LINES IN SPACE

Exploring ruled surfaces with cans, paper, Polydron, wool and shirring elastic

Introduction

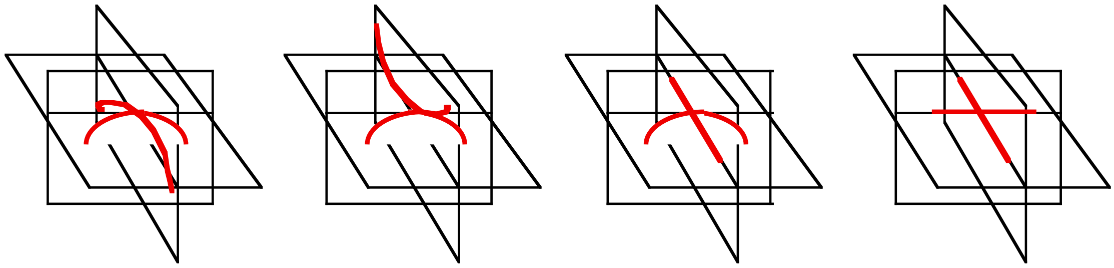
If you are reading this journal you almost certainly know more than the author about at least one of the virtual means by which surfaces can be generated and displayed:
Autograph, Cabri 3D, Maple, Eric Weisstein's Mathematica, … and all the applets in his own Wolfram Mathworld, at Alexander Bogomolny's Cut-the-Knot or at innumerable other sites. Having laboriously constructed one of my 'real' models, your students will benefit by complementing their studies in that wonderful world. Advocates John Sharp and the late Geoff Giles stress the serendipitous dimension of hands-on work [note **1**]. Brain-imaging techniques are now sufficiently advanced that we can study how the hand informs the brain but I fear it may be another decade before this work is done and the results reach the community of teachers in general and mathematics teachers in particular [note **2**]. You may find one curricularly-useful item under the heading 'Ruled surface no.1': the geobox. The rest is something for a maths club, an RI-style masterclass, a parents' evening or an end-of-summer maths week.

Surfaces in general

Looking about us, what we see are surfaces, the surfaces of leaves, globes, tents, boulders, lampshades, newspapers, sofas, cars, people, … . What we *look* at are 3-dimensional solids, but what we *see* are their 2-dimensional outer layers.

Studied in depth, the topic lies beyond the mathematics taught in school, but on a purely descriptive level we can draw distinctions between one surface and another which help children answer the question 'How is that made?'. Why is a car body not folded from steel and a chocolate box not pressed from card, but the other way round? Some simple geometry precedes considerations of materials science and business economics.

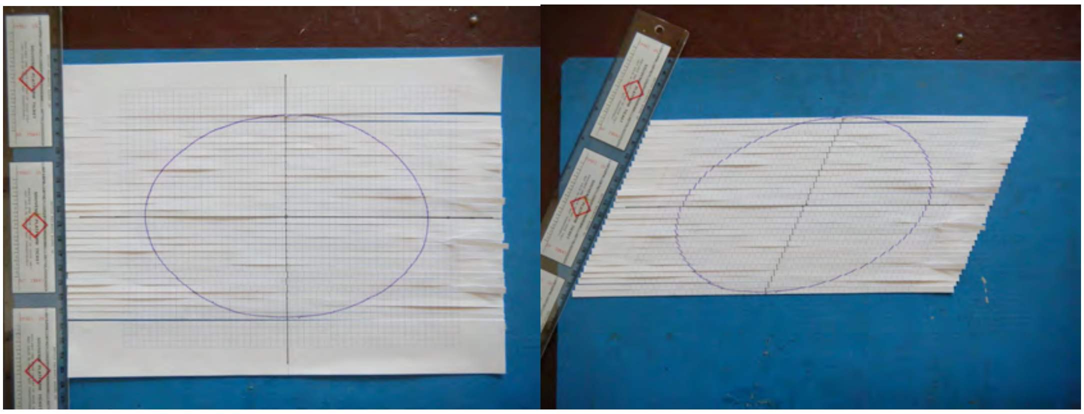
We can characterise any point in a surface by its extreme (*principal*) curvatures in planes normal to the tangent plane. They can be arranged in these 4 ways:



A **B** **C** **D**

As you see, these planes are perpendicular. Why should they be? Euler’s 1760 proof moves from calculus through algebra to trigonometry [note **3**]. All we shall do here is a little experiment to show that the assertion is at least plausible.

Imagine looking down on a hilltop which meets this condition at the start. The easiest ascent and descent is east-west; the steepest, north-south, so that the contour very near the summit (the *Dupin indicatrix*) is an ellipse. An earthquake shears the landscape in plan:



Though the original ellipse has rotated, it is still an ellipse. We imposed half-turn symmetry but the original two mirror lines, the major and minor axes, marking the planes containing the extreme curvatures, have miraculously survived.

A finite distance from the hilltop the contour can be any shape, but look what happens as we take parallel slices of this egg closer and closer to a point:

The muddy print of a rugby ball on the pavilion wall would still be elliptical even if the ball were egg-shaped.

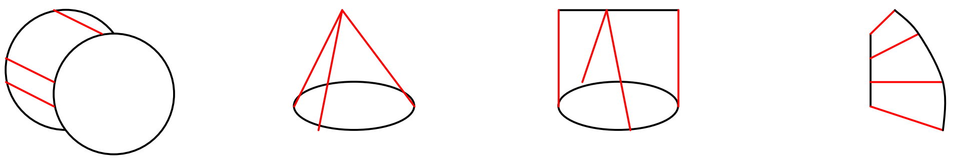
Points on the hilltop, the egg and the rugby ball are all of type **A**. The two principal curvatures share the same sense (positive by convention). In case **B** (the saddle) the sense is opposite. In the special case that they are both opposite and equal at every point (so that the *mean* curvature is zero) we have a surface of least area for the boundary it spans (whether a closed plane curve or a closed space curve), a *minimal* surface. There is a physical correlate to this condition which means that such a surface is modelled by an elastic film like soap. To the curvature there corresponds a proportional opposing pressure. Since the mean curvature is zero at every point, so is the net pressure on the surface there. Any attempt to increase the area of the surface creates an opposing force against which work must be done; that is to say, the minimum area represents a minimum energy state: the film is in equilibrium with its surroundings.

Slicing an egg can be a messy business but *tomography*, the idea of taking parallel, equally-spaced sections of a solid to reveal its shape is an important one. We have already mentioned contour lines on a map to show relief, but the technology behind 3-D printing and how oncologists monitor the progress of a dangerous tumour also depend on it . Happily for maths teachers, a simple means is to hand. Think of the internal cardboard dividers protecting bottles in a supermarket box. Now imagine the top surface not to be flat but to follow some interesting surface. The result, exploited by John Sharp and christened by his editor at the time, Gerald Jenkins of Tarquin, is a 'sliceform' [note **4**].

Ruled surfaces in particular

A ruled surface is the trace of a straight line moved through space. We give an orchestral conductor a baton – in fact a telescopic one so that he can lengthen or shorten it at will – and observe him over the course of the evening. The ends of the baton trace curves in space; the baton itself, a ruled surface. Every point on one curve is joined to a corresponding point on the other by a straight line. Through every point on the surface a straight line passes. Look up from the orchestra pit to the stage and observe the swish of the curtain, a ruled surface.

These 4 cases by no means exhaust the possibilities:



P **Q** **R** **S**

P is a cylinder. The rulings are parallel. They are therefore also all parallel to a particular plane. Such a surface is called a *Catalan* surface. **Q** is a cone. The rulings all pass through a single point. **R** is a *conoid*. The rulings all pass through a single line. They are also all parallel to a particular plane. This too is therefore a Catalan surface. It is a familiar 'puzzle' object because, in orthogonal views, it presents a square, a triangle and a circle. **S** is a *helicoid*. The rulings join a helix to its axis. They lie in parallel planes perpendicular to that axis so again we have a Catalan surface. We shall have more to say about this one.

Back in the 1920s it was the ruled surfaces on display in the Mathematics Gallery of the Science Museum which turned the young artist Henry Moore to think about space and produce some of the most remarkable sculpture of the twentieth century.

We shall describe some common ruled surfaces, and some which bound solids with interesting symmetry. Of our 4 cases, **A** to **D**, we must abandon **A** at this point, but not **C** or **D**, or even **B** if the ruling, while not being a principal curvature, persists across the surface. This is the case as we shall see with the only ruled surface apart from the plane which is minimal: the helicoid.

We shall give the name ‘R-forms’ [note **5**] to five of our seven examples because of the way they are made: we take two congruent shapes in parallel planes, turn one in its plane and rule between them. There are two features of interest:
(a) Projected in plan the overlapping lines produce an envelope curve: we have a piece of ‘curve stitching’.
(b) We can also plot points a chosen fraction of the way along each line. The resulting locus is the cross-section of the R-form at that height. Again we can determine this by projecting the figure in plan.

As promised in the subheading to the piece, we shall use several materials:

(A) Two materials to follow individual rulings:
1. white craft wool (no. 2).
2. white shirring elastic (1 mm).
(B) Two materials which do not stretch or shear and therefore model developable surfaces:
3. paper (any).
4. the alloy from which drinks cans are made (any).
(C) A kit of interlocking polygonal panels to show how a surface of such materials can curve in 3 dimensions while resisting all deformation in 2 [note **6**]:
5. Polydron (‘Frameworks’).
(D) A kit of interlocking curved panels from which to build developable surfaces:
6. Polydron (the quarter-cylinders and quarter-cones of ‘Sphera’).

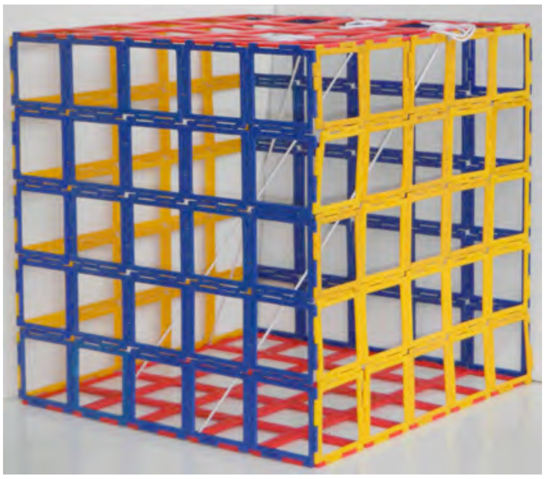
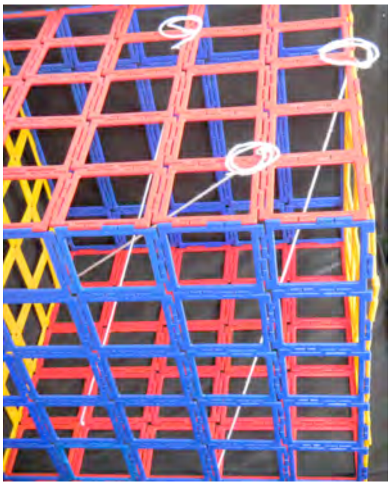
1. and **2.** require a curve at each end to span. In most cases these will be plane curves of the same perimeter picked out in parallel laminae, either **(i)** as series of equally-spaced holes through which wool can be threaded, or **(ii)** as magnet-and-keeper pairs anchoring lengths of shearing elastic. **(i)** forces a ‘discrete’ arrangement; **(ii)** allows a ‘continuous’ one. But our ‘Ruled surface no. 1’ will use an example of **(i)** requiring the least DIY.

Our type **(ii)** arrangement takes advantage of the cheap, small, strong magnets now available to create variable point positions and allow the use of different acetate grids (square, isometric, polar, …) to locate them [note **7**].

Ruled surface no. 1

This is the *plane*, but our concern is not the plane in isolation. The following ‘geobox’ (by analogy with the 2-dimensional geo*board*) is designed to help A-level students investigate lines, planes and the relations within and between them.

Build this cube from Polydron ‘Frameworks’ squares. These are 7 cm on an edge, giving overall dimensions of 35 cm. The red planes are xy planes for $z = 0,5$; the blue, yz for $x = 0,5$; the yellow, zx for $y = 0,5$. Each wool thread should be a bit longer than the cube’s space diagonal. Secure one end of each to a small washer with a round turn and two half hitches; secure the other using a round turn alone, coiling the surplus thread for neatness.

The A-level student must give meaning to the idea of an angle between two lines which do not meet. We shall take such an example:

'Line L_1 joins points (1,0,0) and (4,4,5); line L_2, (5,3,0) and (2,5,5). Find the angle between them.'

To get inside the cage we can open the face $x = 0$ like a door. We have stretched the two threads but we now give one of our students a third thread and a protractor (with the warning to expect a result correct only to within $5°$), and say, "Translate the line L_2 till it cuts L_1 and measure the smaller angle between them." On paper the student has shown the angle to be about $56\frac{1}{2}°$ and now gets $55°$ with the protractor.

Ruled surface no. 2

This is **S**.

Look back at diagram **B**. Picture not just the two principal planes but all the ones in between. You will realise that, as you swing round between two planes in which the curvatures are opposite, you must hit one where it is zero (an observation dignified by the title *the intermediate value theorem*). Where this plane does not change its orientation from point to point but continues indefinitely across the surface, we have a *ruling*. And where there is a ruling through every point we have a ruled surface (*singly* ruled if this is true of one of the two possible planes, *doubly* ruled if true of both. The plane is the only surface with more than 2 rulings.) This condition holds for **S**. To trace the helicoid with your pen: Hold it horizontal. Keep the point in the same vertical line while you lift and turn it at uniform rates.

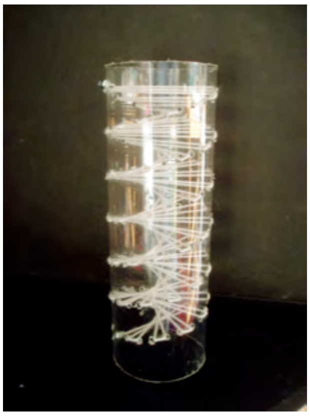
The senses of the principal curvatures only become apparent when you try to run up a corkscrew staircase. Keep to the middle of each step and the gradient is constant; move in towards the axis and it rises; move out towards the wall and it falls. (Thus, the more tired you get, the greater the temptation to move to the wall - if not stop altogether and use it for support).

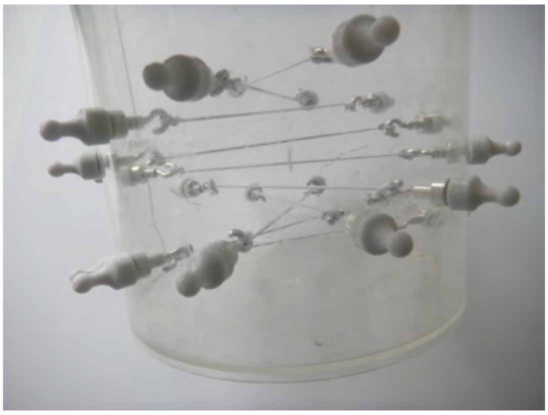
Now it happens that, for the helicoid, the mean curvature at every point is zero. It is therefore a minimal surface, in fact the only ruled surface apart from the plane to be such.

If, in the previous demonstration, you were to follow the helix's axis with the middle of the pen, you would get a second helix a half-turn out of phase with the first. This makes possible a simple soap film simulation. Take an open perspex cylinder and run two rods through diameters to represent a pair of rulings. Immerse the model in soap solution and withdraw. The film will attach itself to the rods and trace a helix along the cylinder wall.

Loop a rubber band round a protruding end of each of the two rods and it will follow the helix. The helix is a *geodesic*: it traces the shortest distance between two points on the cylinder.

Here is our ‘double’ helicoid realised by methods **(i)** and **(ii)** respectively:

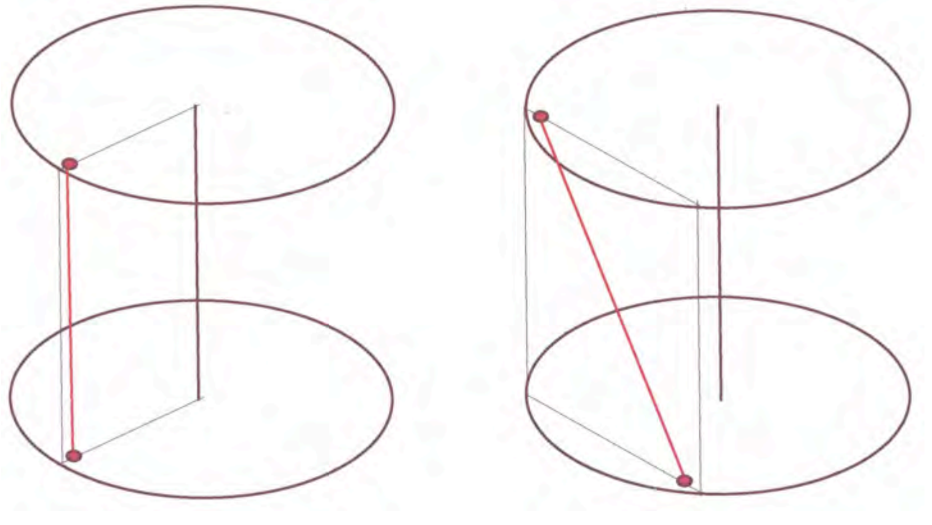




Ruled surface no. 3

Our first R-form is the simplest but one of the most interesting. Here it is, realised in both **(i)** and **(ii)** arrangements:

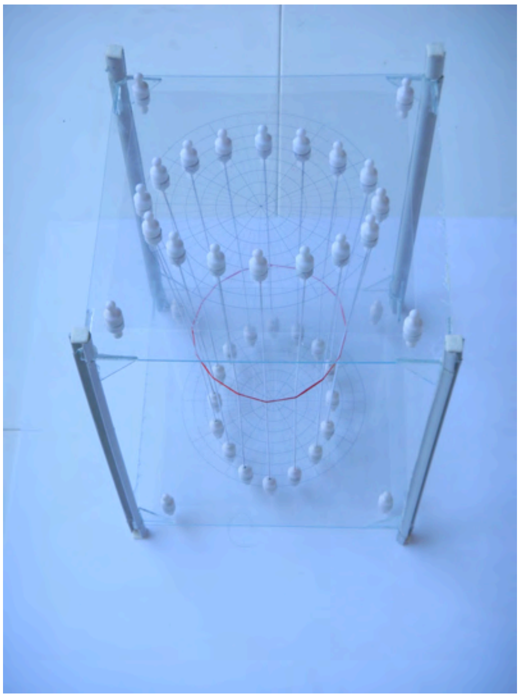
Our plane closed curves are two equal circles forming the end faces of a cylinder. As this diagram shows, we have the choice of joining holes which lie in a plane shared with the axis, producing the cylinder itself, or, as shown on the right, leading to our surface, the hyperboloid of one sheet [note **8**].

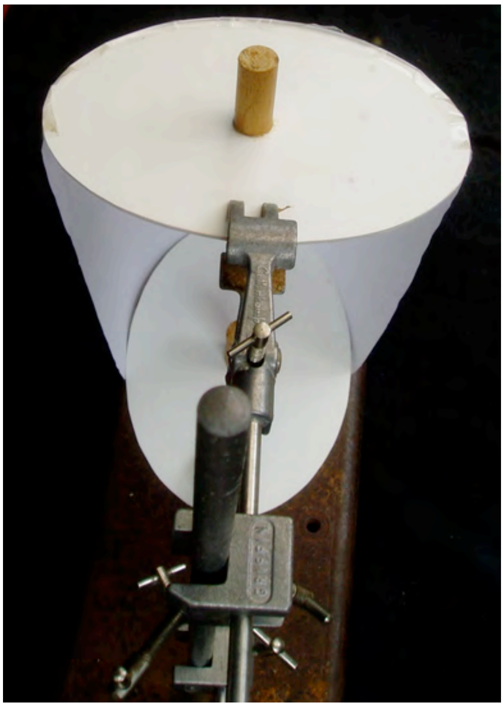


This is the apple-core shape familiar from lampshades and power station cooling towers. (The cooling towers are laced with steel rods in just this way.) The symmetry of the figure tells you that, if the thread is elastic (or each thread is separate and has a weight on the end), you can achieve the right-hand figure by rotating the bottom with respect to the top. Because you can do this in opposite senses, two lines run through every point: the surface is *doubly* ruled. If you give the top a half-turn with respect to the bottom you get a double cone. (This example dramatises the point that in general neither the perimeter nor the area of cross-section remains the same all the way down an R-form.) There are many ways to realise the surface. Here are three. Look at a box of Q-tips after a number have been removed. Bend a length of wire in an 'L' (corresponding to the axis and a radius in the right-hand figure above) extended by a piece bent out of its plane (corresponding to the red line) and put it in a power drill. Spin a skeleton cube about a space diagonal. (The last adds a cone top and bottom.)

Ruled surface no. 4

Replace the disks by ellipses, set their axes at right angles and thread from the end of a major axis to the end of a minor axis. The result is a surface important in geometrical optics. What intuition tells you is in fact the case: the cross-section halfway down is a circle, the ‘circle of least confusion’ [note **9**]. It is a developable surface. The right-hand picture shows it half completed in paper.

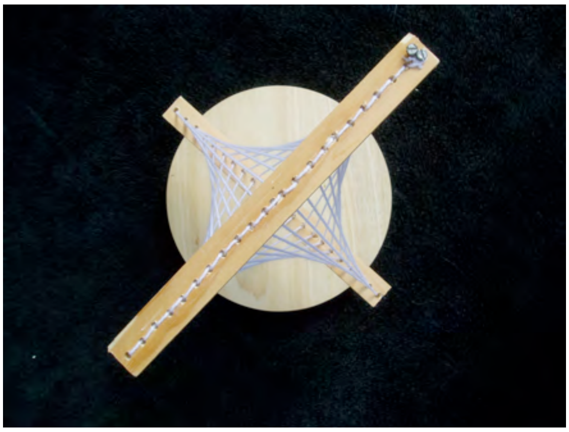




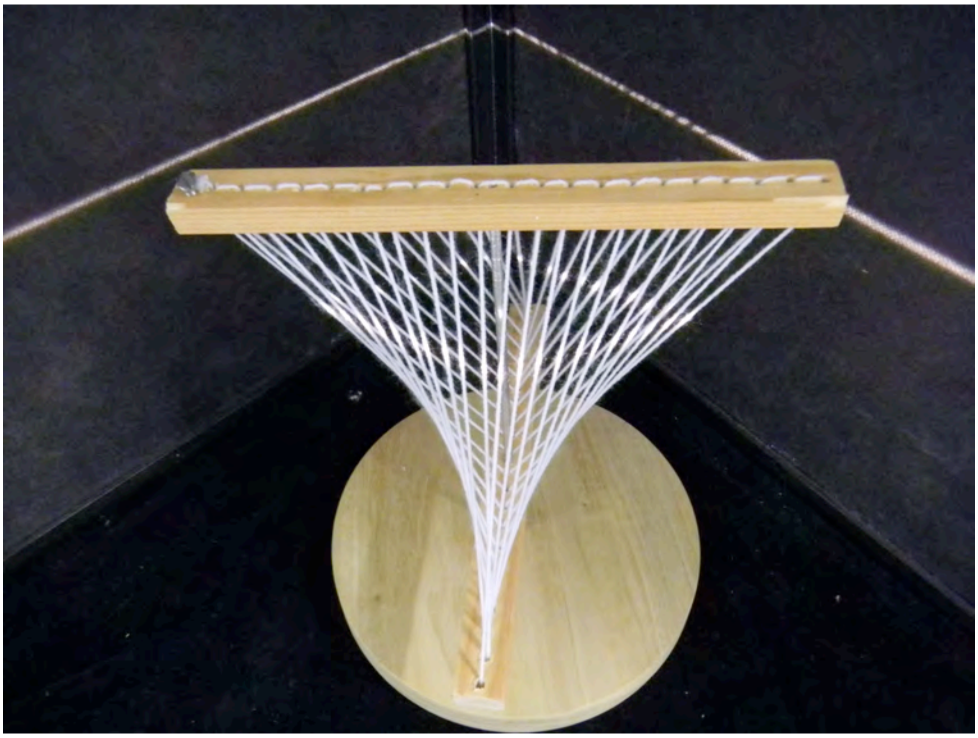
Ruled surface no. 5

What happens if we shrink the minor axes of our two ellipses to zero so that the ellipse becomes a line segment?

In the plan view we see the ‘stitching curve’, a parabola reproduced 4 times [note **10**]. In projection the spacing of the holes is different along the two sticks but roughly constant along each. Though this is not true in the oblique view the changing aspect of the threads across the picture plane compensates.

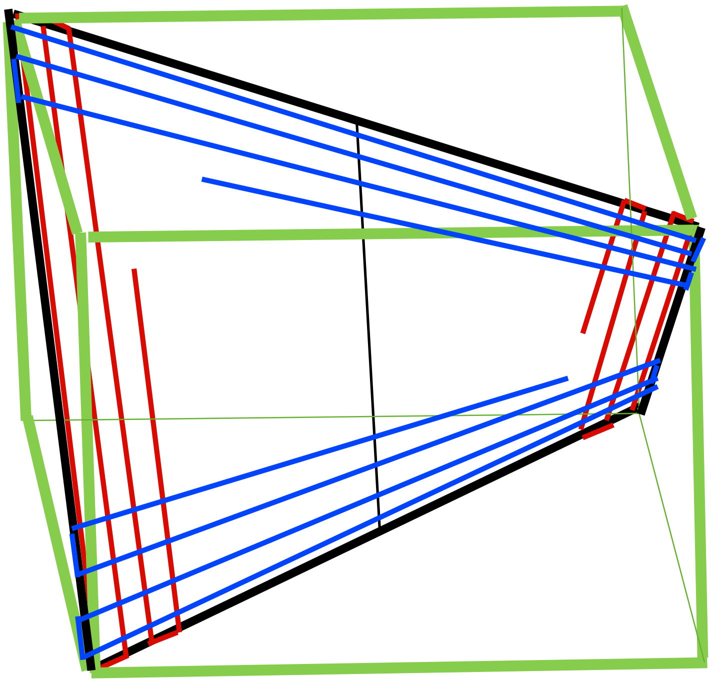
We lay a card slit on an OHP and move the card so that the plane of light issuing from the slit sections the surface parallel to the two sticks.



As you see, the section is a rhombus in this general position. In surface no. 3 the section halfway between the crossed ellipses was a circle. Here, as you can confirm by constructing the corresponding 2-dimensional locus, it is a square.

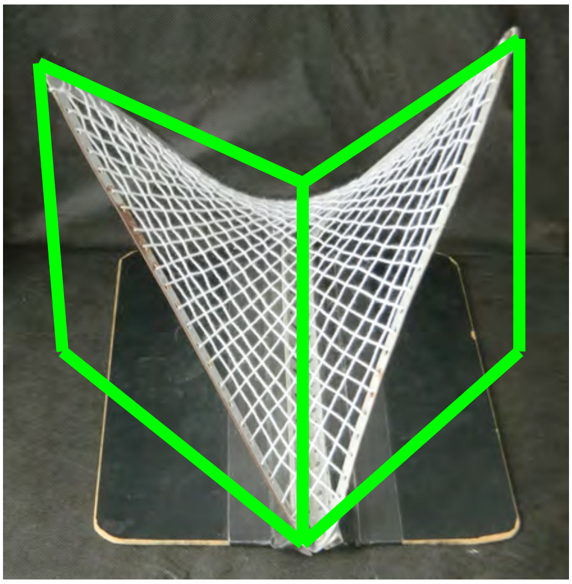
Ruled surface no. 6

In surface no. 5 our stitches ran from the end of one stick to the middle of the other and ran right round to complete a double length of each line. In no. 6 we stitch from end to end and only complete a single length.



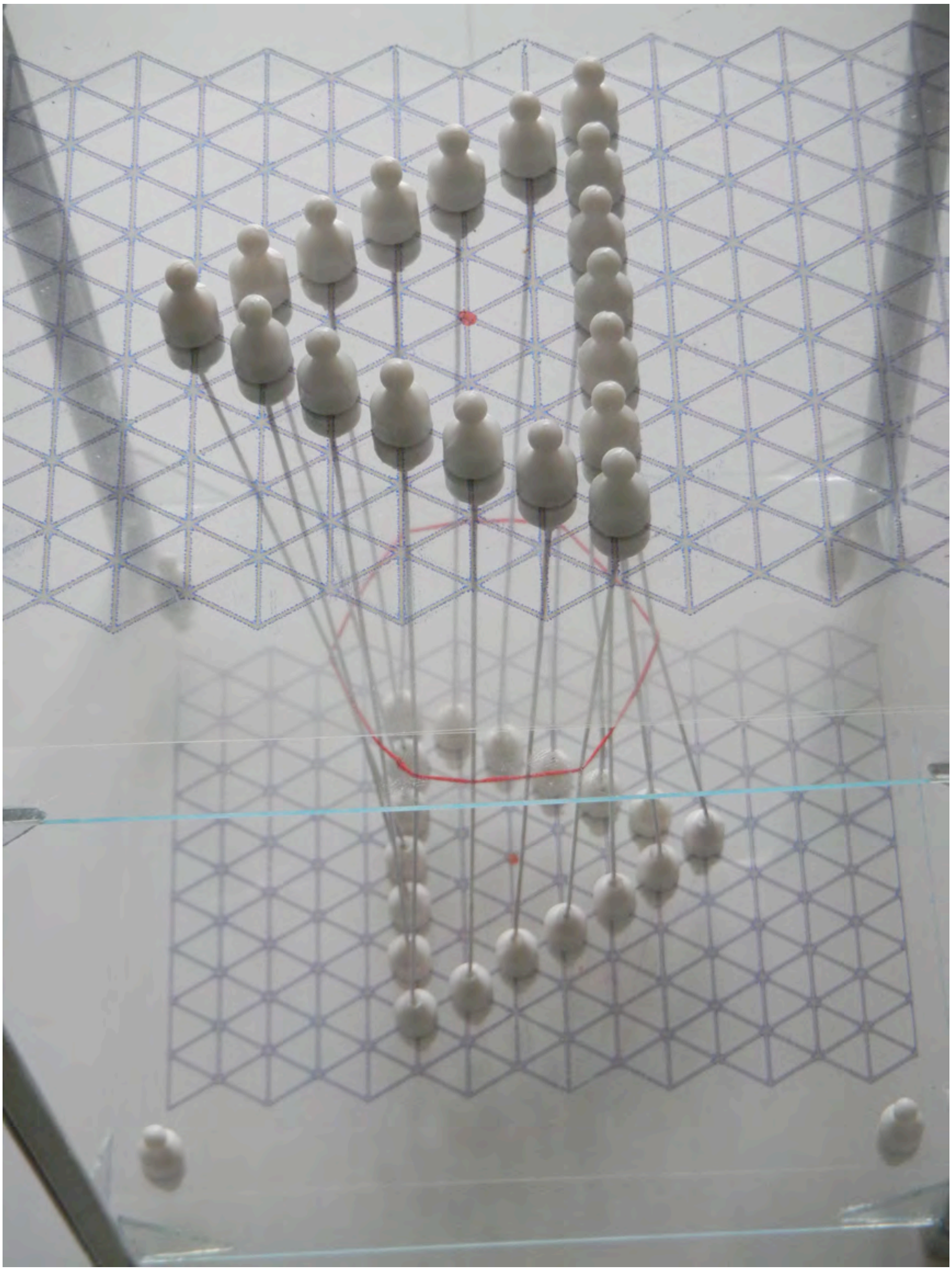
As you see, the line of sight in the photograph passes through opposite edges of the green cube. However, front and back faces in the figure form base and top in the photograph. Note the tetrahedral symmetry: swap top and base and you swap one pair of cube edges for a parallel one.

This is the ‘saddle’ surface familiar from roofs over churches and restaurants. The figure defines a skew quadrilateral, whose opposite sides are the diagonals of parallel cube faces. We start with the red stringing. Now we use two more sticks to complete the quadrilateral, (diagonals of another pair of cube faces) and thread strings in the ‘blue’ direction also. Like the hyperboloid, then, the surface is doubly ruled. In this view of a type **(i)** model you see hyperbolae running across the picture, parabolae up and down. These give the surface its name, the *hyperbolic paraboloid*. They may be displayed by the OHP-&-slit method described for the previous example.

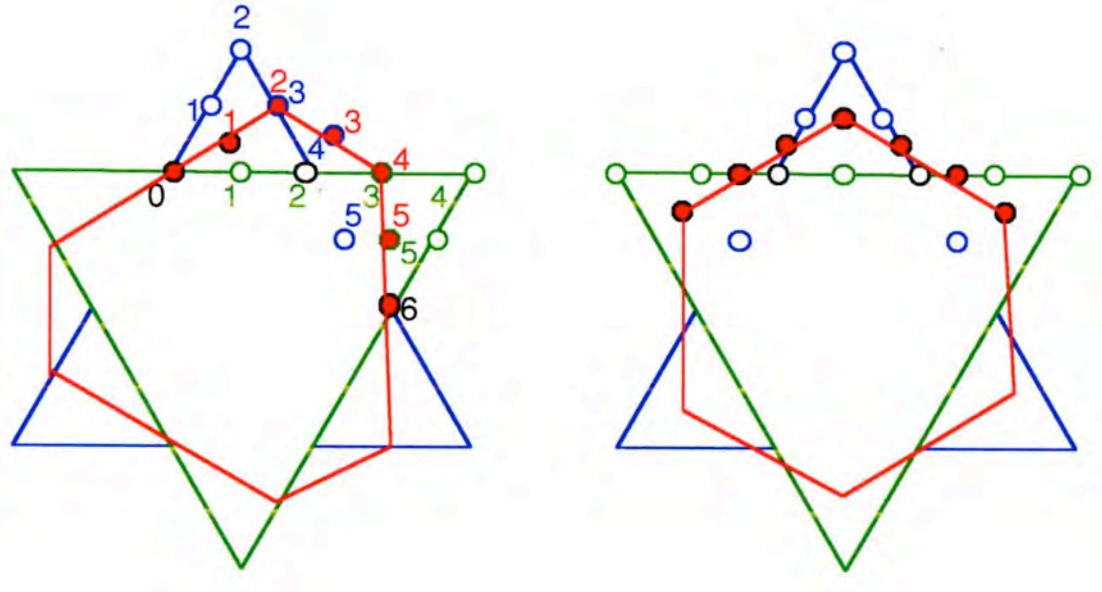


Ruled surface no. 7

When the paired laminae are regular polygons, attractive R-forms result. Here we take a pair of equilateral triangles, separate them along an axis through their common centroid, rotate one so that an axial plane runs through the vertex of one and a side mid-point of the other and thread them correspondingly.



As you see (and perhaps expected from our previous examples) the halfway section is a regular hexagon. Projected on to the plane, the locus is shown in the right-hand figure below. But here is a nice little paradox, resolved by considering rotational symmetry. Imagine we had started at one of the 6 points where one triangle lies directly above the other and proceeded round the figure clockwise from our viewpoint. The result would be as shown in the left-hand figure.

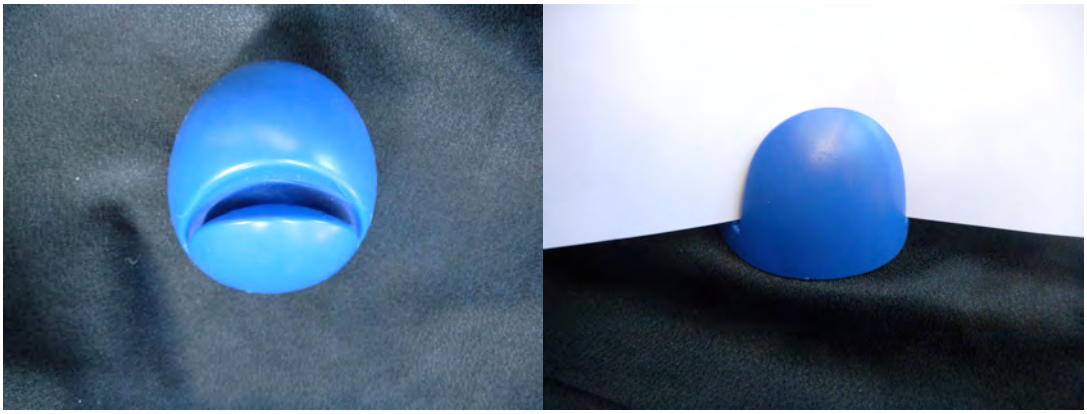


As you see, this hexagon has 3 symmetry axes, not 6.

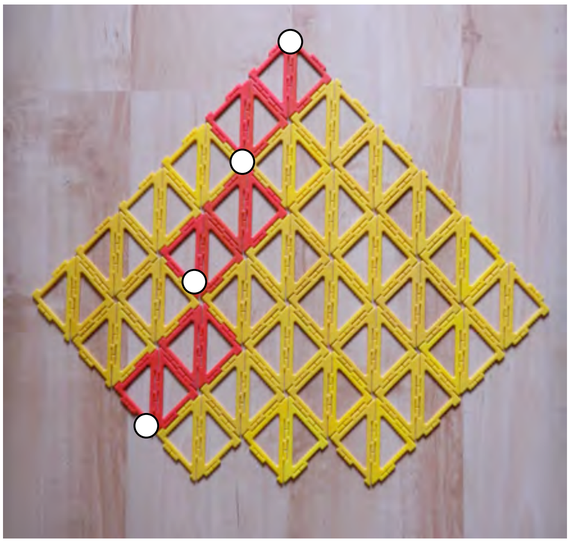
Developable surfaces in particular

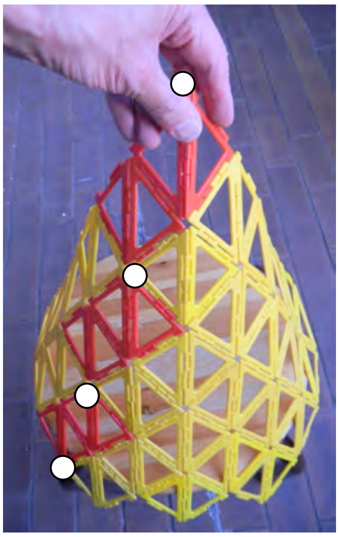
Look back at ruled surfaces **P** and **Q**. They share a particular property. They can be obtained from the plane without deformation. Paired points on the plane and the surface stay the same distance apart. Because of this point-for-point correspondence, the surfaces can be rolled on the plane. Such surfaces are said to be *developable*. All developable surfaces are ruled but not all ruled surfaces are developable. (Most of those we shall go on to describe are not.) Ask 3 children to stand in line facing the class. The first displays a card which says ‘This surface is developable’; the second, ‘therefore’; the third, ‘This surface is ruled’. If that statement is true, is it logical to reverse it, for children numbers 1 and 3 to swap places? Answer: ‘No’. A theorem may be true but its converse false. The developable surfaces are singled out by having one or both principal curvatures zero. Thus their product (the *Gaussian* curvature) is zero. Look right back to our picture series **A** to **D**. Case **C** includes the cylinder, **P**, and the cone, **Q**. Case **D** is the plane. The notion of cylinder and cone can be generalised to include all forms with parallel and convergent rulings respectively. **A**, which, as we know, does not allow rulings let alone developable cases, is unfortunately one of the most important. It includes our planet. The history of the atlas is the history of the mathematical problem of how to represent the round earth on a flat piece of paper.

A developable surface, then, if it bends at all, only bends in one principal plane.
You read out your speech from a sheet of paper held out in front of you. “Ladies and …” Oops, the top has folded out of sight. You start again. This time you bend the sheet slightly in a horizontal plane. “… gentlemen.” A sheet of paper models a developable surface. You have pre-empted a bend in a vertical plane. Here, shown in plan and frontal view respectively, is a paper holder using that principle:



Because distances between points and angles between lines stay fixed, we can imagine a developable surface divided into infinitesimal rigid elements. For example, in the plane, a square must stay a square. If we are to distort the square we must come out of the plane, we must make a fold. The square must therefore become a skew quadrilateral consisting of two right-angled isosceles triangles hinged along their common hypotenuse. Here we use Polydron ‘Frameworks’ pieces to make a mesh and drape it over a round stool to form part of a cone. We pick out points along one particular ruling to show what happens to the elements coloured red.





This cone is a pretty rough approximation but, if our net had had 60 units on a side instead of 6, you can imagine that we could have obtained quite a good one; if 6,000,000, what we actually get with a sheet of filter paper [note **10**].

An important fact about developable surfaces is that, whatever you do to them, they remain developable surfaces (or tangential sets of them). This has the status of a theorem but it follows immediately from the definition.

The most striking example children meet is a cylinder becoming a tetrahedron:

Go back to the speech you were making. I claimed that the curve made you safe. But a drinks can is curved and can be deformed quite easily:

We have used Polydron 'Sphera' pieces to model half of the crushed can. The shapes available are necessarily restricted. The can's conical parts are probably nearer one third than a half of a cone and the triangular faces are not likely to be equilateral, but, qualitatively, you see the sort of thing that happens.

In the next case, the 'squash' line is oblique. Our model shows that the bottom half-cylinder is unaffected but the upper part has become a pair of half-cones arranged apex-to-base.

In our last case we have squashed the can along two lines perpendicular to the cylinder axis and to each other:

The resulting solid has the symmetry of a tetrahedron whose faces are isosceles triangles (like the one we folded from the paper cylinder). We have not tried to make a Polydron model but, given a suitable range of flat, cylindrical and conical pieces we could make a fair imitation.

We've said that a developable surface rolls on a plane. Other than the plane, can one developable surface roll on another?

We can think of 'rolls on' as a relation, **R**. So we can write:

Cylinder **R** *Plane* - but also, since the motion is relative - *Plane* **R** *Cylinde*r.
Similarly,

Cone **R** *Plane*, and *Plane* **R** *Cone*.
The relation is symmetrical.

So we have: *Cylinder* **R** *Plane, Plane* **R** *Cone*.
Do we have *Cylinder* **R** *Cone*?
Yes: the relation is also transitive.

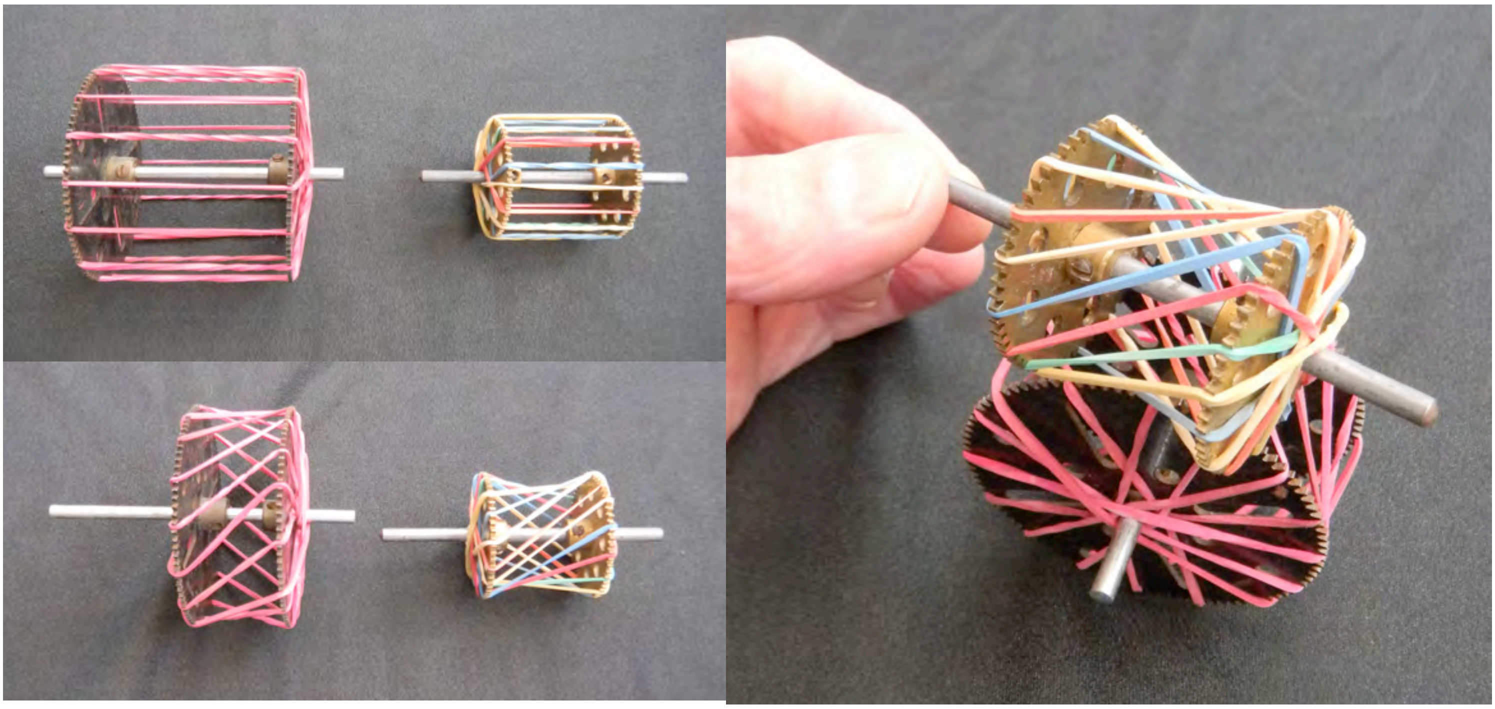
In this experiment a pair of 'feet' has been marked on each of 5 equally-spaced rulings on a cylinder. This has then been rolled on a cone and the corresponding 'footprints' marked. (Note however that, though we have been able to set the cylinder so that the first pair of footprints lies on a ruling of the cone, they necessarily diverge thereafter.)

To see why the relation must be transitive, imagine we adapt the above experiment in the following way. Instead of the little feet, we stick a strip of double-sided tape around the cylinder. As the cylinder rolls on the cone, we pull the cylinder away from the tape so that the tape remains on the cone. The tape represents the plane. We have physically telescoped the two statements
Cylinder **R** *Plane*, *Plane* **R** *Cone* into *Cylinder* **R** *Plane* **R** *Cone*.
The plane acts as the identity in a binomial operation.

There is a case of a ruled surface which is not developable, rolling on another:

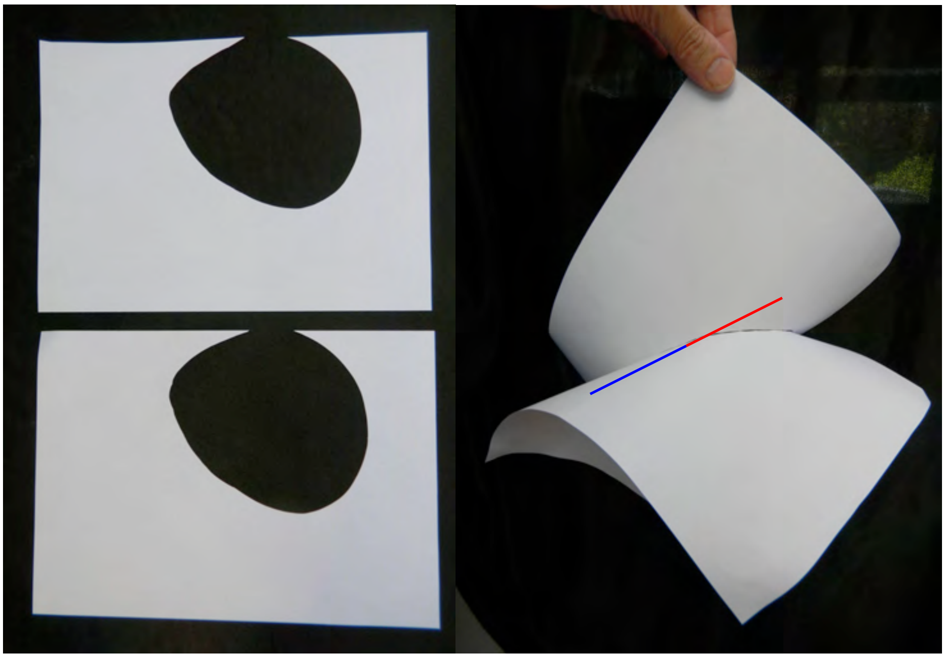
Put two cylinders on skew axes in contact. Imagine one end of each cylinder is turned until a ruling in one lies on a ruling in the other. At that point the cylinders will have become hyperboloids (Our

‘Ruled surface no. 3’). We can now turn the one and the other will mesh with it. This is the principle of hyperboloidal gears [note **12**]. (Other gear types can only accommodate shafts which are either parallel or perpendicular.)



I shall end this section on developable surfaces with a simple experiment from ‘Thomson and Tait’ [see note **3**] which produces a rather unexpected case called the *tangential developable*.

Lay one sheet of paper on another. Cut out and discard a shape which only curves positively and tape together the matching edges. Take one corner of one of the sheets and lift your model off the table:



What was originally a plane curve becomes a space curve. But the beautiful thing is that, at any point on it, the tangent extends to one side in one sheet (red) and the other side in the other sheet (blue).

Acknowledgement

The first thing I did when I completed my first draft was to send it to John Sharp, who inhabits this particular mathematical world. Of the 31 (!) recommendations he made, I hope I have incorporated most.

Notes

1. For a 14-minute video interview with John Sharp go to: http://www.lkl.ac.uk/cms/index.php?option=com_content&task=view&id=69&Itemid=48 or, more simply, google 'John Sharp Sliceforms LKL' and click on the top entry.

2. Since Stanislas Dehaene's landmark publication *The Number Sense* (English edition 1997), a lot has happened which has gone unremarked in the maths teaching literature. Thank goodness then for Gemma Richardson's piece in our junior journal:

Richardson, G., 'The Brain and Primary Maths', *Primary Mathematics*, Vol. 15, issue 3 (Autumn 2011), pp. 15-16.

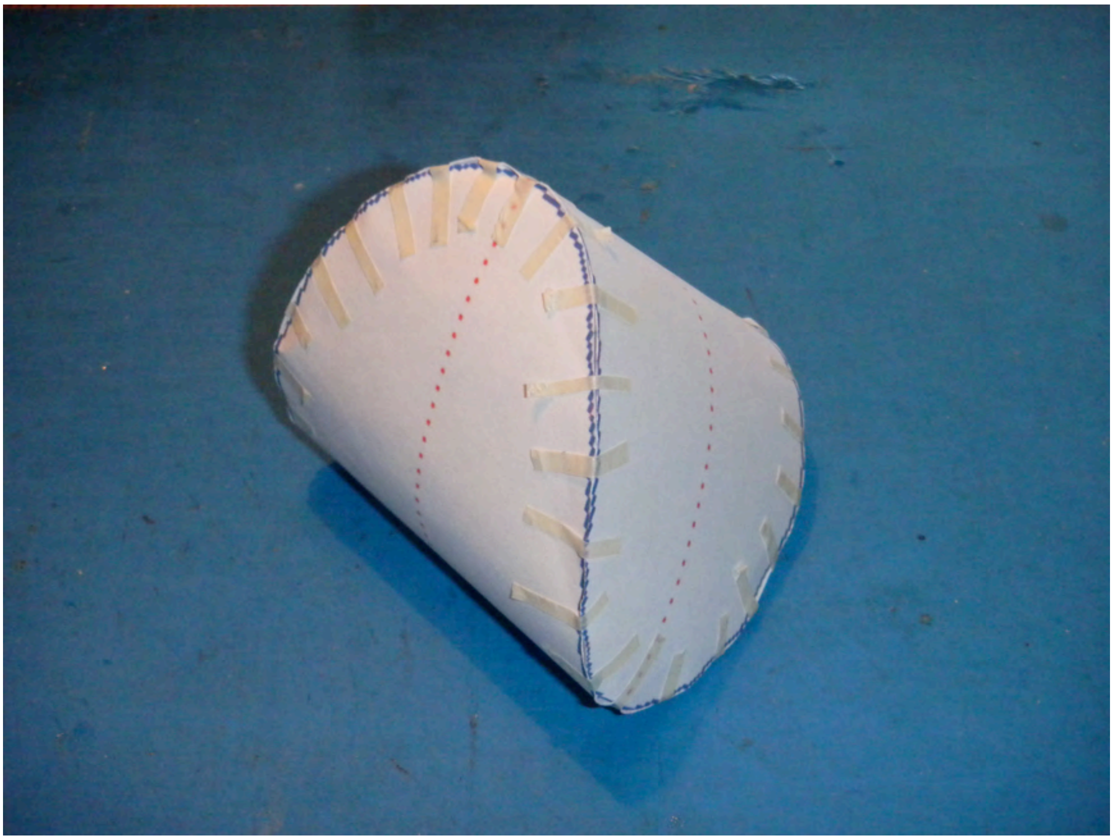
3. There cannot be many textbooks a century and a half old which are still consulted today but 'Thomson & Tait' (1867) is one. Google that title, then the two remarkable collaborators. I hear through my medium that they are taking a great interest in the technology which allows you now to download Volume 1 of their *Treatise on Natural Philosophy* as an e-book. Go to section **130** for their proof of Euler's Theorem.

4. Following his first Tarquin book with the title *Sliceforms*, John has published a much fuller work:

Surfaces: Explorations with Sliceforms, Tarquin 2004.

Go to www.tarquingroup.com and click on 'Books'.

5. The name is a nod to the product designer Tony Wills and his 'D-forms'. There is no mathematical connection between the two, just the following association. Imagine that, instead of keeping the two crossed ellipses parallel in 'Ruled surface no. 4' and threading between them, we had stitched the edges together point-for-point. Think of two pitta bread halves joined at right angles and baked. The result comes out of the oven like this:



Given a material which allows the necessary distortion, and swapping the ellipses for dumbbell-shaped laminae, we have an example which is neither developable nor ruled but a lot more familiar: the tennis ball.

The D-forms, however, are both ruled and developable. They are the subject of John Sharp's 2009 Tarquin book with the same title, and extend the range of surfaces one could ever imagine to be developable.

6. With reluctance we omit the steel balls and stick magnets of Geomag because the resulting surfaces do not preserve angle. (What it could model is a fabric, where the *warp* and the *weft* run at right angles, allowing a square to deform into a rhombus.)

7. Use the Internet. For the company I use, go to www.supermagnete.de . For 'Ruled surface no. 3' I used identical, paired magnets and cut and drilled one set to receive one end of the elastic thread, held with a dab of 'Araldite'. But this is not necessary. In the corresponding picture for 'Ruled surface no. 2' you see magnets already fitted with hooks.

8. The hyperbola has two lines of symmetry: one through the foci and a perpendicular one through the centre. Use the first as axis of revolution and you get two separate pieces (*sheets*); use the second and you get the hyperboloid of one sheet. The latter surface is ruled; the former, not.

9. An astigmatic lens focuses one line at one distance, a perpendicular line at a different distance. In the interval (the *conoid of Sturm*) the blur changes from an ellipse in one orientation to one at right angles. There is an intermediate distance at which it is circular.

10. At www.cut-the-knot/ctk/Parabola.shtml you can use Alexander Bogomolny's interactive applets to investigate every aspect of the parabola's geometry, in particular for our purposes why the curve-stitching procedure and the paper-folding procedure, where you keep folding one edge of a sheet on to an arbitrary point, produce the same figure.

11. To avoid confusion, I should stress here that I am not *triangulating*, a process through which computer animators can simulate any curved surface, whether developable or not.

12. Google 'Ch06J Spiral Gear Geometry' and click on the top entry.

Keywords: Curvature; Developable; Ruled.

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