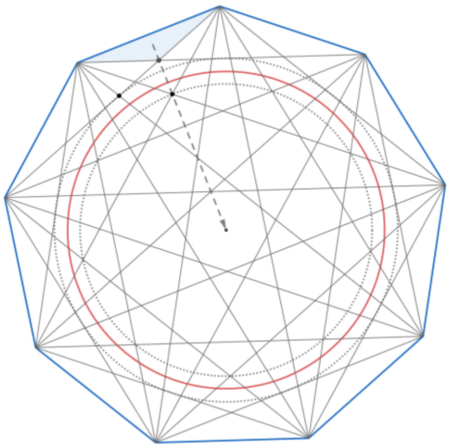
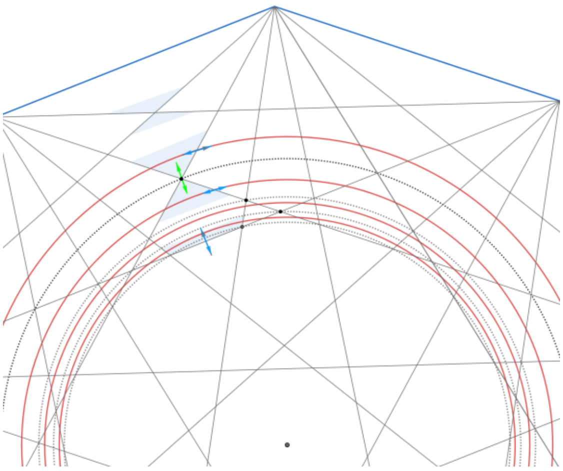
 

On the left we have an even-sided rose; on the right, an odd-sided one.

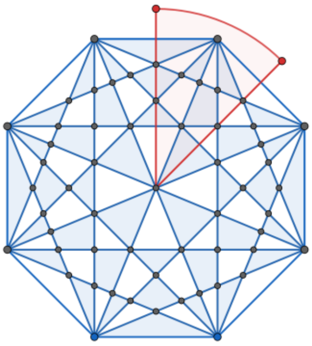
Whatever the number of regions the red line on the left passes through within the part shaded, the total is multiplied by the number of sides. Since this number is even, the grand total must be even.

Within the part shaded on the right, the symmetry axis shows that we have a central region, flanked by $2s$ regions and 2 ½-regions, that is $\left[1 + 2s + 2\left(\frac{1}{2}\right)\right] = 2(s+1)$ regions, an even number. Therefore, although we must multiply by an odd number to obtain the grand total, we are multiplying an even number so, again, the result is even.

We are therefore guaranteed a consistent sequence of alternating colours along each red circle. Since the full set of red circles pass through all regions of the rose - more than once in many cases - we shall have allocated a colour to every region, and done so consistently.

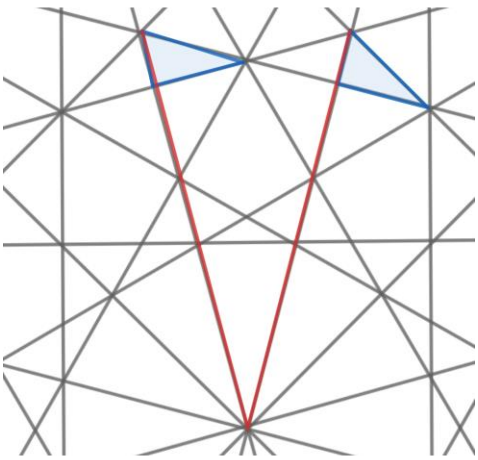
 

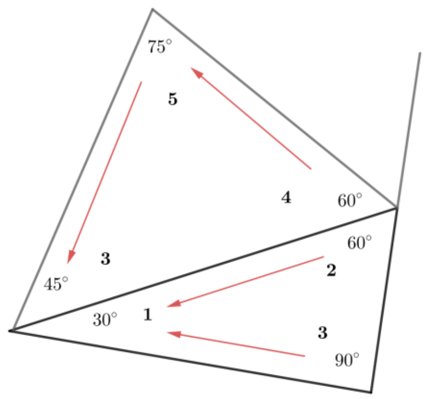
In the right hand figure for **13.**, the total area of the blue regions equals the total area of the white regions. *Prove this.*



This harks back to the previous item. 1/8 of the figure maps to an adjacent 1/8 so that what was blue becomes white and vice versa. Thus, in two adjacent 1/8s, totalling 1/4 of the whole, the blue and white areas are equal. Multiplying up by 4, the same proportion is true for the whole octagon.
A similar argument can be made for any regular polygon with an even number of sides.

14. From left to right we zoom into a rose. Can we identify the whole from a part? On the left we see lines which look as if they're symmetry axes radiating from a centre below at angles that look like 30°, clearly suggesting a 12-gon. In the middle, it's again fairly clear, focusing on the blue triangles, that a rotation of 30° is involved.





On the right we must fall back on **2.**, where we noted that every angle is a multiple of $\frac{\pi}{n}$. The straight angle on the right is made of three equal angles, of 60° therefore. We also recognise the right angle in the lower triangle. Moving out from those angles, we deduce the 30° bottom left, and estimate first the 75° then the 45° above. The angles of the lower triangle have greatest common divisor 30°, suggesting that *n* = 6. But, for the upper triangle, we see that the angles stand in the ratio 3:4:5, suggesting that *n* is not $1 + 2 + 3$ but $2 + 4 + 6 = 3 + 4 + 5 = 12$.

(b) Level: KS5

Mystic rose 1: The rose symmetry

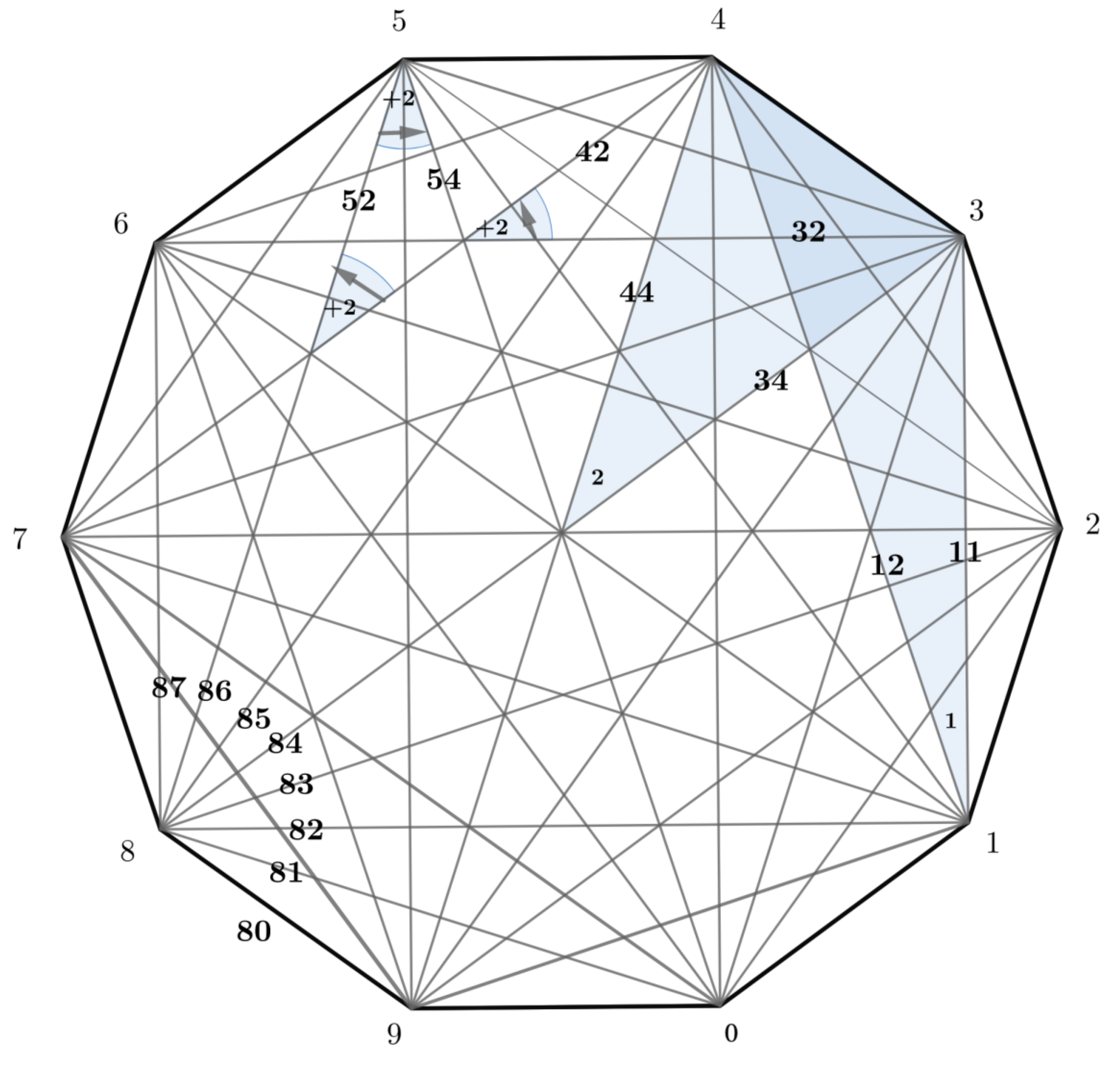
It belongs to the symmetry group D_n.

Mystic rose 2: Navigating the mystic rose

The rotation symmetry of the mystic rose reduces the calculation of angles to arithmetic with integers.

We shall code a diagonal/side with two numbers. The first is the vertex label, v ; the second gives its position among the diagonals, ordered anticlockwise from *vo*. (Because a side/diagonal has two ends, you may like to confirm that an alternative code to ab is $(a+b+1)(n-b-2)$, mod n.)

Every angle between diagonals/sides is a multiple of $\frac{\pi}{n}$. By always measuring angles in the same sense, anticlockwise say, we can work out the angle between any pair of diagonals from their codes.



From the figure we see that unit difference in the first digits represents an angle of size $\frac{2\pi}{n}$; unit difference in the second digits represents an angle of size $\frac{\pi}{n}$. Say we require the angle between 32 and 54, which we shall show with an arrow indicating the anticlockwise sense.

Using $\frac{\pi}{n}$ as our unit, we have

$$\begin{aligned} 32 \to 42 &= (4-3) \times 2 = +2. \\ 42 \to 52 &= (5-4) \times 2 = +2. \\ 52 \to 54 &= (4-2) \times 1 = +2. \end{aligned}$$

Therefore:

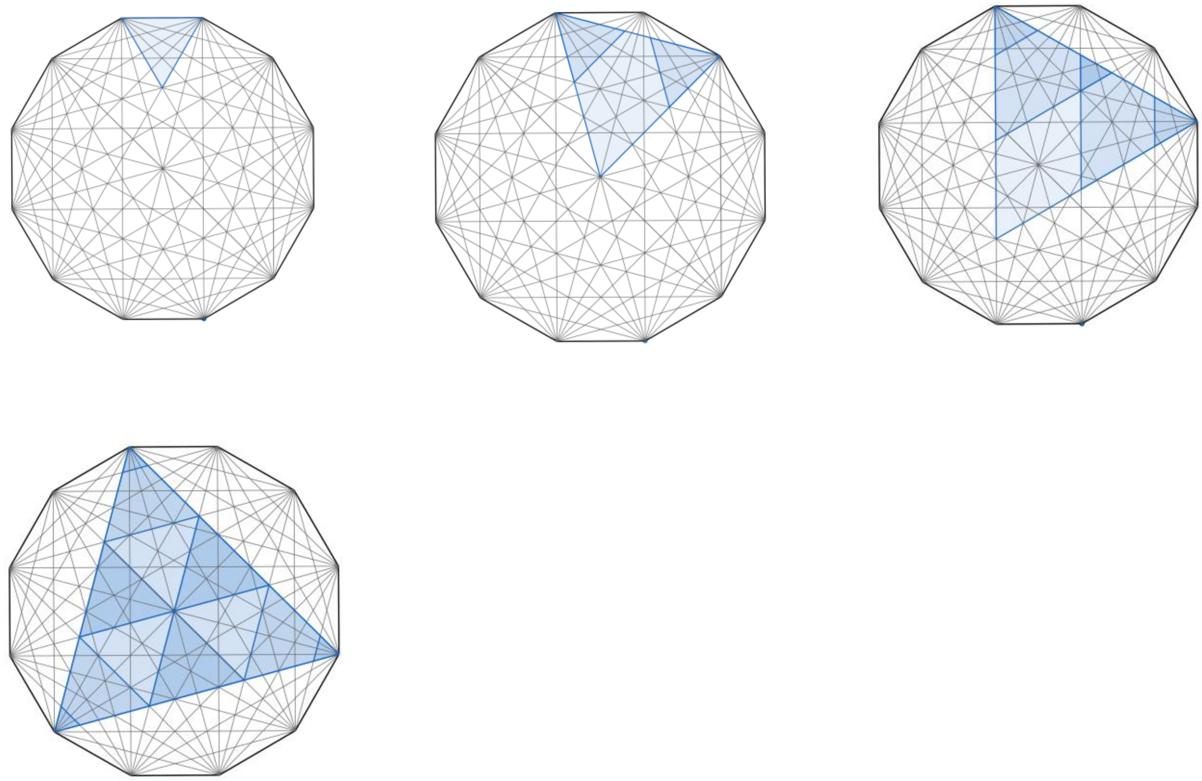
$$32 \to 54 = \qquad\qquad\qquad +6,$$

an angle of $\frac{3\pi}{5}$ therefore.

Mystic rose 5: Counting equilateral triangles in the order 12 rose

In making any count, we must avoid the dangers of, on the one hand, omitting examples, on the other, double-counting. It is therefore advisable to make the count in more than one way as a check. We shall use two.

1. We treat our iris blade as a pole with flags hanging from it - our equilateral triangles - and sweep it round a chosen vertex. This gives us equilateral triangles with their bases in all 4 possible orientations. There would be 12 but the symmetry of the triangle reduces these by a factor 3.

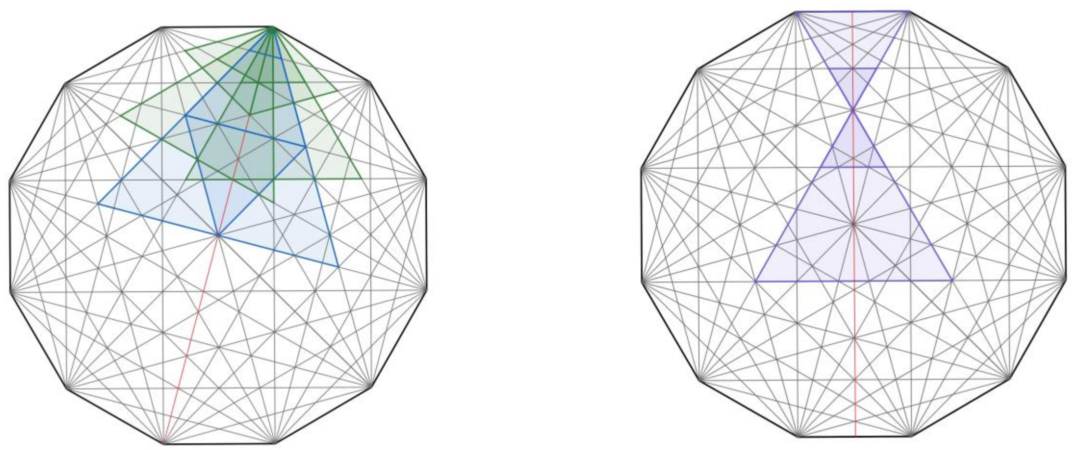


The triangles in the fourth case are shared by three distinct vertices. We must therefore divide that count by 3. Counting the triangles with their bases along the ‘poles’ we therefore have: $1 + 3 + 6 + \frac{13}{3}$. We must multiply this total by the order of rotation symmetry, 12, giving $\frac{43}{3} \times 12 = 172$.

2. A $2k$-sided regular polygon has k symmetry axes which pass through opposite vertices, and k which pass through the midpoints of opposite edges, so here, 6 of each.

Below left we have collected in blue half the triangles (4) which sit astride a vertex-vertex axis and in green half the triangles which lie off to the right (3) and half which lie off to the left (a matching 3). So that's 10 in total.

Below right we have collected in lilac half the triangles which sit astride an edge-edge axis (4).



We must take those $10 + 4 = 14$ and multiply by the order of rotation symmetry, 12: $14 \times 12 = 168$. Finally we must add 4 for the triangles which share three vertices with the 12-gon. So the grand total is 172.

Mystic rose 10: for odd *n*: (a) How many interior vertices, *i* ? How many vertices in total, *c* ? (b) How many line segments (edges), *e* ? (c) How many regions, *f* ?

We must assume the following lemma: If n is odd, just two lines cross in every interior vertex and every exterior vertex.

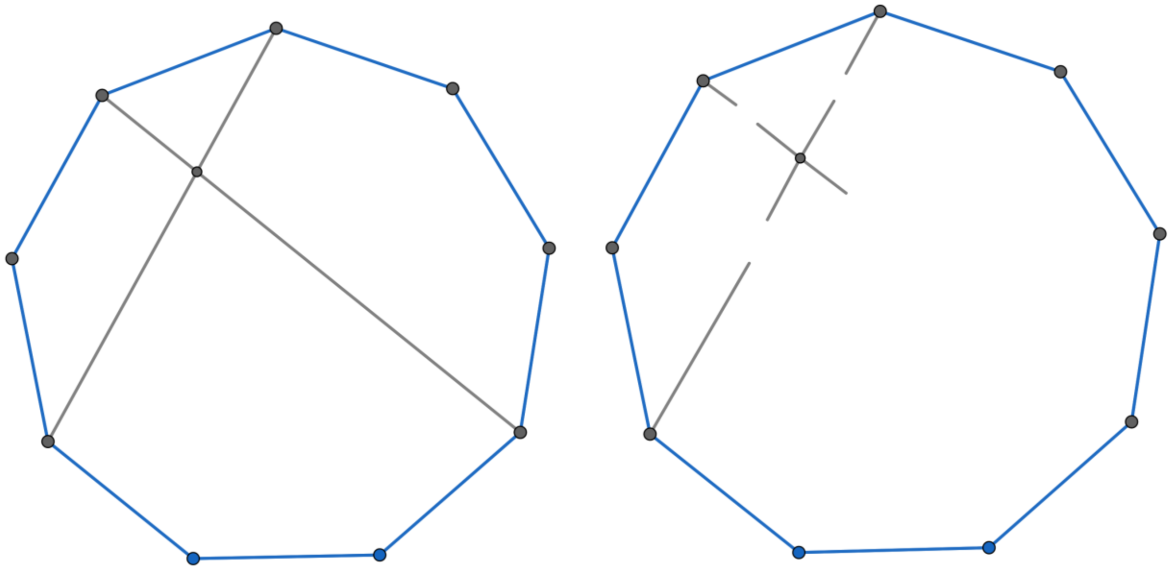
(a) This makes our first count easier since every particular intersection marks a crossing of two particular diagonals, and each diagonal is defined by two polygon vertices - see left hand figure. Therefore the number of intersections, i, is the number of choices of four vertices from the polygon's $2k + 1$,

$$i = \binom{2k+1}{4} = \frac{(2k+1)(2k)(2k-1)(2k-2)}{4!} = \frac{(2k+1)(2k-1)k(k-1)}{6}.$$

Hence the '35' for our 7-gon: $\frac{7\times5\times3\times2}{6}$.

For the total number of vertices, c, we must add in the $v = 2k + 1$ exterior vertices, so

$\boldsymbol{c = i + v} = \frac{(2k+1)(2k^3-3k^2+k+6)}{6}$.



(b) $2k$ half-edges come out of each of the $2k + 1$ exterior vertices, so the number of complete edges from that source, $e_v = k(2k + 1)$.

4 half-edges come out of each interior vertex, so the number of complete edges from that source,Type equation here. $e_i = 2i$.

So $\boldsymbol{e = e_v + e_i} = \frac{k(2k+1)(2k^2-3k+4)}{3}$.

(d) To use the Euler formula, we need to exclude the exterior region, so we have:

$\boldsymbol{f = e - c + 1}$
$= (e_v + e_i) - (v + i) + 1$
$= e_v - v + i + 1$
$= \frac{k(2k-1)(2k^2-k+5)}{6}$.

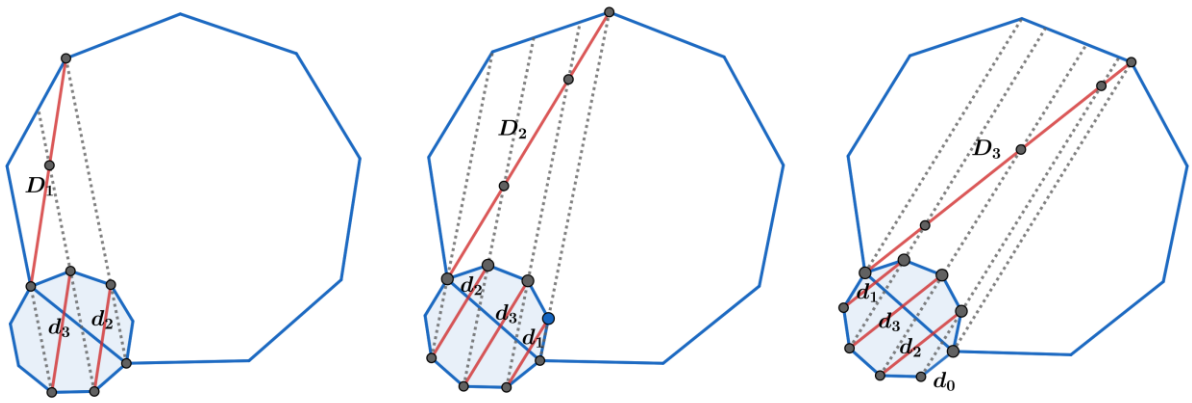
Mystic rose 11: (a) Relations between the lengths of complete diagonals, (b) The lengths of segments between interior vertices

Part 1

The topics **(a)** and **(b)** are related by a visualisation due to Anne Fontaine and Susan Hurley, which takes us back to our figure of overlapping similar polygons.

We take a 9-gon as our example. We do what we did before: we fit it to a similar figure so that a diagonal of the smaller coincides with a side of the larger.

We work with absolute lengths. For the smaller polygon, d_k is the length of the k^{th} diagonal, counting anticlockwise, say, from a vertex, where d_0 is the length of a side. For the larger, we use capitals so that the corresponding lengths are $D_0, D_1, D_2, \ldots$ We match D_0 to d_3.



Note that, by symmetry, $d_4 = d_3, d_5 = d_2, d_6 = d_1$.

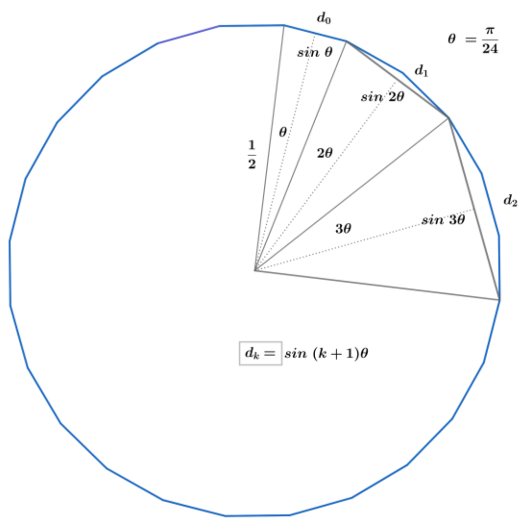
By parallel projection we see that:	By scaling, we also have:	So that:
$D_0 = d_3$ (the given condition),	$D_0 = d_0 \frac{d_3}{d_0}$,	
$D_1 = d_2 + d_3$,	$D_1 = d_1 \frac{d_3}{d_0}$,	$d_1 d_3 = d_0(d_2 + d_3)$,
$D_2 = d_1 + d_2 + d_3$,	$D_2 = d_2 \frac{d_3}{d_0}$,	$d_2 d_3 = d_0(d_1 + d_2 + d_3)$,
$D_3 = d_0 + d_1 + d_2 + d_3$.	$D_3 = d_3 \frac{d_3}{d_0}$.	$d_3 d_3 = d_0(d_0 + d_1 + d_2 + d_3)$.

In order to generalise the result we need to distinguish all the diagonals from a given vertex, d_0 to d_6 in this case. The last three equations then look like this:

$d_1 d_3 = d_0(d_2 + d_4)$,
$d_2 d_3 = d_0(d_1 + d_3 + d_5)$,
$d_3 d_3 = d_0(d_0 + d_2 + d_4 + d_6)$.

The features of significance are these:
1. There is one more term in the bracket than the suffix of the first 'd'.
2. The terms in the bracket are centred on the second 'd'.
3. The suffices in the bracket go up in 2s.

The general statement of the relation, Steinbach's 'diagonal product formula', is this:
Given a regular n-gon, if $k \leq \left\lfloor \frac{n-2}{2} \right\rfloor$, $h \leq k$, then $d_h d_k = d_0 \sum_{i=0}^{i=h} d_{k-h+2i}$.
It can be shown by centring the polygon in a unit circumcircle on the origin of an Argand diagram and using the fact that the vertices are roots of unity. But we will show it by direct substitution in a trigonometric identity.



Inscribing the n-gon in a circle of radius ½, we see from the figure that $d_k = \sin(k+1)\theta$, where $\theta = \frac{\pi}{n}$. (Compare **3.**)

The sum on the right of the formula is therefore a sum of sines, whose arguments run in arithmetic progression, from $d_{k-h} = \sin(k-h+1)\theta$
to $d_{k+h} = \sin(k+h+1)\theta$,
in steps of 2θ, $h+1$ terms in all.

The identity we need is:

$$\sum_{i=0}^{i=n} \sin(\varphi + i\alpha) = \frac{\sin\frac{(n+1)\alpha}{2} \sin\frac{2\varphi+n\alpha}{2}}{\sin\frac{\alpha}{2}}.$$

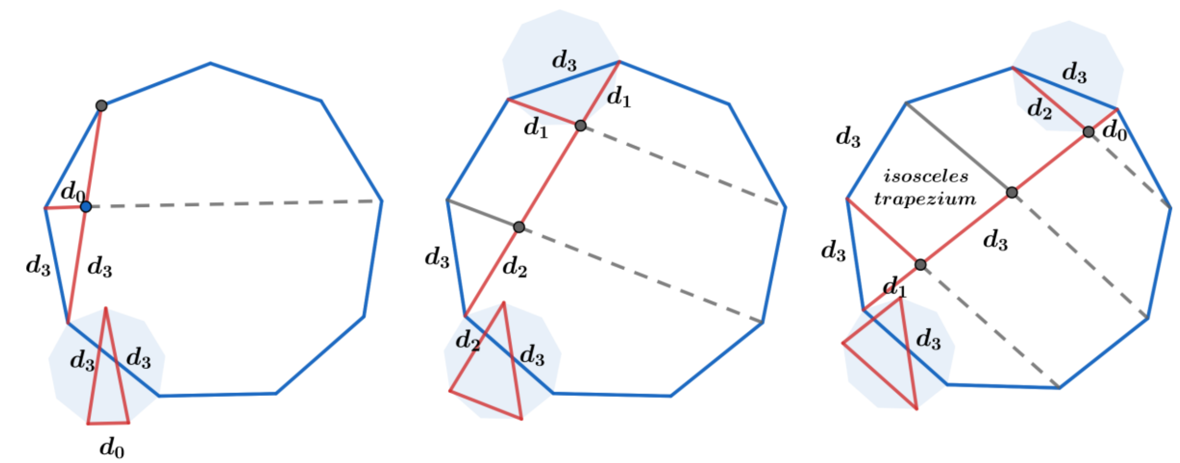
The substitutions: $n = h$, $\varphi = (k-h+1)\theta$, $\alpha = 2\theta$, yield: $\frac{\sin(h+1)\theta \sin(k+1)\theta}{\sin\theta} = \frac{d_h d_k}{d_0}$.

Substitution in the RHS of the identity yields the LHS, as required.

We can derive the special case of the diagonal product formula where $h = 1$, i.e. $d_1 d_k = d_0(d_{k-1} + d_{k+1})$, by taking figure **4.** In each case the ratio between the base and side of an isosceles triangle is $\frac{d_1}{d_0}$, so we have:
$d_0 + d_2 = \frac{d_1}{d_0} d_1$, whence $d_1 d_1 = d_0(d_0 + d_2)$, similarly $d_1 d_2 = d_0(d_1 + d_3), d_1 d_3 = d_0(d_2 + d_4)$, …

Part 2

What can we say about the points marking the divisions between the lengths marked in the three figures above?



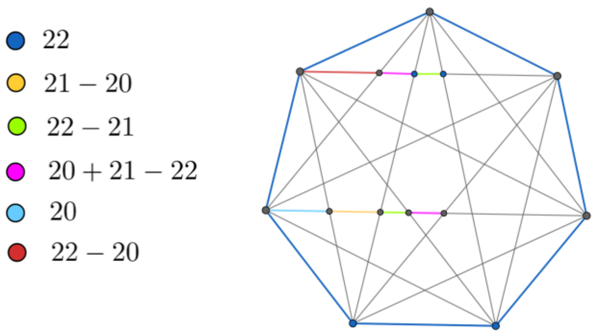
Through the small polygon, whose diagonals and sides parallel corresponding diagonals and sides in the larger one, we can identify congruent triangles mutually rotated by half a turn revealing parallels on the larger figure. In this way, or by identifying lengths and angles by other means, we find that the points in question are intersections of diagonals on the main figure, that is, interior vertices.

The distances will not generally be those between *adjacent* vertices. But we can always write distances between adjacent vertices as differences of these lengths.

Part 3

If we divorce the big polygon from its small partner, we must specify the latter when labelling lengths. In the above example, in place of $'d_k'$ we could write $'d_{3k}'$ to acknowledge this, but in the next figure we shall just write the suffix for shortness, '$3k$'. The new figure is a regular heptagon with the polygons set so that $D_0 = d_2$. A difference of two lengths will then appear as '$2k - 2l$'. (If we change the position of the auxiliary polygon so that it becomes a different size, we shall have a different first digit, and all the lengths will receive different codes.)

By parallel projection and the use of symmetry, we find there are just six distinct lengths.



The three diagonal lengths are:

$$D_0 = 22,$$
$$D_1 = 21 + 22,$$
$$D_2 = 20 + 21 + 22.$$

Note that 20, 21, 22 code the diagonal lengths of the auxiliary polygon. In these sums we see the same pattern we observed in the case of the 9-gon.

(If we set the auxiliary polygon so that $D_0 = d_1$, we find:

$$D_0 = 11,$$
$$D_1 = 10 + 12,$$
$$D_2 = 11 + 12.$$

But the former pattern can be restored by creating signed sums as we did for the individual segment lengths above.)

Part 4

Peter Steinbach found that what is true of all diagonals of a given regular *n*-gon is that, given a side length 1, their ratios constitute an extension field to the rational numbers, that is to say, all binary operations on linear combinations of the ratios yield a linear combination also. Particularly interesting is the case where $n = 2k + 1$ is prime. There the k distinct ratios form the basis for the extension field.

This fact is only well known in the case of the regular pentagon, where the basis has two elements, 1 and the golden ratio. But, for the regular heptagon, giving the ratios in order as $1, \rho, \ \sigma$, we can use the diagonal product formula to confirm these relations:

$$\rho^2 = 1 + \sigma, \rho\sigma = \rho + \sigma, \sigma^2 = 1 + \rho + \sigma.$$

From those in turn, we an derive these expressions:

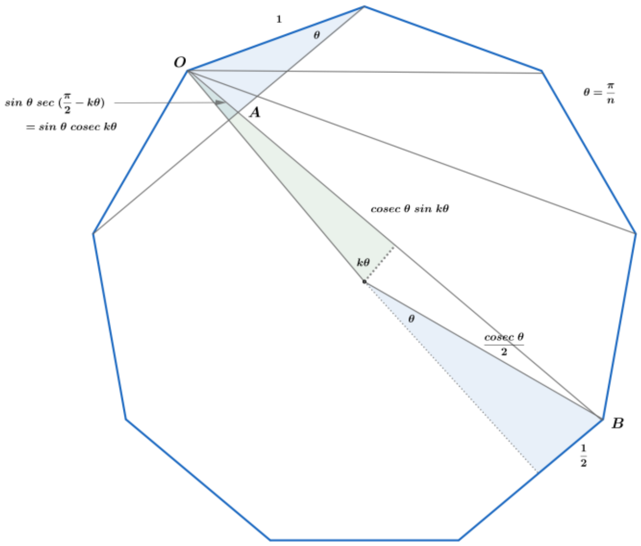
$$\frac{1}{\sigma} = \sigma - \rho, \frac{1}{\rho} + \frac{1}{\sigma} = 1, \frac{1}{\rho} = \rho - \sigma + 1, \frac{\sigma}{\rho} = \sigma - 1, \frac{\rho}{\sigma} = \rho - 1.$$

Part 5

Because the two overlapping polygons are similar:
(1) Any '*d*' ratio can be written as a *'D'* ratio.
(2) Any '*d*' value can be written as a '*D*' value by multiplication by the scale factor, here $\frac{d_0}{d_2} = \frac{D_0}{D_2}$.
Thus, for example, $22 = d_2 = \frac{d_0 D_2}{d_2} = \frac{D_0 D_2}{D_2} = D_0$.
In this way we can write all the individual segment lengths for the 7-gon in terms of D_0 by using the ratios found above. This gives us their relative lengths, which are as shown in this table. Below right we show how the diagonals are made up from these.

22	1		f	$a + b + c = c + d + e = 1$
21 – 20	$2\rho - \sigma - 1$		e	$2(a + b) + c \quad = \rho$
22 – 21	$2 - \rho$	$\}1/\sigma$	c	$2(c + d + e) + b \quad = \sigma$
20 + 21 – 22	$\sigma - 2$		b	
20	$\sigma - \rho$	$1/\sigma$	d	
22 – 20	$\rho - \sigma + 1$	$1/\rho$	a	

We note that the length of the first segment of a diagonal is the reciprocal of the diagonal's length. This is true for the general *n*-gon as we now show.

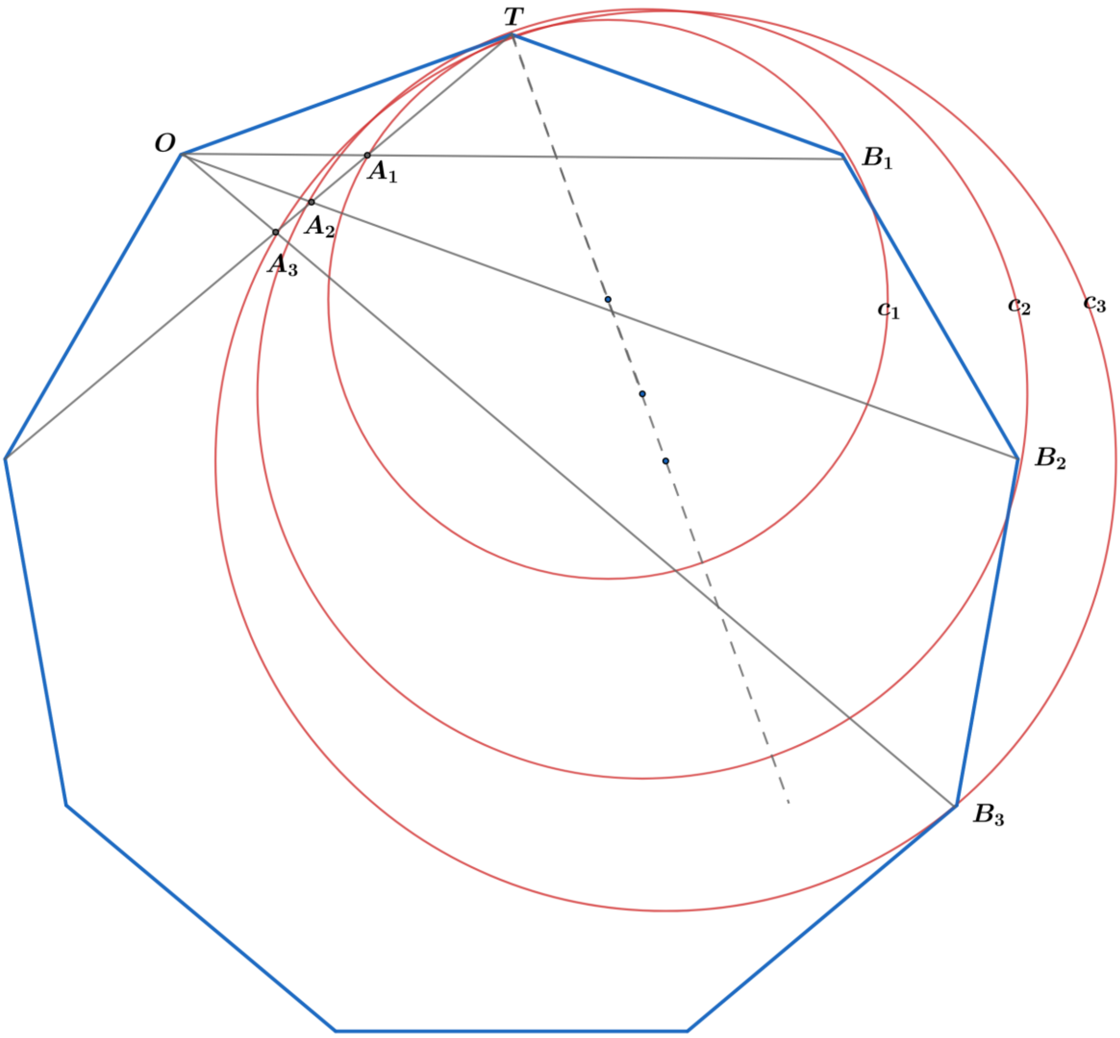


We take the odd case but a similar figure could be drawn for even *n*.

Similar right triangles have been given the same colour. (Note that the large and small green triangles have the opposite orientation.)

The trigonometry shows that $|OA| \times |OB| = 1$.

This has a geometrical consequence. Let the diagonal from T cut a diagonal from an adjacent vertex O in its first intersection, A. Let that second diagonal terminate at polygon vertex B. We have just shown that $|OA| \times |OB| = |OT|^2$. By the converse of the tangent/secant theorem, we can draw a circle which passes through A, B and is tangent to the polygon side OT at T. We thus obtain a family of such circles, c_1, c_2, c_3, ..., whose centres are collinear.



Mystic rose 12: Counting intersections

Let i_n be the number of interior intersections, o_n the number of exterior intersections.

Let $n = 2k + 1$, $k > 2$.
Four vertices, numbered in order (anticlockwise, say) 0, 1, 2, 3, define the two quadrilaterals 02, 13, which in turn define an intersection, 02/13. Therefore $i_n = \binom{2k+1}{4}$, the number of ways of choosing 4 from the number of vertices of the polygon. This is also the total number of quadrilaterals.

Type **(2)** quadrilaterals are isosceles trapezia. Their number up to symmetry is the number of pairs of parallel diagonals in the polygon, $\binom{k}{2}$. The total number, q_2, is therefore $(2k+1)\binom{k}{2}$.

o_n exceeds i_n by the number of type **(1)** quadrilaterals, q_1, which is $i_n - q_2$.
Therefore $o_n = 2i_n - q_2 = 2\binom{2k+1}{4} - (2k+1)\binom{k}{2} = \frac{(4k-5)(k-1)k(2k+1)}{6}$.

(One of $(k-1), k$ divides by 2. If neither divides by 3, then $(k-2), (k+1)$ must both divide by 3. In that case $(4k-5)$, which = $3(k-2)+(k+1)$, divides by 3. In all cases, therefore, the numerator divides by 6, confirming that o_n is an integer.)

We note therefore that the excess, e_n, demonstrated in part **(a)** = $o_n - i_n = \frac{(k-2)(k-1)k(2k+1)}{3}$.
(From the three consecutive terms, we see that 3 divides the numerator, confirming that e_n is an integer.)

For reference, $i_n + o_n = (k-1)^2k(2k+1)$; the total numbers of intersections including the polygon vertices = $(2k+1)(k^3-2k^2+k+1)$.

References

Anne Fontaine, Susan Hurley, 'Proof by picture: products and reciprocals of diagonal lengths in the regular polygon', Forum Geometricorum, **6** (2006), pp. 97 - 101.

Peter Steinbach, 'Golden fields: a case for the heptagon', Mathematics Magazine, **70**, 1 (February 1997), pp. 22 - 31.

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