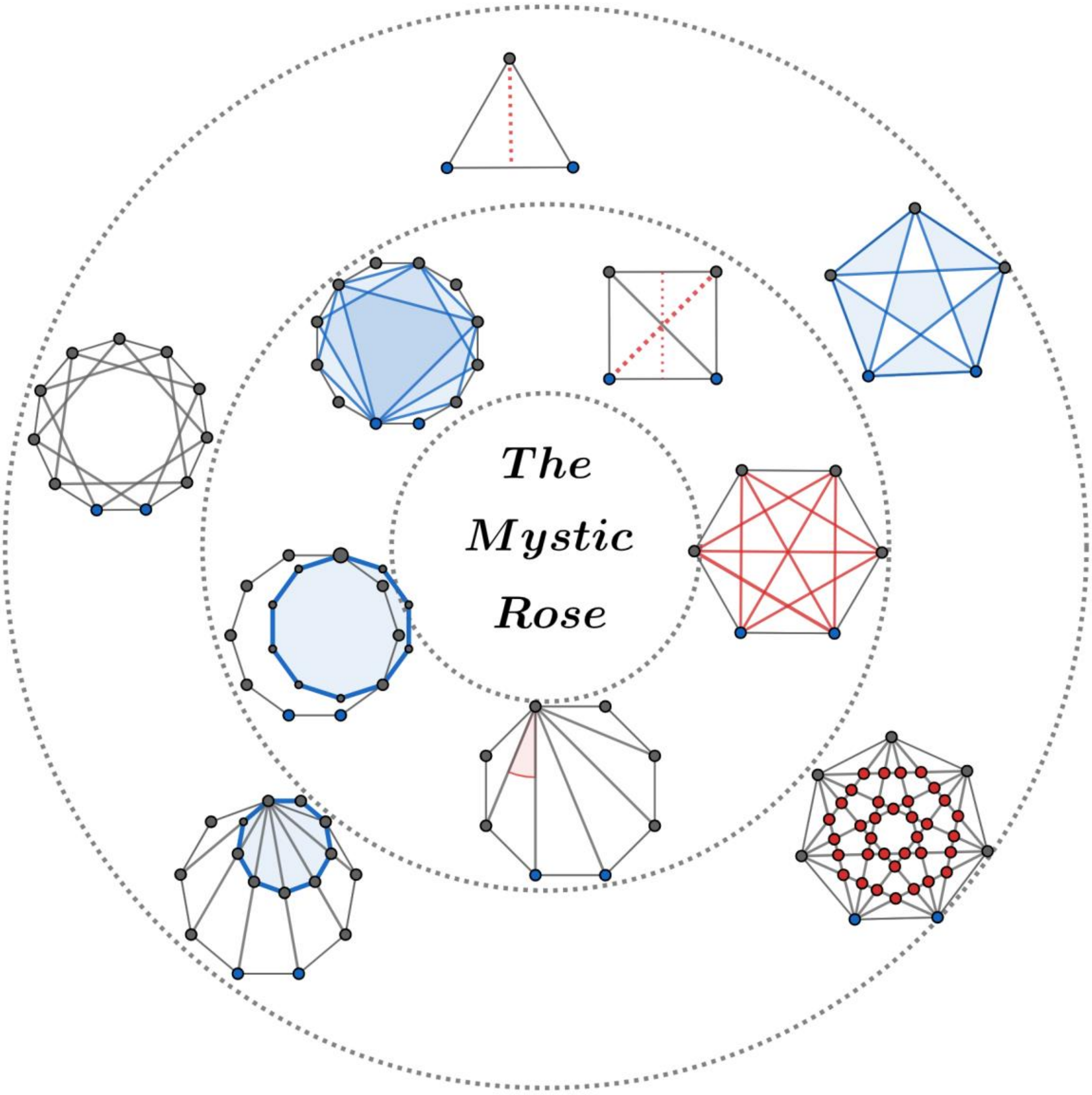
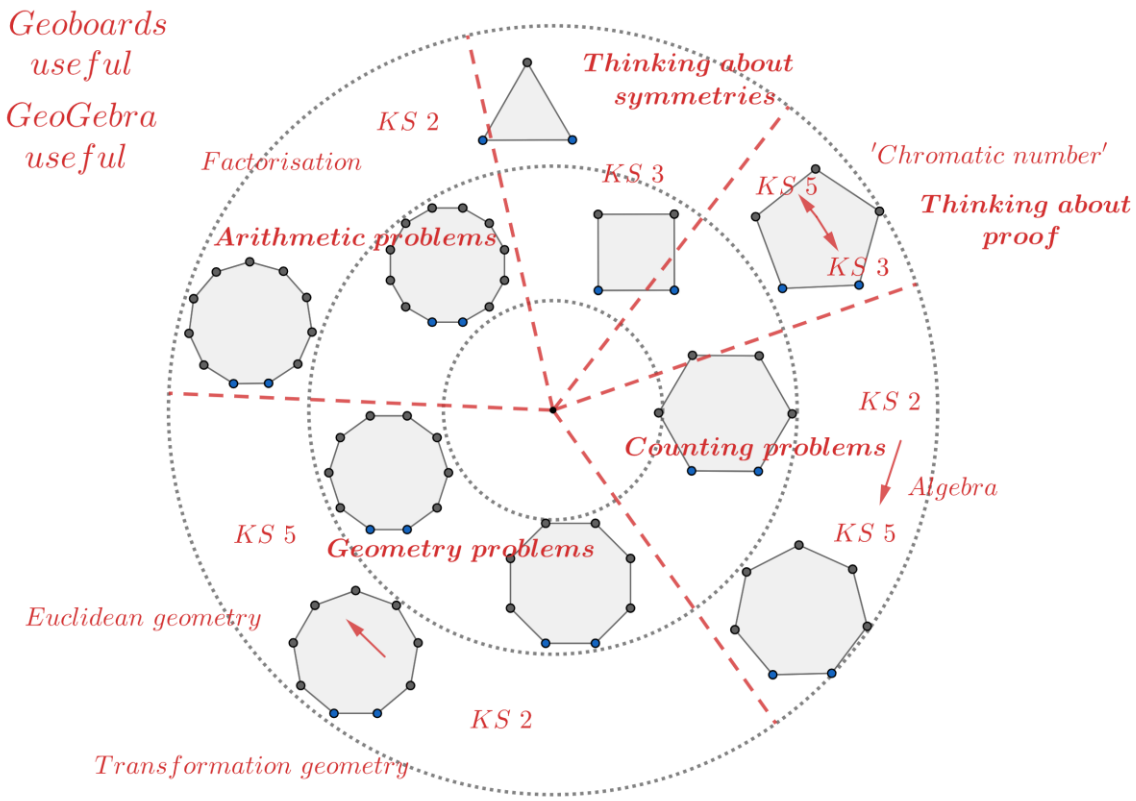

The
Mystic
Rose

A mystic rose is a regular polygon together with all its diagonals. We shall call a rose based on a regular n-gon a rose *of order n*. We shall call page 1 the *summary chart.*

Here is an overview of the work we shall do with the rose, the means we shall employ and the levels of difficulty we shall encounter.

In the figure, even-sided roses lie in the inner ring, odd-sided roses in the outer ring.
We find that some properties hold for both types; some do not.



In the absence of physical geoboards, nrich provides virtual ones: nrich.maths.org/2810 .

The GeoGebra instructions assume the children are working two to a computer.

Unless stated otherwise, the figures which appear in the text are to be projected.

Here are some questions we shall try to answer.

How can you generate a rose?

What are its symmetries?

1

What other regular polygons
lie in a rose of order *n* ?

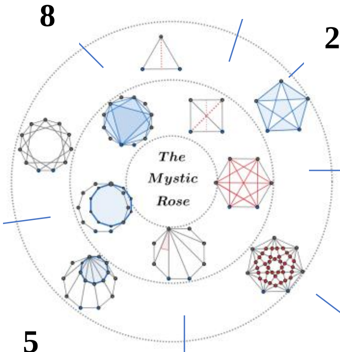
8

2

How many colours are
needed to map-colour
a rose?

How do you trace a
star polygon in the
diagonals of a rose?

7



How many diagonals
and sides has a polygon
of *n* sides?

3

When do interior
vertices belong to
a similar polygon?

6

5

4

How many interior
vertices has an odd-sided
rose?

How many edges?

How many regions?

How are the lengths
of adjacent diagonals
related?

What is the angle between
two adjacent diagonals ?

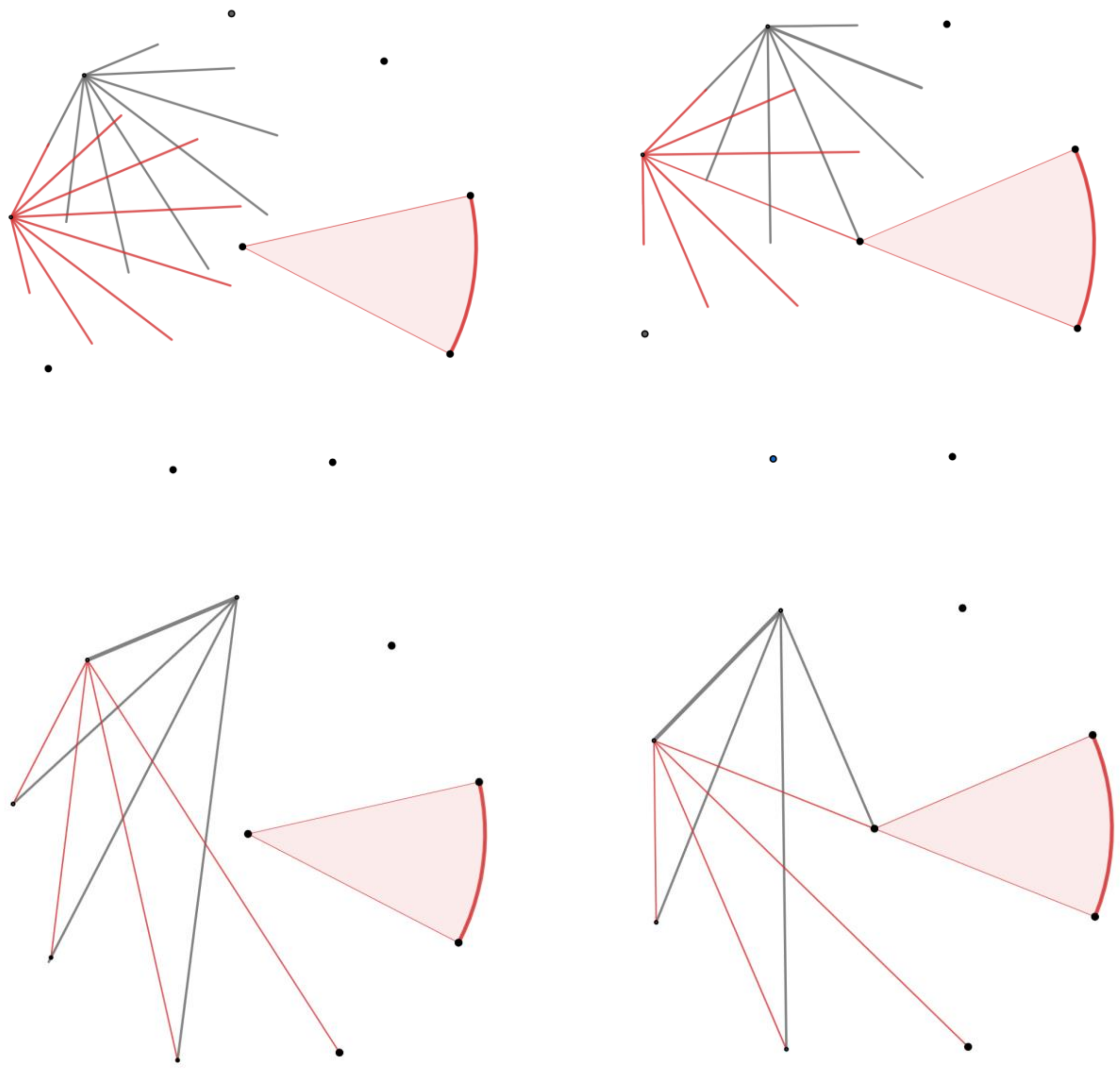
What angles between
diagonals are possible?

In the following pages we give the mathematical background to each topic, then suggest an activity. Instructions for the teacher are in bold **black**, instructions for the student in blue. I give these in adult terms for you to interpret appropriately. The GeoGebra instructions assume familiarity with the basic tools and are therefore brief. Again, supply the detail appropriate to the version of GeoGebra you are using and your children’s expertise with it. Of course, you may use other software packages: Cinderella, Autograph, ...

1

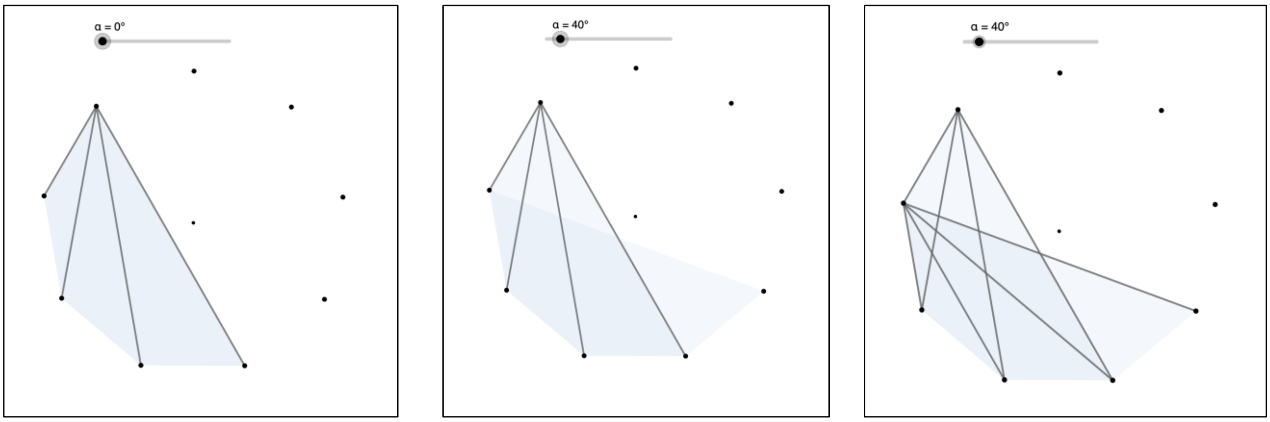
We begin at the beginning by generating a rose. Our aim is to gain an appreciation of the symmetry. This symmetry is key to everything we shall do.

A rose has the symmetries of the parent *n*-gon. We shall use the rotation symmetry to generate it from *n* congruent pieces. We generate an order 9 rose on the left, an order 8 rose on the right. We keep rotating the piece by the same angle, $\frac{2\pi}{9}$ on the left, $\frac{2\pi}{8}$ on the right. Here are two pencils of line segments we can choose. In the first we use complete lines; in the second, half-lines. *Any combination of lines and line-segments is possible which contains exactly 1/n of the rose.*



Use GeoGebra. Choose your n value (here 9). Decide on either the first or the second arrangement of lines (here the second). Use the **Circle with centre** tool to locate the centre. Use the **Regular polygon** tool to construct the 9-gon within it but hide all but the circle centre and the 9 vertices. Draw 4 line segments from a chosen vertex as above. Use the

Polygon tool to define your pencil of lines as an object, removing the boundary. Set up a **Slider** with intervals of $\alpha = 40°$. Set that angle (anticlockwise, say) for the **Rotation** tool. Each time you move the slider on one position, draw in the black lines:



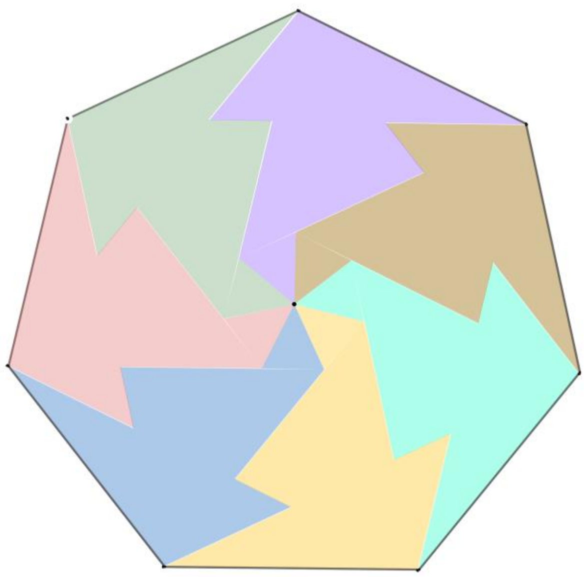
Project the summary chart.

On the screen you see a selection of roses, odd-sided on the outside, even-sided on the inside. How do the symmetry axes lie in the two cases? What do we know from the construction you have just made about their rotation symmetry?

Summary:

For an even-sided regular polygon, the symmetry axes alternately bisect a pair of opposite sides and pass through a pair of opposite vertices. For an odd-sided regular polygon the symmetry axes bisect a side and pass through the opposite vertex. In both cases the order of rotation symmetry is equal to the number of sides or vertices.

Using GeoGebra, draw an order 7 rose. Using the **Polygon** tool, trace and colour the most complicated shape you can which generates the rose by rotation. Here's an example:



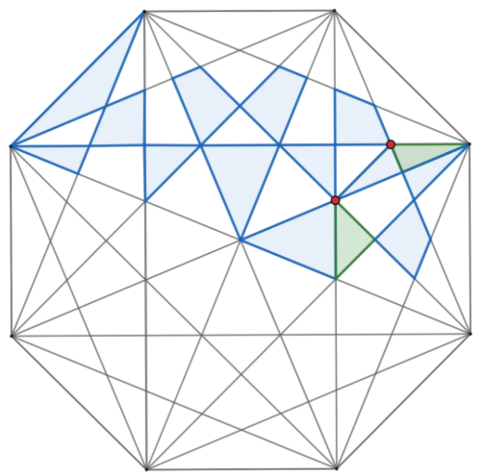
2

We map-colour a graph when we allocate colours so that two regions sharing an edge do not share a colour. We use the fewest colours possible. The ‘four colour map theorem’ tells us that we never need more than four colours.

Distribute paper copies of order 8 roses.

Choose a quadrilateral formed by the rose diagonals somewhere on the right and draw a single diagonal across it, i.e. join two rose diagonal crossing-points (interior vertices). Mark the points in red. Note therefore that this diagonal is *not* a rose diagonal.

Use one colour, plus white as your second colour. Keep a third colour in reserve. Advancing from the left, try to colour the rose using just your two colours.



Discuss the result of the experiment.

Points which should emerge:

An ‘odd’ node (one with an odd number of edges coming out of it) needs three colours (like the false nodes in red we introduced.)

Because lines cross in each interior vertex, all interior rose nodes are even.

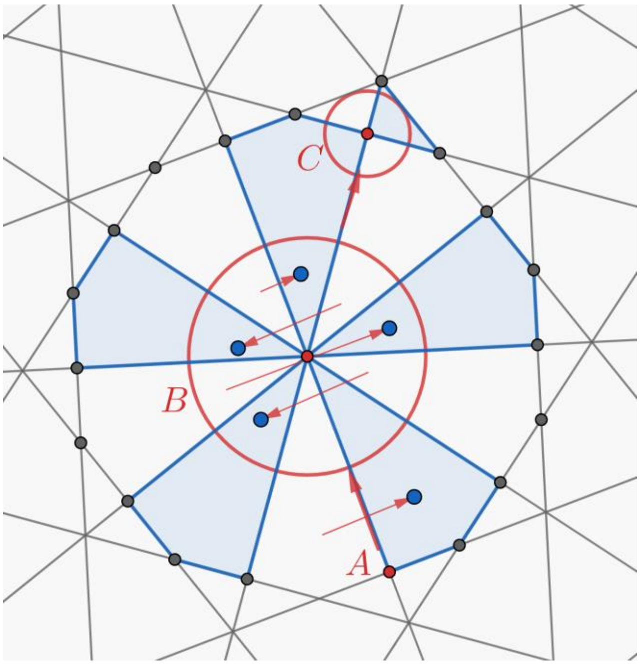
Even nodes need just two colours.

What needs to be shown is that this applies to the whole graph.

All we can say provisionally is that a rose needs *at least* two colours.

Working in pairs or small groups, compose a proof that two colours are enough, and prepare a presentation to the whole class.

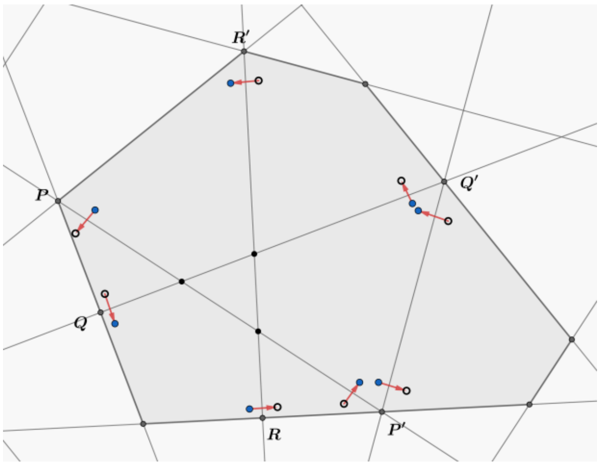
Many approaches are possible. We’re trying to establish a global property, which is ambitious, so it’s probably best to ‘think local’. Here is one inductive argument.



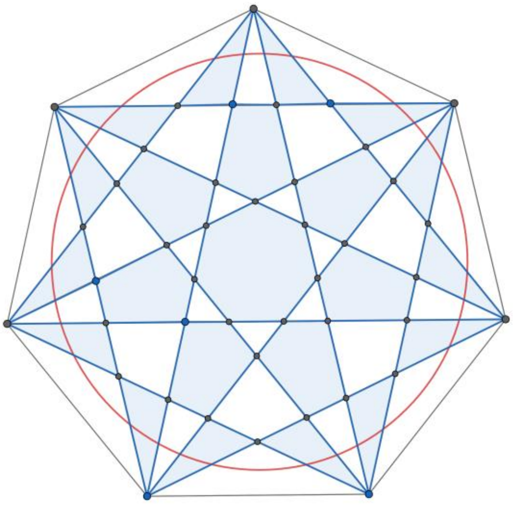
We start at a random vertex *A* and proceed along an edge, choosing blue for the region on our right. When we reach *B*, we know that we can allocate colours consistent with that choice because it is an even node. We proceed along a new edge from *B* and argue in the same way when we reach *C*. And so on.

What happens if we complete a circuit and arrive back at a vertex around which the colours have already been allocated? What guarantees that the colour scheme will be consistent with colours allocated on the new tour?

The next figure shows a patch around which we intend to make a circuit. Because a line like *PP'* crosses the whole region, we cross it twice. On completing our circuit we therefore make an even number of crossings, ensuring a consistent colouring.



The exterior vertices may be odd or even but are irrelevant to our argument since we shall already have chosen colours for the regions meeting there. That is to say, having completed our tour of vertices within the red circle, we shall have a complete colouring of the rose.



3

There are many instructive counts we can make. We shall advance from the familiar 'handshakes' problem to numbers requiring several preliminary counts. These problems are not conceptually more difficult. I give a difficulty level of KS 5 to these only because of the amount of algebra required.

3.1 How many lines (sides plus diagonals) are there?

Without requiring a generalisation in algebraic form, this problem is accessible to KS 2 children.

Use a 12-pin circular geoboard, ideally with removable pegs, and a set of rubber bands. How many lines are there joining 3 pegs to all the others? 4 pegs? 5 pegs? 6 pegs? (Space your pegs as far apart as possible.) Make this table. Predict and check the number for 7 pegs.

Number of pegs	Number of lines	Difference
1	0	
		1
2	1	
		2
3	3	
		3
4	6	
		4
5	10	
		5
6	15	
7		

1st order triangle numbers

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

Children can be introduced to Pascal's triangle, which we shall refer to as we proceed through the activities. Able KS3/4 students can be shown the young Gauss' discovery and led to the formula for the n^{th} triangle number, $T_n = \frac{n(n+1)}{2}$. More simply, keen programmers can use the recurrence relation $T_n = T_{n-1} + n$ to generate the triangle numbers.

3.2 How many interior vertices are there in odd-sided roses?

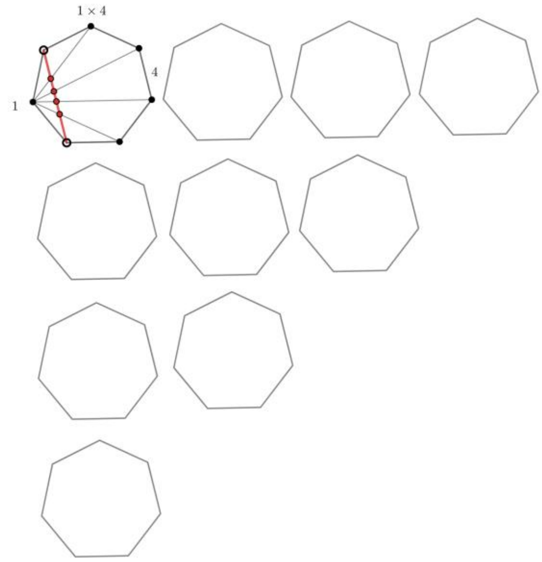
We shall come at the number of interior vertices from two directions: the first extends our work with triangle numbers; the second (more direct but more advanced) solution characterises the r^{th} entry in the n^{th} row as 'n-choose-r'. But first, an activity for everyone. **Project the summary chart.**

What differences do you notice between the odd- and even-sided polygons as regards their diagonals?

In the even-sided ones sometimes more than two lines pass through a single vertex.

The property of odd-sided roses that only two lines pass through one interior vertex makes it easier for us to find the number we want, so we shall deal only with those.

(a) Project, and distribute copies of this worksheet for the order 7 rose.



Explain the principle of the task and how the students are to proceed.

The first diagonal is drawn from the bottom left vertex. From this vertex four diagonals can be drawn. These will occupy the top row.

All the cuts on this first diagonal are caused by diagonals from the one vertex on the left to the four on the right. There are therefore 1×4. We've written 1×4 above the figure.

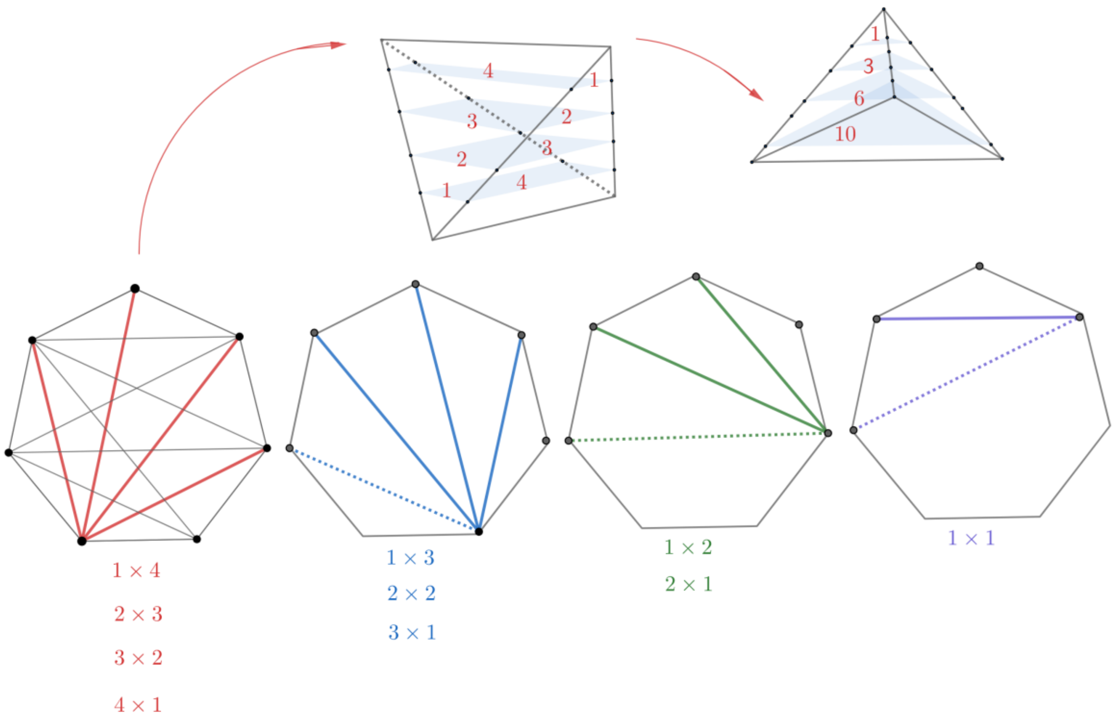
All the cuts on the second diagonal are caused by diagonals from the two vertices on the left to the three on the right. This means 2×3 cuts.

Complete the row in red.

The first row has given us the cuts on all the diagonals from the bottom left vertex so we can exclude that vertex from the second row, which is devoted to diagonals from the bottom *right* vertex. This leaves only three diagonals to consider. Try to complete the row in blue.

Use green for the third row, devoted to diagonals from the next vertex round anticlockwise.

Use lilac for the fourth row.



In this figure each 7-gon shows the results for a complete row. Take the red row, summarised by the first figure. What pattern do you see in the numbers?

[The numbers on the left go up from 1 to 4 as the numbers on the right go down from 4 to 1.]

The figure above shows that the products are the dimensions of rectangular layers of spheres packed in a tetrahedron set on edge. Laying the tetrahedron flat so that the horizontal layers are triangular, we see that we have the sum of the first 4 triangle numbers, 20.

We call 1, 3, 6, 10, ... the triangle numbers of the *first order*.
1, 1+3 = 4, 1+3+6 = 10, 1+3+6+10 = 20, ..., are the triangle numbers of the second order.

But we must total the red, blue, green and lilac numbers. In other words we must add consecutive triangle numbers of the second order. This gives us the 4^{th} triangle number of the *third* order:

1+4+10+20 = 35.

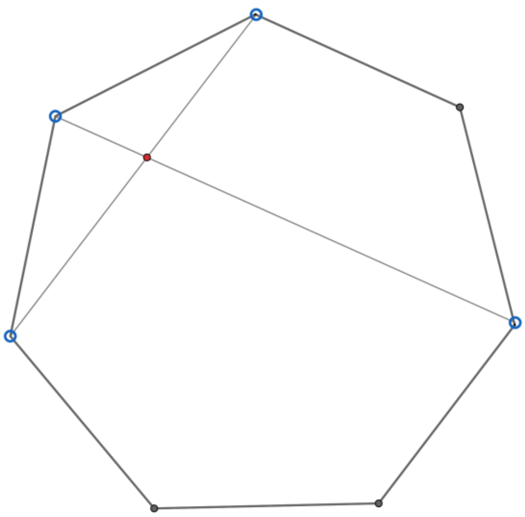
The answer to the question: ‘How many interior vertices has a 7-gon?’ is therefore 35.
How many interior vertices has a 9-gon? an 11-gon? a 13-gon?, ...

If we make it big enough, we can read off the answers from Pascal’s triangle. Here anyway is our 35:

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1
\end{array}
$$

The first 4 **second** order triangle numbers summing to the 4^{th} **third** order triangle number

(b)



Here is our 7-gon again.

Each interior vertex is defined by a single pair of diagonals, and each pair of diagonals is defined by four exterior vertices.

The number of interior vertices is therefore 7-choose-4.

This provides an opportunity to introduce the way we work that out, which may be sketched somewhat as follows.

From 7 there are 7 ways of choosing the first item, 6 ways of choosing the second, 5 ways of choosing the third, 4 ways of choosing the fourth. That gives $6 \times 5 \times 4 \times 3$ possibilities. But consider those 4 choices. There are 4 ways to choose the first item, 3 ways to choose the second, 2 ways to choose the third, 1 way to choose the fourth. These $4 \times 3 \times 2 \times 1$ possibilities all represent the same selection. So the number of distinct ways of choosing 4 from 7 is $\frac{7\times6\times5\times4}{4\times3\times2\times1} = 35$, written $\binom{7}{4}$.

All the entries on Pascal’s triangle are numbers of this type.

$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

This is how the first order triangle numbers appear In this notation.

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$

$$\binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6}$$

$$\binom{7}{0} \quad \binom{7}{1} \quad \binom{7}{2} \quad \binom{7}{3} \quad \binom{7}{4} \quad \binom{7}{5} \quad \binom{7}{6} \quad \binom{7}{7}$$

Pascal's triangle has a vertical symmetry axis: $\binom{7}{4} = \frac{7\times6\times5\times4}{4\times3\times2\times1} = \binom{7}{3} = \frac{7\times6\times5}{3\times2\times1}$.

To summarise, for an odd-sided rose of n sides there are therefore $i = \frac{n(n-1)(n-2)(n-3)}{4\times3\times2\times1}$ interior vertices. If $n = 2k+1$, this is $\frac{(k-1)k((2k-1)(2k+1)}{6}$.

3.3 How many line segments (edges) are there in odd-sided roses?

Set older students this challenge, pointing out the following approach.

Work in half-edges. $(n-1)$ half-edges come out of each of the n exterior vertices; 4 half-edges come out of each of the i interior vertices.

The edge sum works out as $\frac{n(n-1)(n^2-5n+12)}{12}$ or $\frac{k(2k+1)(2k^2-3k+4)}{3}$.

3.4 How many regions are there in odd-sided roses?

Set older students this further challenge, giving them Euler's formula with the exterior region excluded:

Total number of vertices (interior and exterior) + number of regions = number of edges + 1.

The number of edges works out as $\frac{(n-1)(n-2)(n^2-3n+12)}{24}$ or $\frac{k(2k-1)(2k^2-k+5)}{6}$.

A rider, which your older students can investigate. Call the number of regions for a rose of order n, where n is odd, r_n, and ,correspondingly, the number of interior vertices, i_n .

$$r_n = \frac{(n-1)(n-2)(n^2-3n+12)}{24}$$
$$= \frac{(n-1)(n-2)[n(n-3)+12]}{24}$$
$$= \frac{n(n-1)(n-2)(n-3)}{24} + \frac{(n-1)(n-2)}{2}$$
$$= i_n + T_{n-2}.$$

Writing $n = 2k + 1$ to acknowledge we're dealing with odd-sided roses:

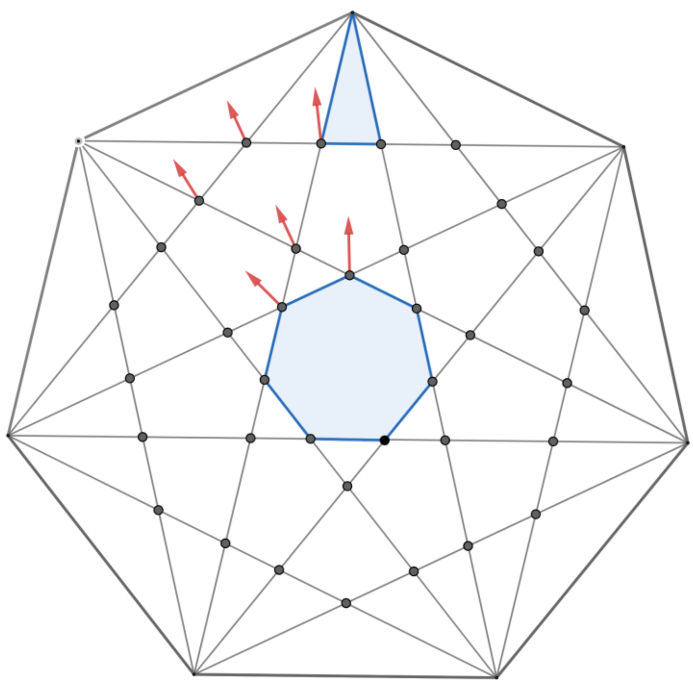
$$r_{2k+1} = i_{2k+1} + T_{2k-1}, k \geq 1.$$

To see the significance of the triangle number, the students should write down a few and see how they relate to (2k+1):

k	$2k+1$	T_{2k-1}	breakdown
1	3	1	$0(3)+1$
2	5	6	$1(5)+1$
3	7	15	$2(7)+1$
4	9	28	$3(9)+1$
5	11	45	$4(11)+1$

Our identity is:

$$T_{2k-1} = (k-1)(2k+1) + 1 .$$



In the figure we have moved out along radii, pairing each interior vertex with the region beyond.

We see that, where we have a diagonal containing an even number of vertices, we have a region (blue) unaccounted for.

There are $(k-1)$ such diagonal types. And symmetry repeats the missing regions $(2k+1)$ times.

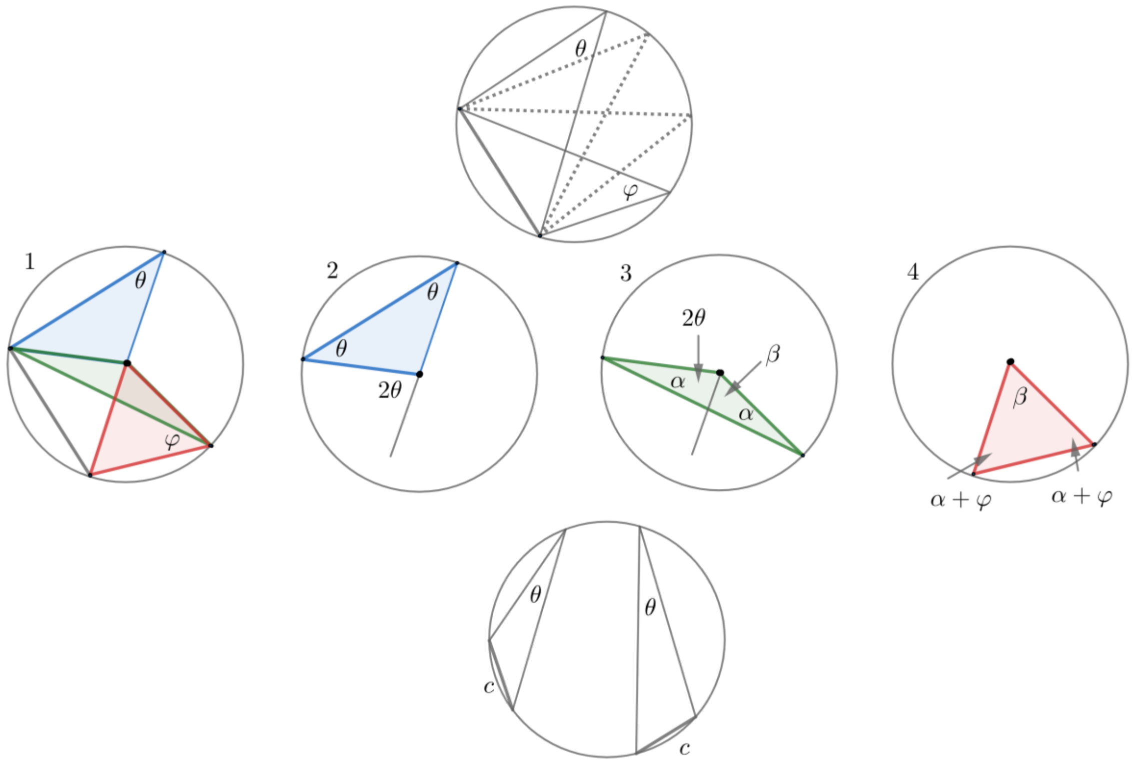
We also have the central region to account for.

This gives us the term needed.

Reversing the argument, we derive r independently of the Euler identity.

4

4.1 The children can use 12-pin circular geoboards with a central pin as you go through the following geometry on this projected figure. With younger children, for the Greek letters denoting angle sizes, substitute coloured dots.



Following is the geometry you need to convey. In the figure above, the upper circle shows two angles subtended by the same chord at the circumference. We wish to compare their sizes. We do so by identifying a series of isosceles triangles whose equal sides are circle radii. We know that in such triangles base angles are equal. The triangles we need are shown in figure 1.

Figure 2: we find the size of the exterior angle, 2θ.

Figure 3: we locate the 2θ in the green triangle, name the base angles α and the remaining angle β, giving a total of π.

Figure 4: we complete the labelling without having to name any further angles. Again we have a total of π.

Figures 3 and 4: we equate the totals:

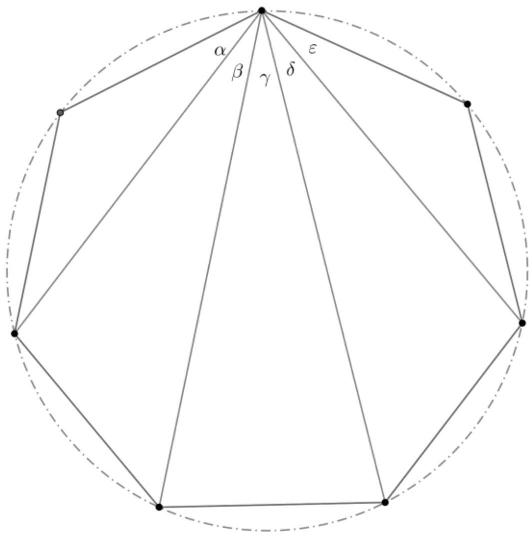
$$2\theta + 2\alpha + \beta = 2\varphi + 2\alpha + \beta,$$

inferring that $\theta = \varphi$.

Discuss with the children the significance of this result.
In the circle below the sequence of four we take a further logical step, which you can present to the children in these terms:

Instead of drawing the two angles on the same chord, we've taken one and rotated it about the centre of the circle. If we now think of the two chords as separate but of the same length, what result have we found?

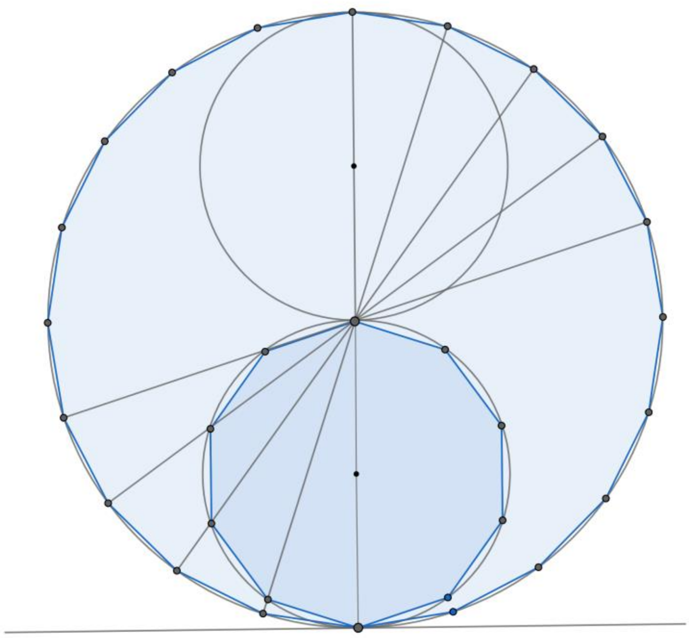
[In the same circle, angles on chords of the same length are equal.]



What can you say about angles $\alpha, \beta, \gamma, \delta, \varepsilon$?

[They're all equal.]

For the next activity, the children work in pairs with GeoGebra.



What does the figure show?

[A regular 20-gon,
A regular 10-gon of half the circumradius
laid along a circumradius,
a circle of the same size above it]

Project the figure with circle figures 1 to 4 again.

Something we found but didn't comment on in Figure 2: the angle at the centre is twice the angle at the circumference.

Return to the main figure. What is the angle at the centre of the 20-gon?

[$360° \div 20 = 18°$]

What would be the angle at the centre of one of the 10-gons?

[$360° \div 10 = 36°$]

What is the angle at the circumference of one of the 10-gons?

[$36° \div 2 = 18°$]

What can you say about the points where the radii cut the upper circle?

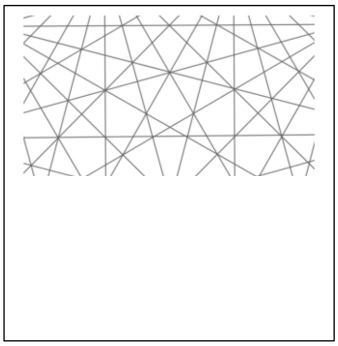
[They're vertices of a regular 10-gon.]

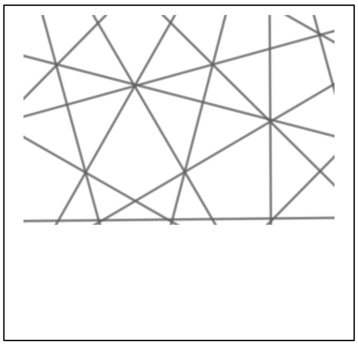
Using GeoGebra, draw a large 16-gon and *construct* a small 8-gon within it, i.e. use the intersections we've just noted not the GeoGebra **Regular polygon** tool.

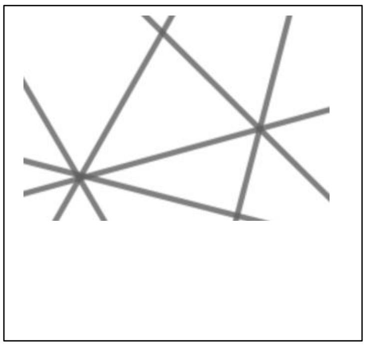
4.2 From the way we generated a rose in activity **1**, we saw that the same diagonals are repeated at equal angles round the rose. This means there is only a handful of angles between any two diagonals, and therefore only a handful of angles in the polygons they form.

The next figure the children can also bring up as a GeoGebra file.

Bearing that in mind, here's a puzzle. In these three pictures we zoom in on a particular rose. Which is it?







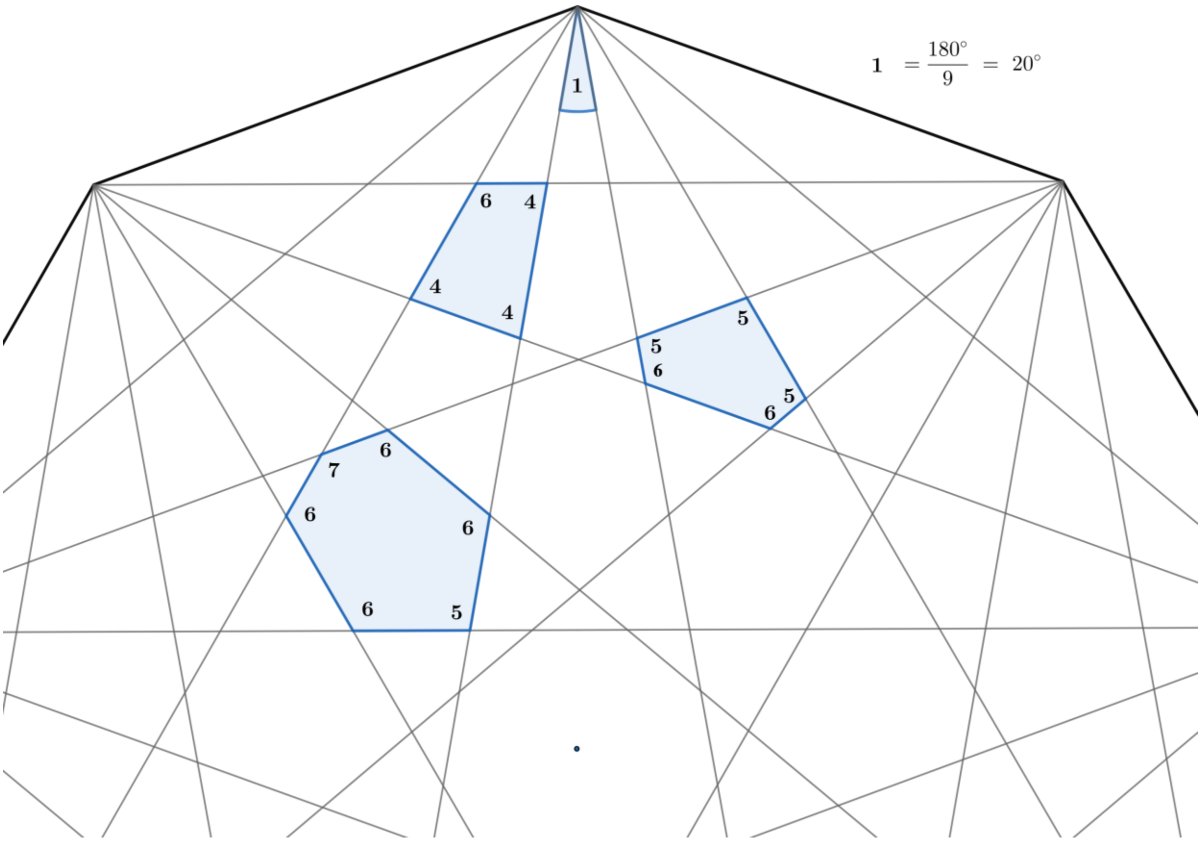
To find out which rose it is, the easiest picture to use is the left one. Why?

[There are lines which look as if they converge on the centre or on a vertex. We can measure the angle between lines which form the smallest angle and divide that angle into 180°.]

There are such lines in the middle picture. How could you check that they are the ones you need?

[Look at the shapes either side. Identify two lines which would carry the same shapes round with them under a rotation.]

The right hand picture is the hardest, but there's a principle we can apply.



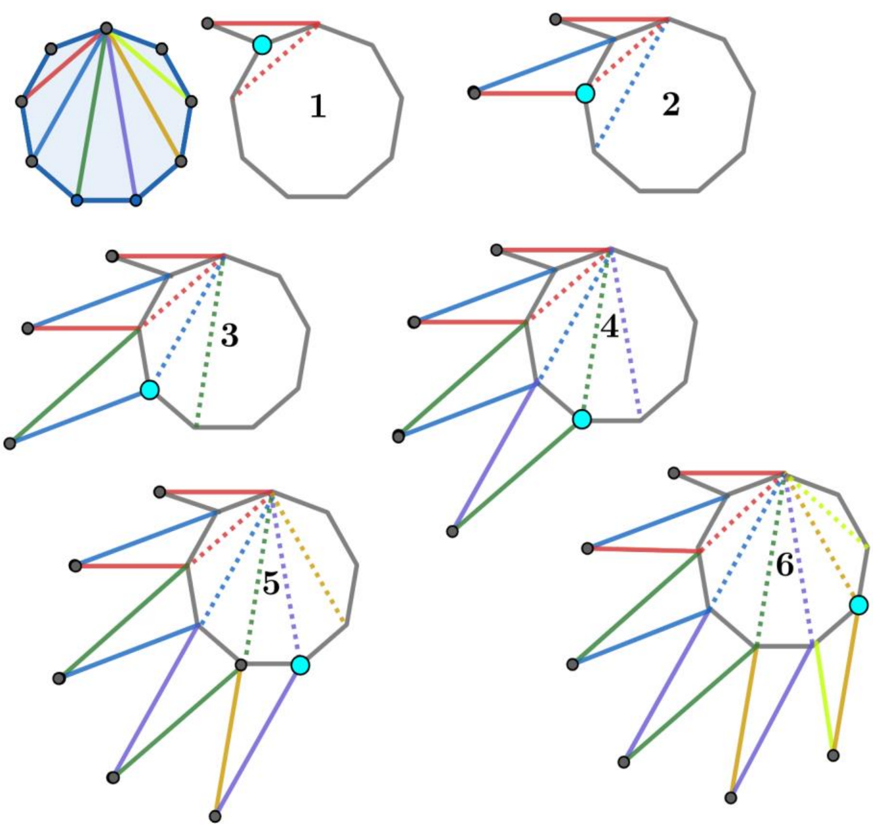
The principle is that every interior angle is a multiple of the smallest one. And the smallest one is the angle between two adjacent diagonals at a vertex. So here we have a 9-gon. Our unit, our smallest angle, is $180° \div 9 = 20°$. We just call this **1**. So, in the figure, **4** means $4 \times 20° = 80°$. The problem is of course that, in the example we are studying, we're working backwards. We can measure the angles, but then we have to work out what the unit is. We do that by finding the numbers' highest common factor. Look at the quadrilateral. We would have measured $80°, 120°, 80°, 80°$. Their highest common factor is $120° - 80° = 40°$. But now look at the hexagon. Here we would have measured $100°, 120°, 120°, 140°, 120°, 120°$. $140° - 120° = 120° - 100° = 20°$. So it looks as if $20°$, not $40°$, is the angle we're after.

See how you manage with the third picture of the three, using GeoGebra.

Use **Measure: Angle**. Click on the two arms of your chosen angle. (You may get the *supplement* of the angle you want and have to subtract from 180°.)

[The right triangle yields an h.c.f. of 30°, the quadrilateral, 15°. So we guess that our polygon has $\frac{180°}{15°} = 12$ sides.]

5



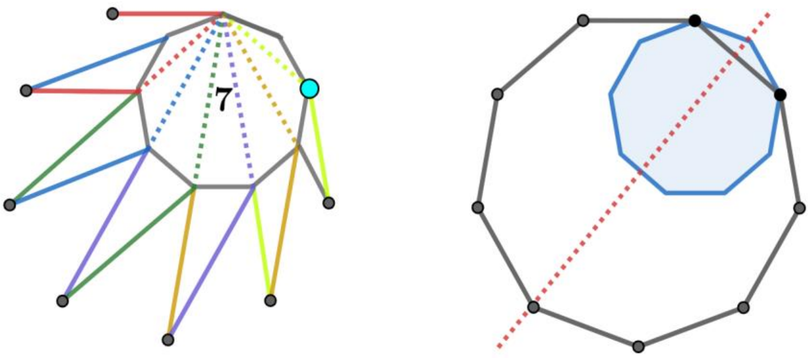
Follow the cyan dots down through the numbers 1 to 7.

Look for example at 3. What happens to the green diagonal and the blue diagonal?

[They're rotated round the cyan dot.]

The anticlockwise Angle of rotation is in fact 140°.

We'll find out why this amount later.



Use GeoGebra. Draw a regular 9-gon in the middle of the screen. Draw the first diagonal. Use the tool **Rotate about point**, enter the angle 140° and the sense anticlockwise. Choose the diagonal and the vertex as centre. You don't need to rotate the side of the 9-gon as well: just use the **Line segment** tool and complete the triangle.

What can you say of the triangle you've drawn?

[Because it has two sides the same length, it's isosceles.]

Do the same with the second diagonal (figure 2).
and the third (figure 3).
Notice our complete triangles are now made of two parts.

How else could you make the construction?

[Rotate a complete triangle.]

Use the **Polygon** tool to draw round the blue-green triangle in figure 4 and try that.

Because the apex angle of all the isosceles triangles you've completed is 140°, what can you say about all the triangles you've made?

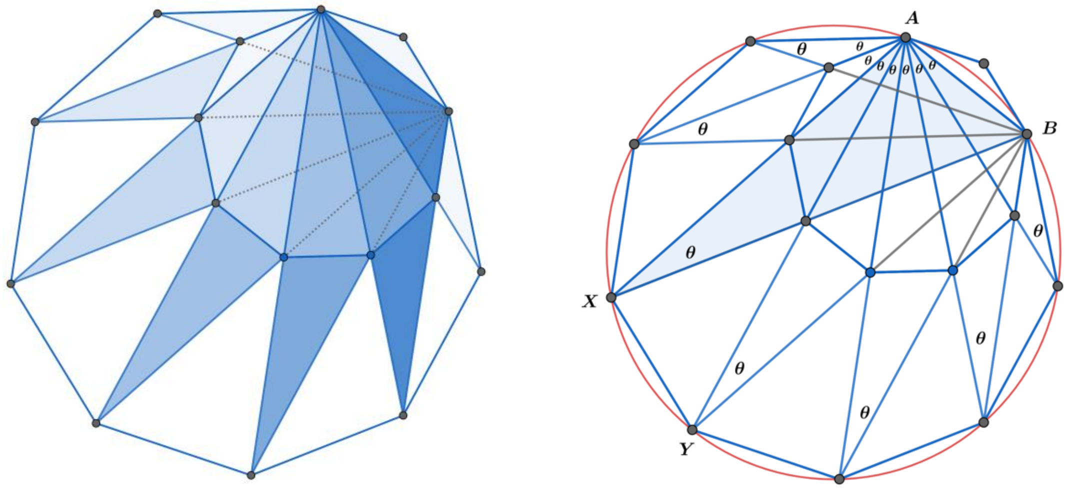
[They're similar.]

Rather than go on to figures 4, 5, 6, 7, what shortcut can we take to complete the big 9-gon shown in the last figure?

[Make a side from two of the outer points and use the **Regular polygon** tool.]

The dotted red line in the last figure is the symmetry axis common to the two roses.

Project this pair of figures:

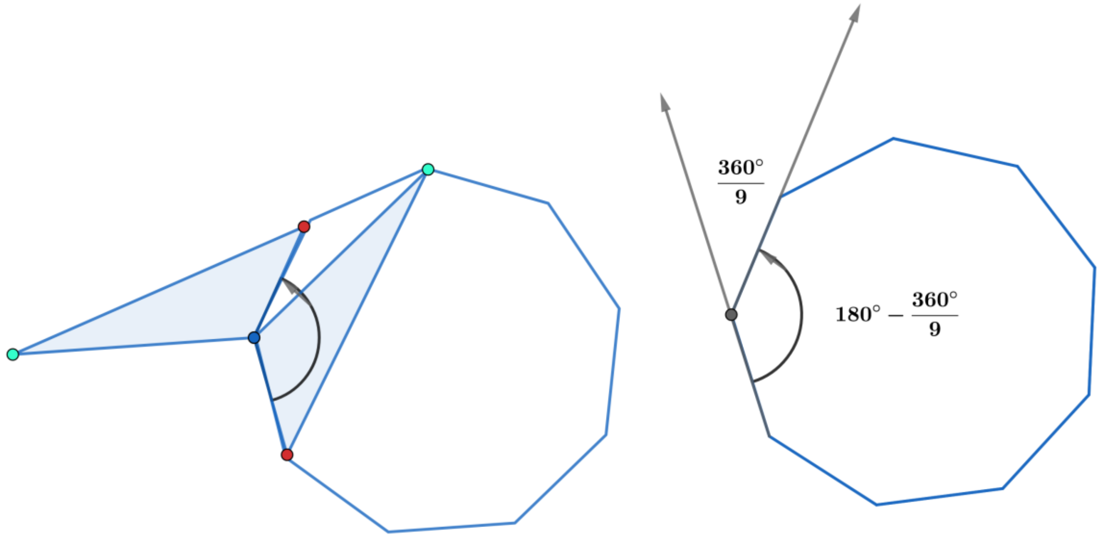


In the left hand figure we've given the triangles we've rotated the same shade.
We've also shown how two sets of diagonals of the big 9-gon go through the vertices of the small one. On the right hand figure, these fan out from *A* and *B*.

We saw in **4** that, in a circle, a chord subtends a constant angle at the circumference. The *converse* is also true. If triangles on a given base have the same apical angle, their apices lie on a circle. The equal angle is shown as θ (40° in fact).

We also know from **4** that adjacent diagonals of a regular polygon are separated by the same angle. Again, the converse is true: if points on a circle subtend the same angle at a point on the circumference, they are the vertices of a regular polygon.

This figure shows where the 140° and the 40° come from.



We now use our discovery of the similar isosceles triangles to write down equal length ratios:

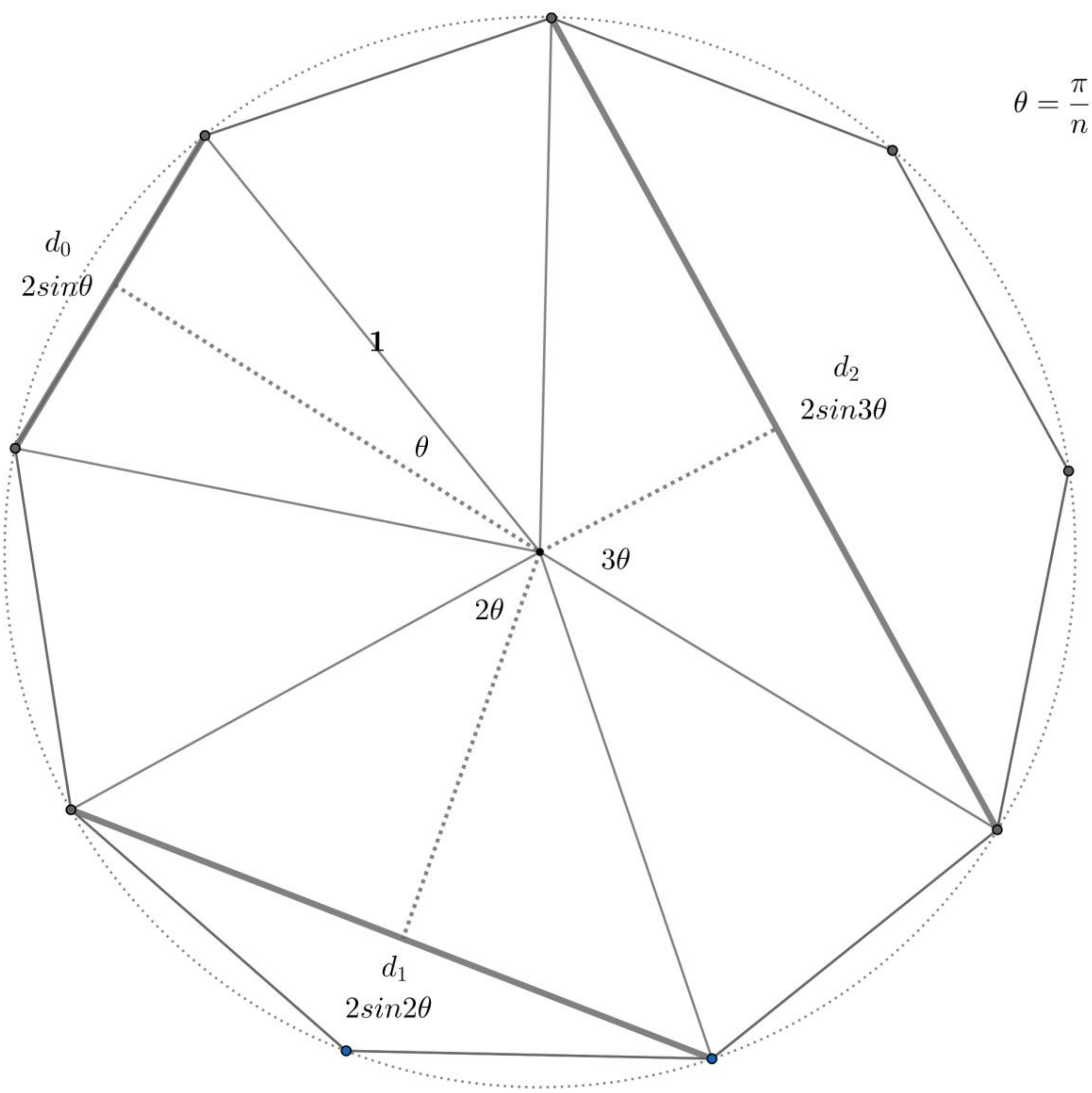
$\frac{black+brown}{light\ green} = \frac{light\ green+lilac}{brown} = \frac{brown+dark\ green}{lilac} = \cdots = \frac{red}{black}.$

In other words, in terms of lengths, the sum of the diagonals either side of a chosen one is to the chosen one as the first diagonal is to a side. We'll 'mathematise' that.

Name the successive diagonals beginning with a side, $d_0, d_1, d_2, \ldots, d_k$. Then:

$\frac{d_k+d_{k+2}}{d_{k+1}} = \frac{d_1}{d_0}.$

Older students can rewrite this relation in terms of sines, thus establishing a trigonometric identity by means of geometry. Here is the geometry needed:



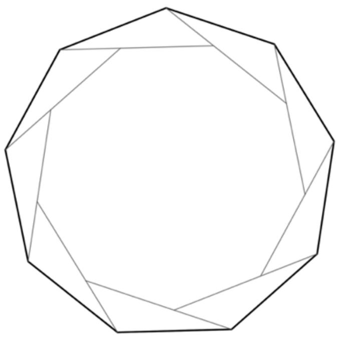
$\frac{d_k+d_{k+2}}{d_{k+1}} = \frac{d_{k-1}+d_{k+1}}{d_k} = \frac{\sin k\theta+\sin(k+2)\theta}{\sin(k+1)\theta} = \frac{d_1}{d_0} = \frac{\sin 2\theta}{\sin\theta} = 2\cos\theta,$

$\frac{\sin k\theta+\sin(k+2)\theta}{\sin(k+1)\theta} = 2\cos\theta$, for k = 0, 1, 2, ...

6

In this section we trace polygons similar to the original in the diagonals of the rose.

6.1 First we trace polygons whose edges lie along diagonals.



Using GeoGebra, draw a regular polygon and all its diagonals, drawn with fine lines. How has this figure been drawn?

[A line segment runs along a first diagonal from a vertex to the point where it cuts the next first diagonal.]

Using heavy lines, draw in the 9-gon formed on the inside.

How can you trace a smaller 9-gon?

[Do the same with *second* diagonals.]

Draw in all these concentric 9-gons.

Try a different polygon to start with.

We shall try to predict how many concentric polygons similar to the original there will be.

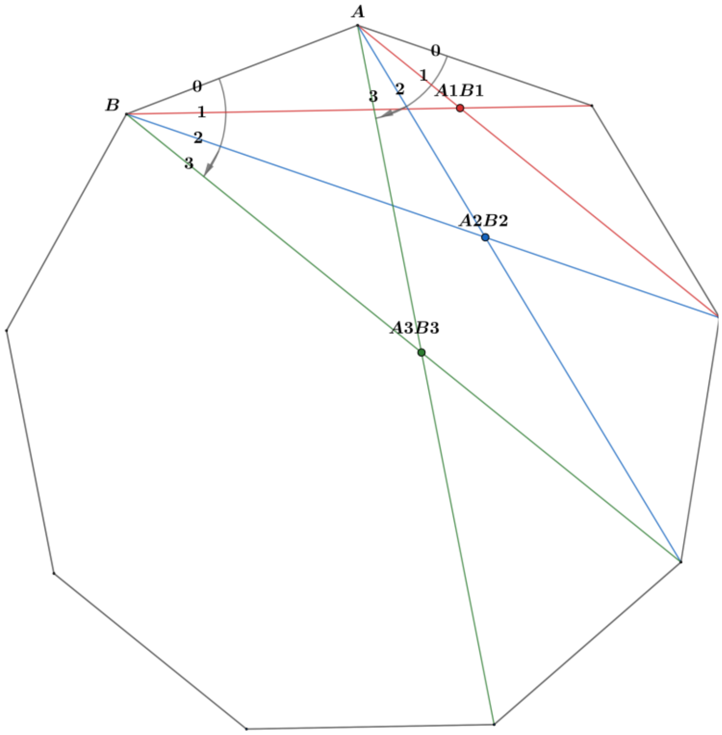
For the two adjacent vertices *A, B*, we've named the diagonals clockwise from 0. The points where matching diagonals cross are vertices of the small polygons.

This polygon gas 9 sides. Including the outer polygon there are 4 concentric polygons in total.

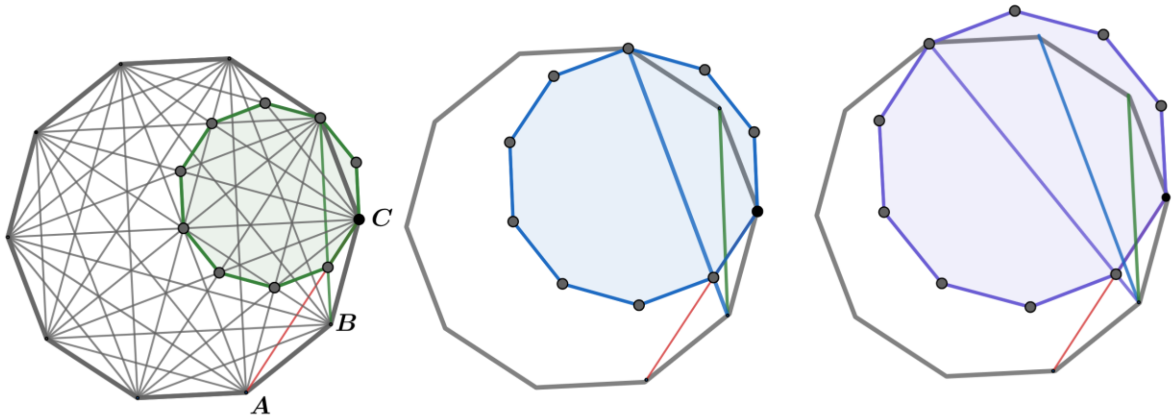
Try another polygon with $2k + 1$ sides and convince yourself that there are k polygons.

Investigate even-sided polygons with $2k$ sides.

[The corresponding number is $k - 1$.]



6.2 Now we trace polygons whose vertices coincide with vertices of the original polygon, both interior and exterior.



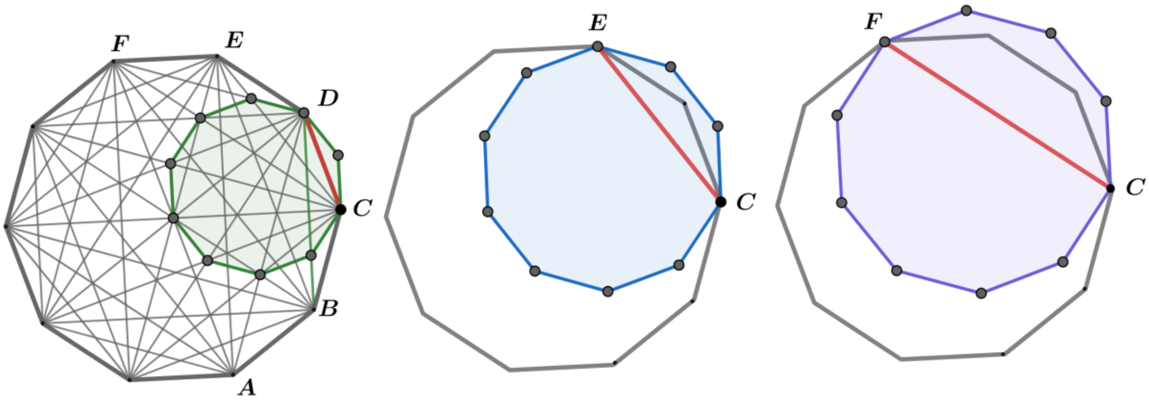
Study these three figures. What's going on?

[The three figures show a large order 10 rose. The red line, *AC*, is a first diagonal.
Diagonals are drawn from the vertex in between, *B*.
These cut off successive segments on the red diagonal: green, blue, lilac.
These form the sides of new 10-gons.]

What we find is that the exterior vertices of the new 10-gons are interior vertices of the old, as shown in the first figure.
Why is this?

[Because the large and small regular polygons are similar and they share a diagonal, all the angles between the diagonals of the large polygon are also angles between diagonals of the large one.]

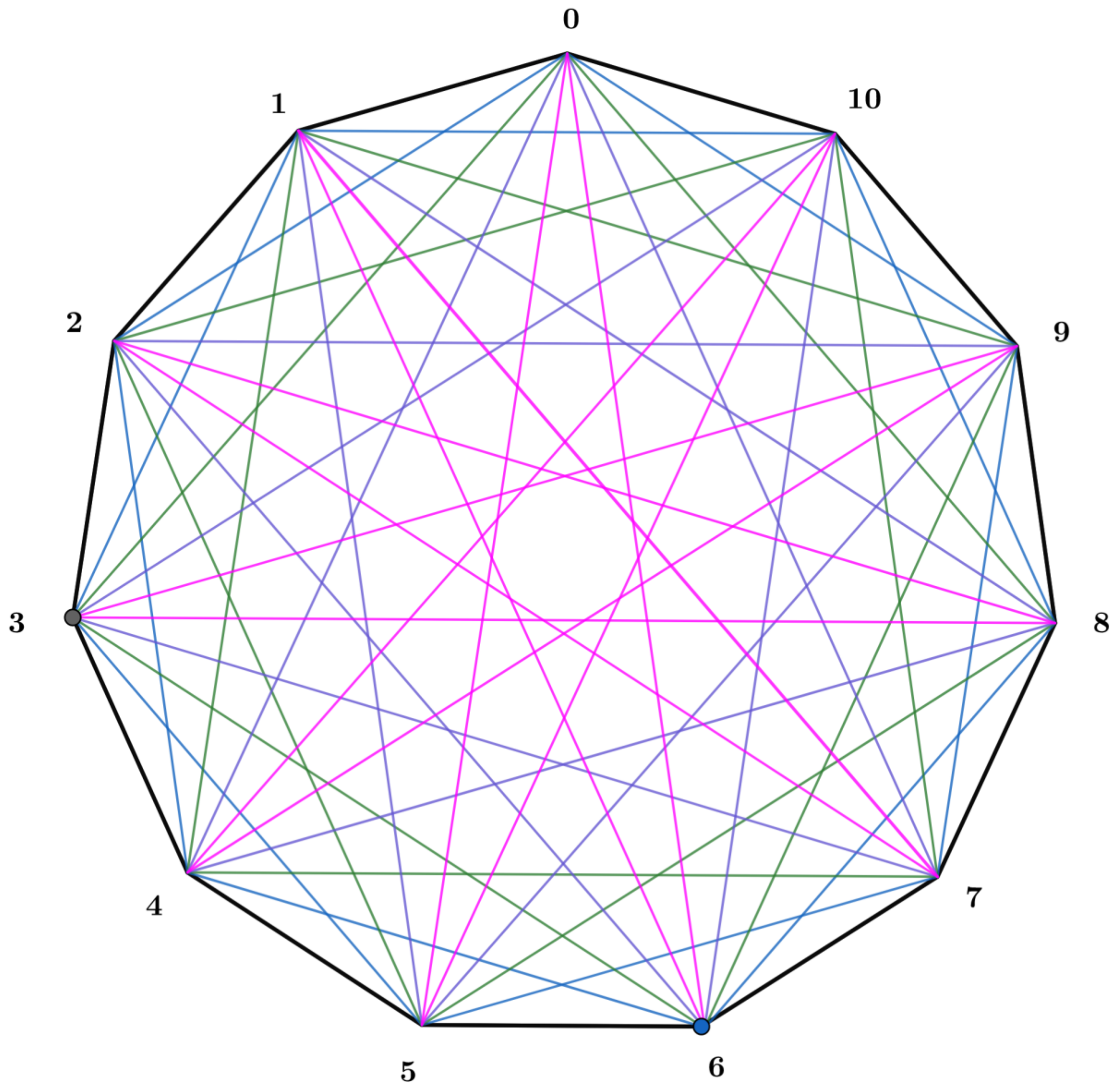
What further property does this figure show?



[A first diagonal of the green 10-gon is a side of the original;
a second diagonal of the blue 10-gon is a first diagonal of the original;
a third diagonal of the lilac 10-gon is a second diagonal of the original.]

7

$\{7\}$ is a regular polygon, $\{7,3\}$ is also a regular polygon. Under the wider definition, $\{7\}$ is an abbreviation of $\{7,1\}$. The first number indicates the number of sides; the second number, how many circuits of the centre are needed to complete the figure. Where the second number differs from 1 we have a *star* polygon. These we shall trace among the diagonals of a rose.



This would be a rose of order 11, but the diagonals are coloured according to a scheme. Fix on one colour at a time. What rule do you think it follows? Discuss your ideas with a partner. What is the overall scheme into which your colour's particular rule fits?

[The secret lies in the vertex numbers. For example, the lilac lines run in sequence 0 - 4 - 8 - 1 - 5 - 9 - 2 - 6 - 10 - 3 - 7 - 0. We add 4 each time but modulo 11. The figure it draws is the *star polygon* {11,4}. The scheme comprises all the star polygons which can be drawn in the order 11 rose: {11,2}, {11,3}, {11,4}, {11,5}.]

Why is there not an {11,6}?

[This would be {11,5} drawn in the opposite sense (anticlockwise).]

What are the matches for {11,7}, {11,8}, {11,9}?

[{11,4}, {11,3}, {11,2} respectively.]

Starting at vertex 0, how many times does a star polygon go round the centre before it arrives back at its starting point? Fix on a particular example and see if your hypothesis stands up for the others.

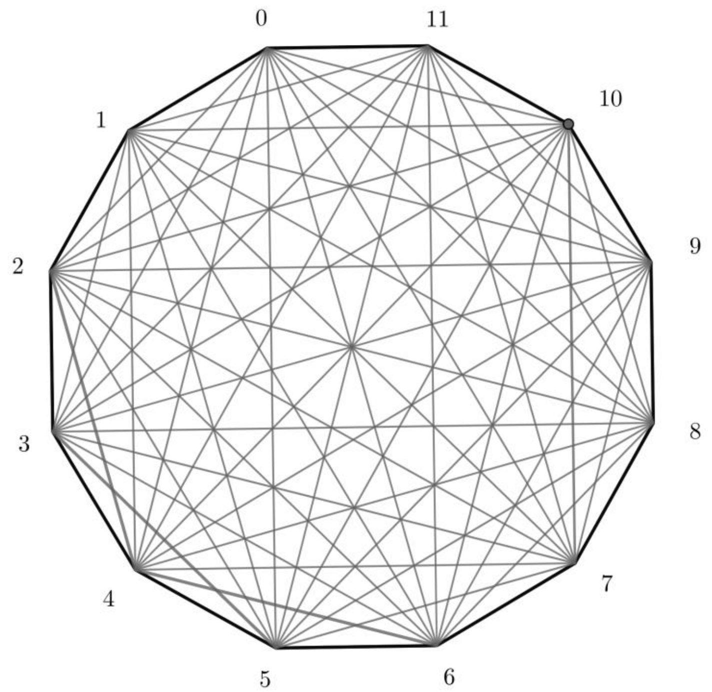
[{11,2} goes round twice, {11,3} three times, etc.]

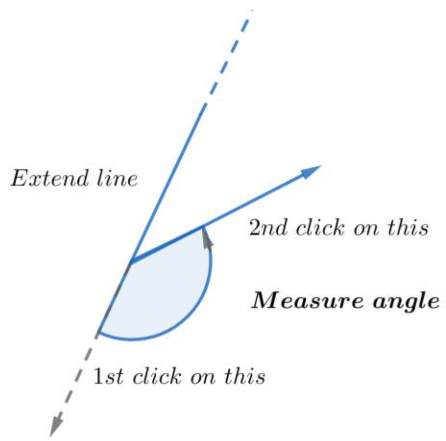
How many vertices do all the star polygons based on the order 11 rose have? Check a few cases.

[11, the same as the regular polygon itself.]

Ordinary regular polygons have an interior angle: the equilateral triangle 60°, the square 90°, etc. But, if we imagine a vector drawing them as it moves along and turns at the vertices, we get the supplements of these angles: 120°, 90°, etc. We call these *turning angles.*

Project this figure:





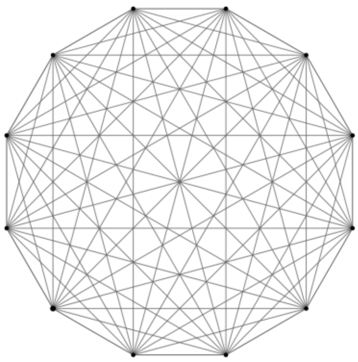
Pick out the star polygon {12,5} in colour in the order 12 rose. Choose **Measure: Angle**.

Measure the turning angle as shown.
Multiply by 12 for the number of vertices.
Divide by 5.
Can you explain your answer?

[Since {12,5} goes 5 times round the centre, the vector makes 5 complete turns of 360°. Each turning angle must therefore be $\frac{360° \times 5}{12}$, which gets us back to the angle we measured. The inverse sequence of operations - multiply by 5 and divide by 12 - is what we've just done.]

8

This children work with this GeoGebra file, the rose of order 12. The task is to pick out and colour segments so as to trace regular polygons with fewer sides than the 12-gon.

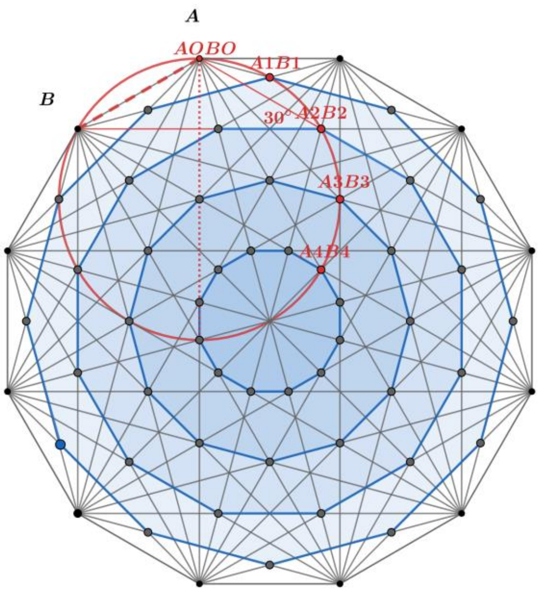


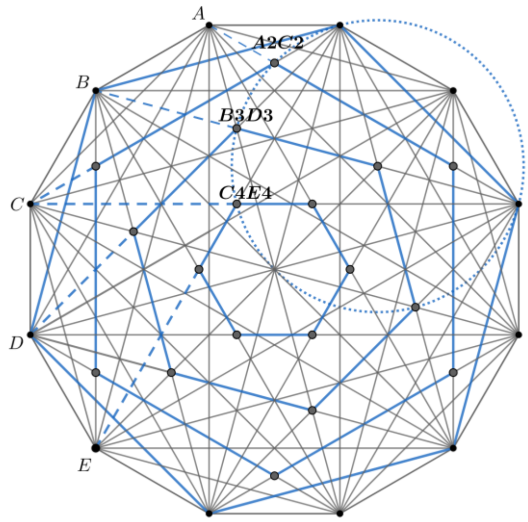
They can use two methods:

1. They add points to the end of the segment, hide the whole diagonal, draw in and colour the segment (preferably in a line thickness greater than the original), and add the ends back.

2. They use the **Polygon** tool.

In **6.1** we drew a regular 9-gon and the smaller 9-gons nested concentrically inside it. On the left we show a 12-gon in the same way. On the right we show the 12-gon again but this time with concentric 6-gons:





What do you notice about the difference in the vertex labels?

[On the left the letters are consecutive; on the right a letter is missed out.]

Notice how the vertices lie on circles. On the left we've marked an angle of 30° subtended at *A2B2* by the side *AB*. We would have found the same angle at the other vertices. Recalling the work we did in **4** and **5**, what does this imply?

[The vertices lie on a circle.]

On the right we've drawn one hexagon of each size. How many of each size could we have drawn?

$[12 \div 6 = 2]$

Use one of your 12-gons to draw in squares in the same way, and use the other for triangles. Because three diagonals pass through each square vertex, and four through every triangle vertex, you will have to decide how to set up your label system in each case. But do use one: it will ensure you catch all the different sizes.

How many squares of each size will there be? $[12 \div 4 = 3]$
How many triangles? $[12 \div 3 = 4]$