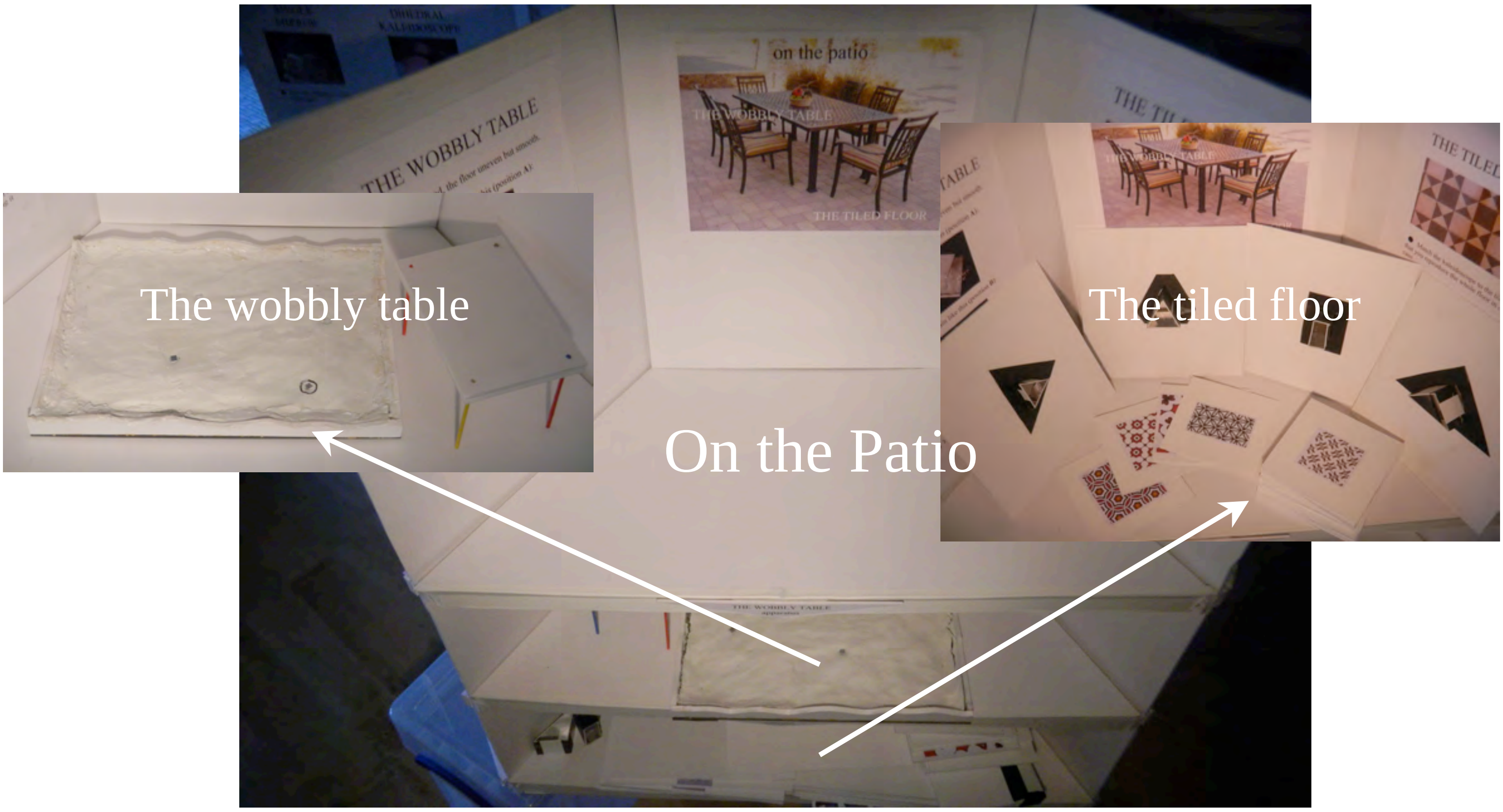
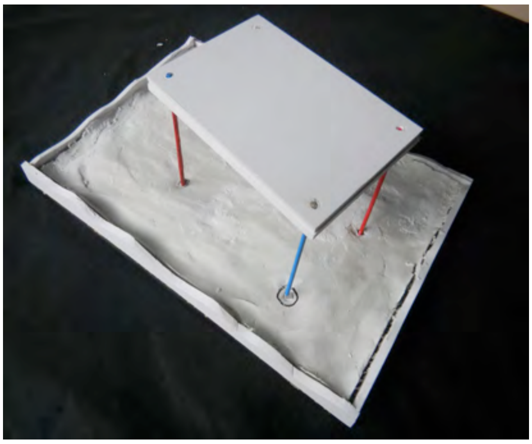
Household geometry



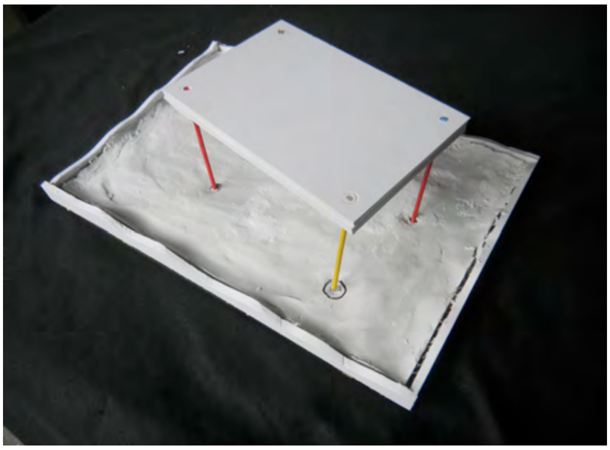
THE WOBBLY TABLE

The table is good, the floor uneven but smooth.

Set the table down like this (position **A**):



Set it down again like this (position **B**)



Now imagine starting the table in position **A** and, keeping 3 legs on the ground, turning it till it arrives at position **B**.

Make the experiment.

Notes

Though part of mathematical folklore long before, the problem became known through Martin Gardner's column in 1975. Gardner's intuitive proof makes no use of the intermediate value theorem but later writers cite the problem as a nice exemplar. In terms of our apparatus, Gardner's argument goes like this:

At some point as we turn the table, the yellow leg leaves the ground. Since there are always at least 3 legs on the ground, the blue leg must be one of them. At some point as we turn the table, the blue leg touches the ground. Since there are always at least 3 legs on the ground, the yellow leg must be one of them. Since no 2 legs are in the air at once, there must be a point where both the blue leg and the yellow leg, and therefore all 4, are on the ground. (You can imagine the yellow leg saying to the blue leg, "Oh, I was just leaving".)

For an explanation using the intermediate value theorem, visit David Eubanks' IVT and Wobbly Table - You Tube (www.youtube.com/watch?v=CD3URqfVyQY)

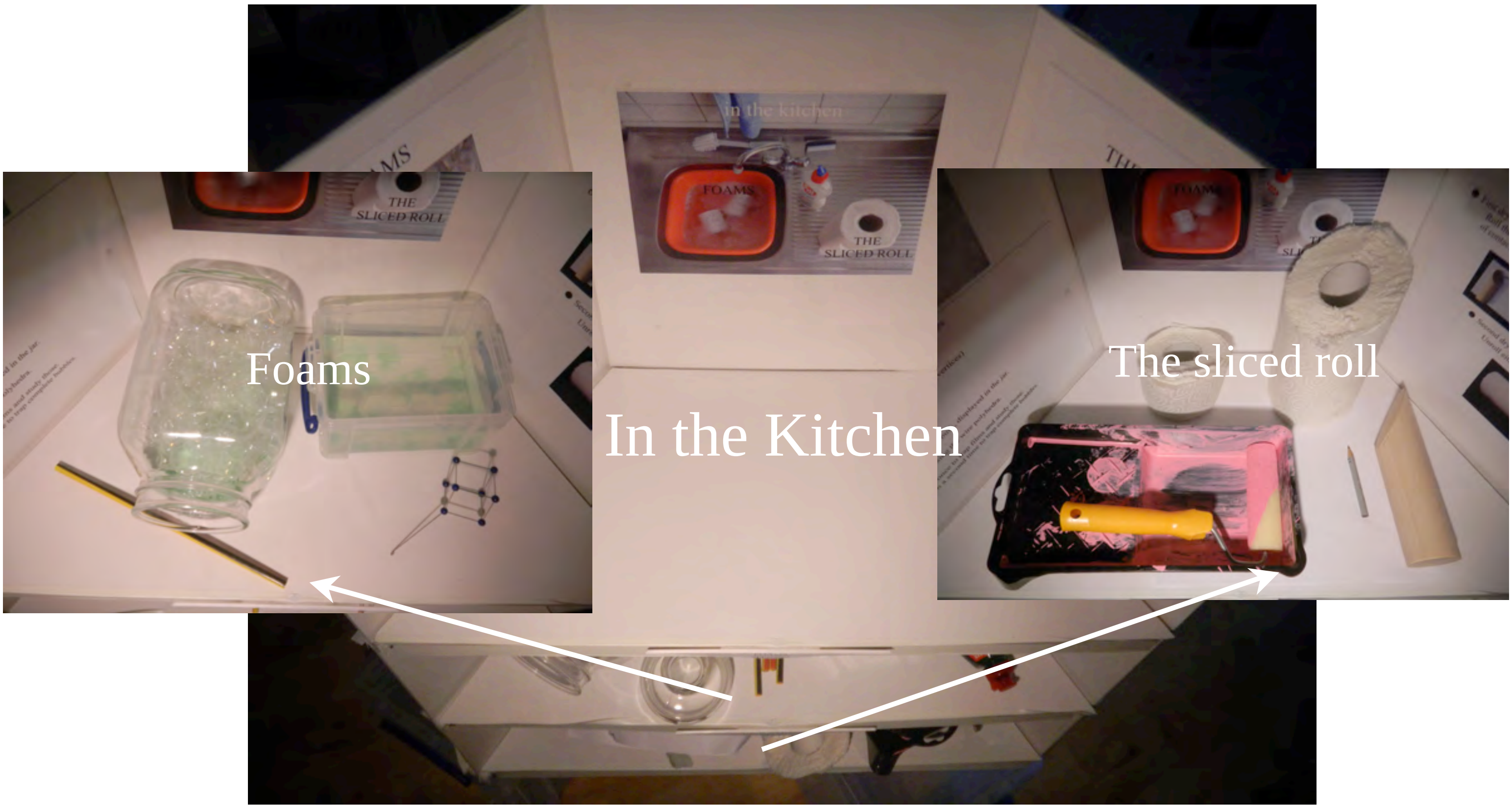
THE TILED FLOOR

Match the kaleidoscope to the tiling so that you reproduce the whole floor in each case.

Notes

If the symmetries of a tiling can be accounted for solely in terms of reflections, it can be recreated by the appropriate polygonal prism lined with mirrors.

The *locus classicus* is 'Tilings and patterns' by B. Grünbaum & G. C. Shephard. For an alternative characterisation of the symmetry groups concerned, see part 1 of 'The symmetries of things' by J. H. Conway, H. Burgiel & C. Goodman-Strauss. As part of the Atractor project, M. A. Chaves has developed the program Simetria, in which the tilings are printed by rolling deformable 'stamps' on the plane.



FOAMS

What seems to be true of the bubbles

- the walls (faces)
- the face junctions (edges)
- the junctions-of-junctions (vertices)

- sometimes?
- always?
- never?

Study them in the bulk, displayed in the jar.

Isolate them using the wire polyhedra.

- Dip them once to trap films and study those.

- Dip them a second time to trap complete bubbles.

Notes

The following features are easier to spot in films spanning the wire frames:

1. 3 walls meet symmetrically in an edge, and are mutually inclined at $2\pi/3 = 120°$ therefore.

2. 4 edges meet symmetrically in a vertex, and are mutually inclined at $\arccos(-1/3)$, $\approx 109\frac{1}{2}°$ therefore.

The third is not apparent to the eye but is a consequence of the films forming surfaces of minimal area:

3. The principal curvatures at every point are equal and opposite and lie in perpendicular planes.

When a film encloses a single bubble in equilibrium, the bubble is spherical and there is a pressure excess on the inside proportional to the curvature, preventing the bubble collapsing. Deep inside a foam, where the bubble is surrounded by others, the pressures are equal on all sides so play no part in determining the bubble's shape. This is governed therefore by conditions **1**, **2** & **3**. If the foam is perfectly uniform, what we have, to quote the title of Lord Kelvin's paper of 1887, is 'The division of space with minimum partitional area'. Kelvin's ideal bubble was a truncated octahedron, distorted. It was more than a century before the Irish physicists Aste and Weaire derived a more economical structure ('A counterexample to Kelvin's conjecture on minimal surfaces', 1994). This is a packing of two bubble shapes. The biologist Matzke made careful observations of actual foams to find the proportions of (distorted) *n*-gons of different *n*. His numbers agree better with the Weaire-Aste than the Kelvin model. Interestingly, neither Kelvin nor Aste & Weaire had set out to study foams. Kelvin sought a structure for the aether, the medium then supposed necessary to carry electromagnetic waves. Aste & Weaire were studying metal alloys.

A good starting point is Rachel Thomas' piece 'Swimming in mathematics' (http://plus.maths.org/latestnews/sep-dec08/watercube . See also ch. 3 of www.magicmathworks.org/maths-club projects/docs/thescottishbubbleandtheirishbubble.pdf .

THE SLICED ROLL

One experiment, three versions

Predict the results before making the experiments.

Wet version (thought experiment):

The roller lies obliquely in the tray.
Paint.

First dry version:

Roll the obliquely sliced wooden cylinder, marking the points of contact.

Second dry version:

Unroll either half of the paper towel roll.

Notes

The border of the painted patch, successive points of contact of the oblique section of the wooden cylinder with the paper, the frayed edge of the paper towel roll, all trace a sine curve.

You can find the paint tray experiment in 'New horizons in geometry' by M. Mnatsakanian & T. Apostol; the sliced roll experiment in Hugo Steinhaus' 'Mathematical Snapshots'. (Steinhaus wrote before there were paper towel rolls. He wrapped a candle in paper.) For a derivation go to www.magicmathworks.org/geomlab3.pdf .

THE UMBRELLA

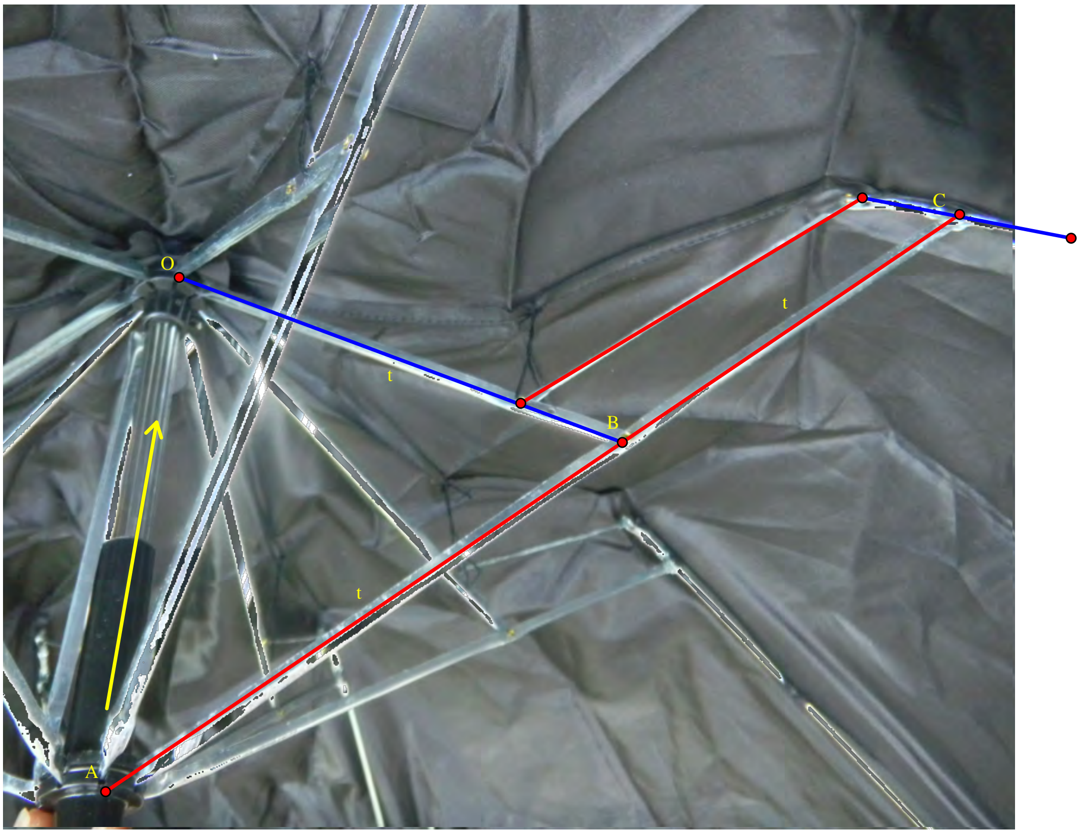
What 2-D shapes can you identify?

How can they change?

How are they connected?

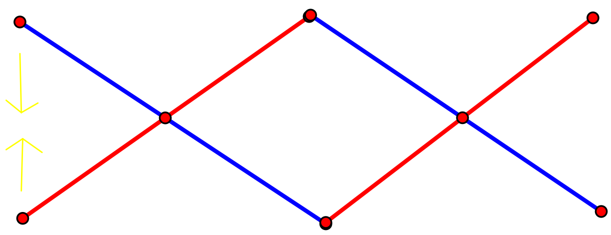
What is their function?

Notes



Parallels are shown by colour. The user pushes the yellow slider in the direction of the arrow. We observe the apex of the isosceles triangle formed on this base to move outwards (describing a circle about the head of the umbrella), the parallelogram to open out, and the blue lines to separate. But we can say more. If the lengths are as marked, we have a circle centre **B**, diameter **AC**, passing through **O**. Angle **AOC** is therefore a right angle, which means that the locus of **C** is a straight line perpendicular to **AO**.

If the other side of the parallelogram was joined to the slider, and the parallelogram was a rhombus, we would have a pair of 'lazy tongs':

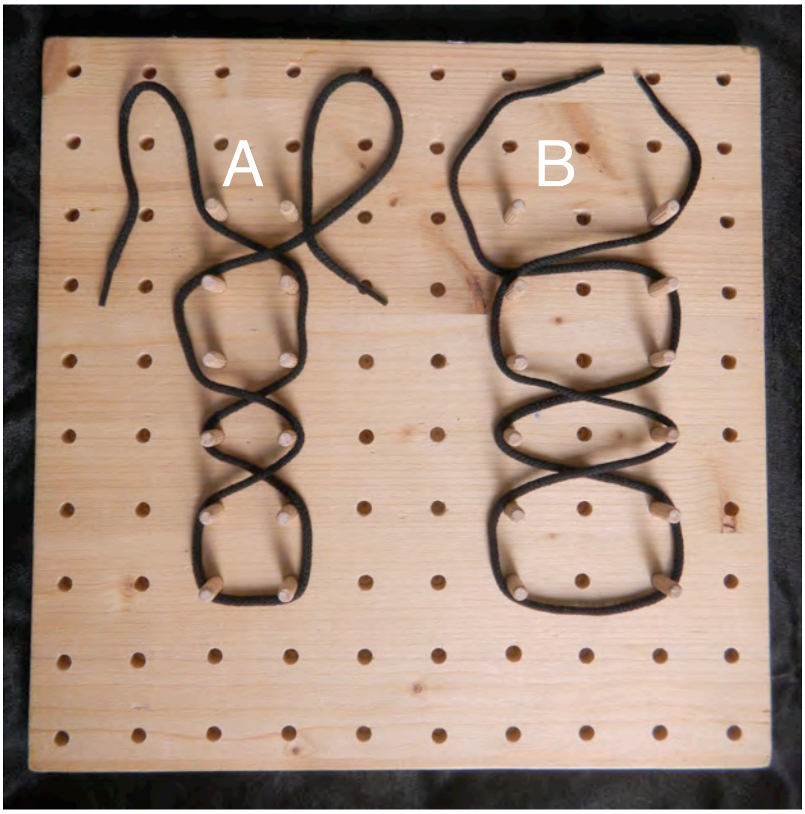


An index of mobility, describing the variety of positions into which it can be moved, can be defined for anylinkage. The 'degree of freedom' is determined for each part, and a sum formed for the whole. Go to en.wikipedia.org/wiki/Linkage_(mechanical) . If we make the calculation for the umbrella, we get a mobility of 1. This makes sense if we consider the point **C**, which is confined to a straight line, representing a single degree of freedom.

For a first sketch of a morning workshop for 13-14-year-olds on the subject, e-mail me at the address below.

SHOELACES

Use the pegboard to help you tackle these questions.



What do laces do?
What do laces do that velcro doesn't?

What makes for a strong lacing pattern? [**1**]

Does it make a difference whether the horizontal spacing is close [**A**] or wide [**B**] compared with the vertical separation?

Which pattern(s) use(s) the shortest lace? [**2**]

You should find that pattern is/those patterns are, symmetrical. Why is this?

You should also find requirements **1** and **2** conflict. What compromise solutions are possible?

What is the special requirement for a soldier's boot? Which pattern(s) meet(s) it?

What is the special requirement for a footballer's boot? Which pattern(s) meet(s) it?

If you're wearing lace-ups, what is the requirement for your shoes today? Look down! Does your lacing pattern meet it?

Notes

What do laces do?

They enable the shoe to conform to the foot within an adjustable tolerance so that, on the one hand, the shoe does not slip when the foot moves; on the other, it is not painfully tight.

What do laces do that velcro doesn't?

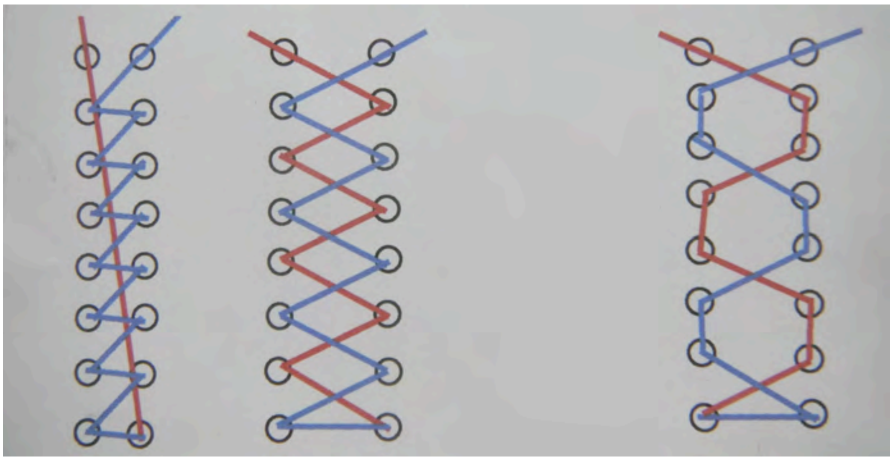
Unless it is hand made, a shoe does not follow the contours of an individual foot. Friction at the lace holes allows the lengths of lace between the holes to vary, achieving that match. Unlike a fastening with buckles or velcro, this variation is to some degree self-adjusting.

What makes for a strong lacing pattern? [**1**]

The tension should have the greatest possible horizontal component.

Does it make a difference whether the horizontal spacing is close [**A**] or wide [**B**] compared with the vertical separation?

*Yes. It turns out that pattern **X** is strongest in case **A**; **Y** in case **B**:*



X ***Y*** ***Z***

Which pattern(s) use(s) the shortest lace? [**2**]

*In the 'trivial case' the laces would just run up each side. **Z** has enough crossing-points to give a strong result while using a reasonably short lace.*

You should find that pattern is/those patterns are, symmetrical. Why is this?

Were the pattern asymmetric, the mirror image would also give the shortest result. For there to be a single minimum the two lacings must be one and the same.

You should also find requirements **1** and **2** conflict. What compromise solutions are possible?

***Z** is one possibility. Of the 42 patterns described on 'Ian's shoelace site', quoted below, almost all are such compromises.*

What is the special requirement for a soldier's boot? Which pattern(s) meet(s) it?

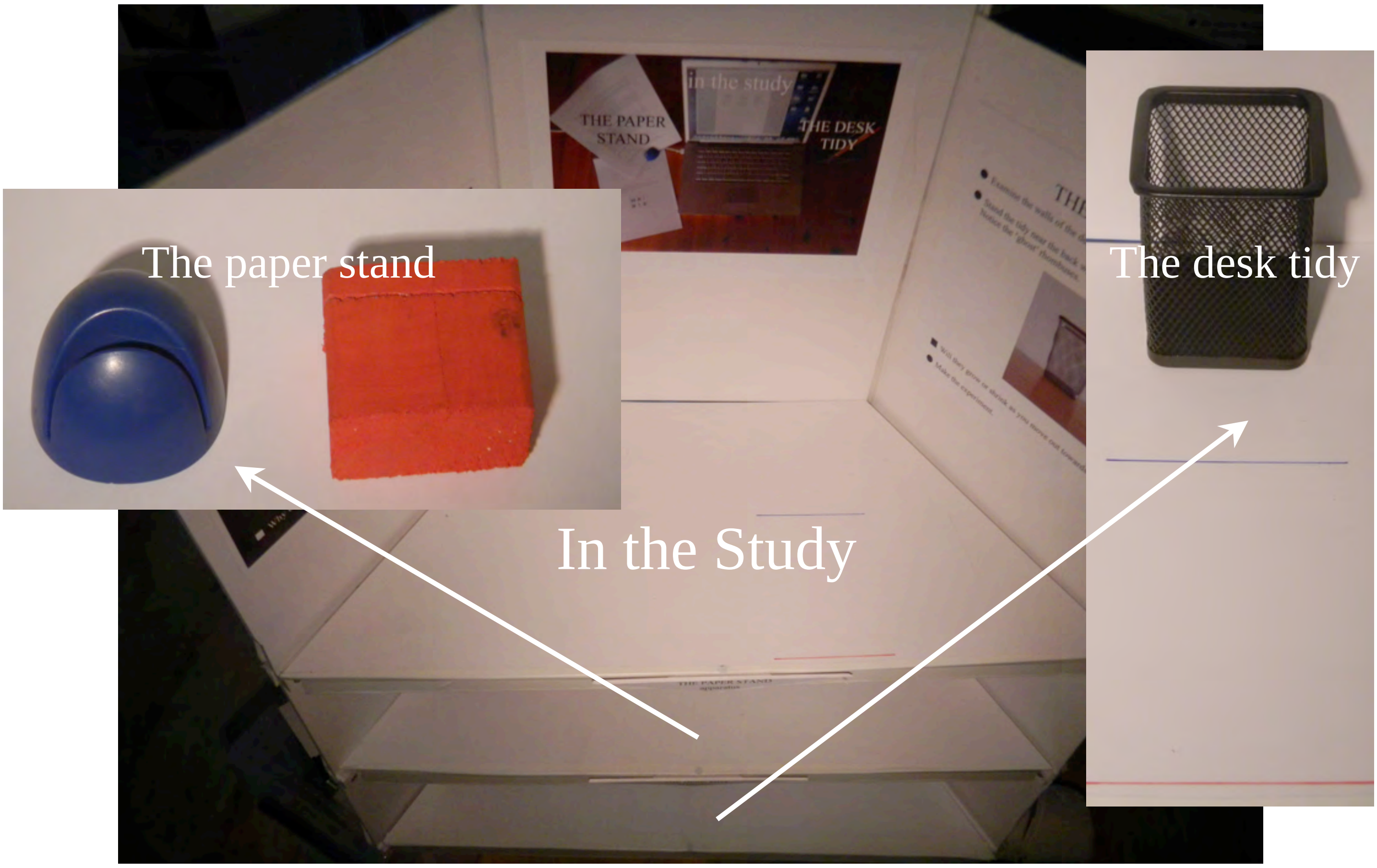
*If the foot is injured in battle, the boot must be removed promptly, preferably by a single cut. **X** is one lacing which allows this.*

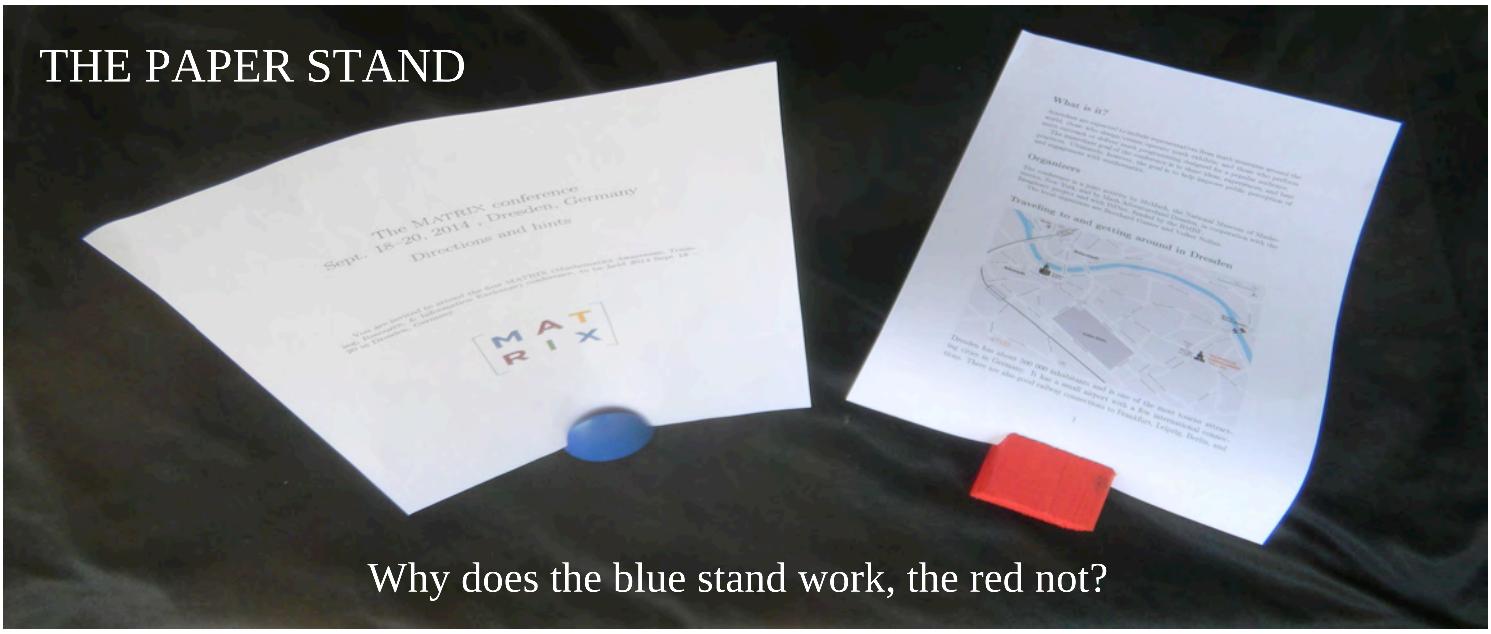
What is the special requirement for a footballer's boot? Which pattern(s) meet(s) it?

The part of the boot which scoops up the ball should be free of lace. See which of Ian's lacings has a gap at this point.

If you’re wearing lace-ups, what is the requirement for your shoes today? Look down! Does your lacing pattern meet it?

'Ian's shoelace site' (www.fieggen.com/shoelace/) contains, or has links to, everything of importance on this topic. This includes reference to a paper which appeared in Nature vol. 420 p. 476, studying the effect of changing all the parameters we have been considering and more. There was a synopsis jn the 4.12.02 issue of New Scientist.





Notes

The blue stand imposes on the sheet a plane of principal curvature. To flop back, as it does in the red stand, it would have to bend in a second plane at the points where it emerges from the stand. But, since the sheet is of paper, not rubber, it can only conform to a surface which rolls out flat, a *developable* surface. This geometry allows only one plane of principal curvature, perpendicular to which straight lines run through the sheet, i.e. it is a *ruled* surface. All developable surfaces are ruled, (though the converse is false).

THE DESK TIDY

Examine the walls of the desk tidy. Notice the rhombuses.

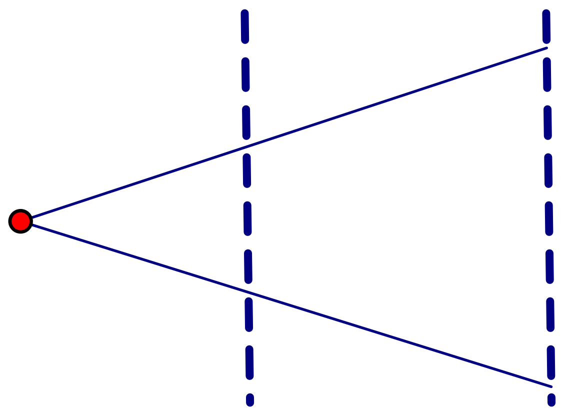
Stand the tidy near the back wall and look from the blue line.
Notice the 'ghost' rhombuses.

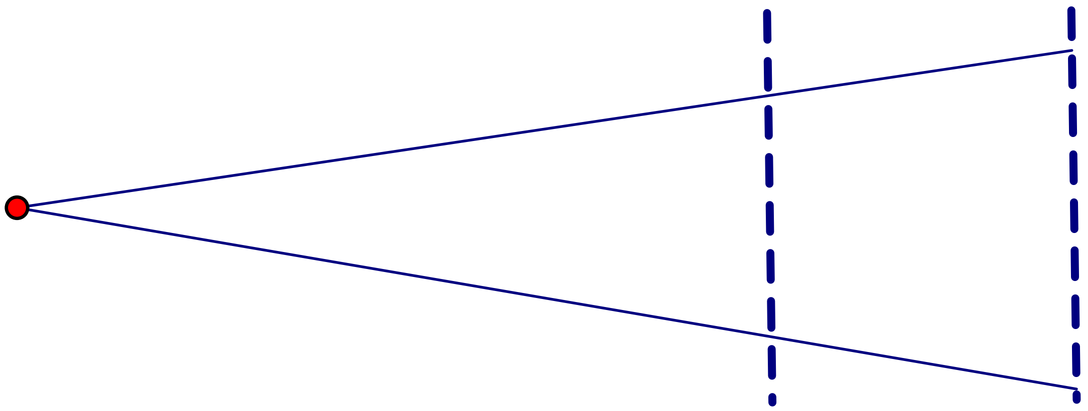
Will they grow or shrink as you move out towards the red line?

Make the experiment.

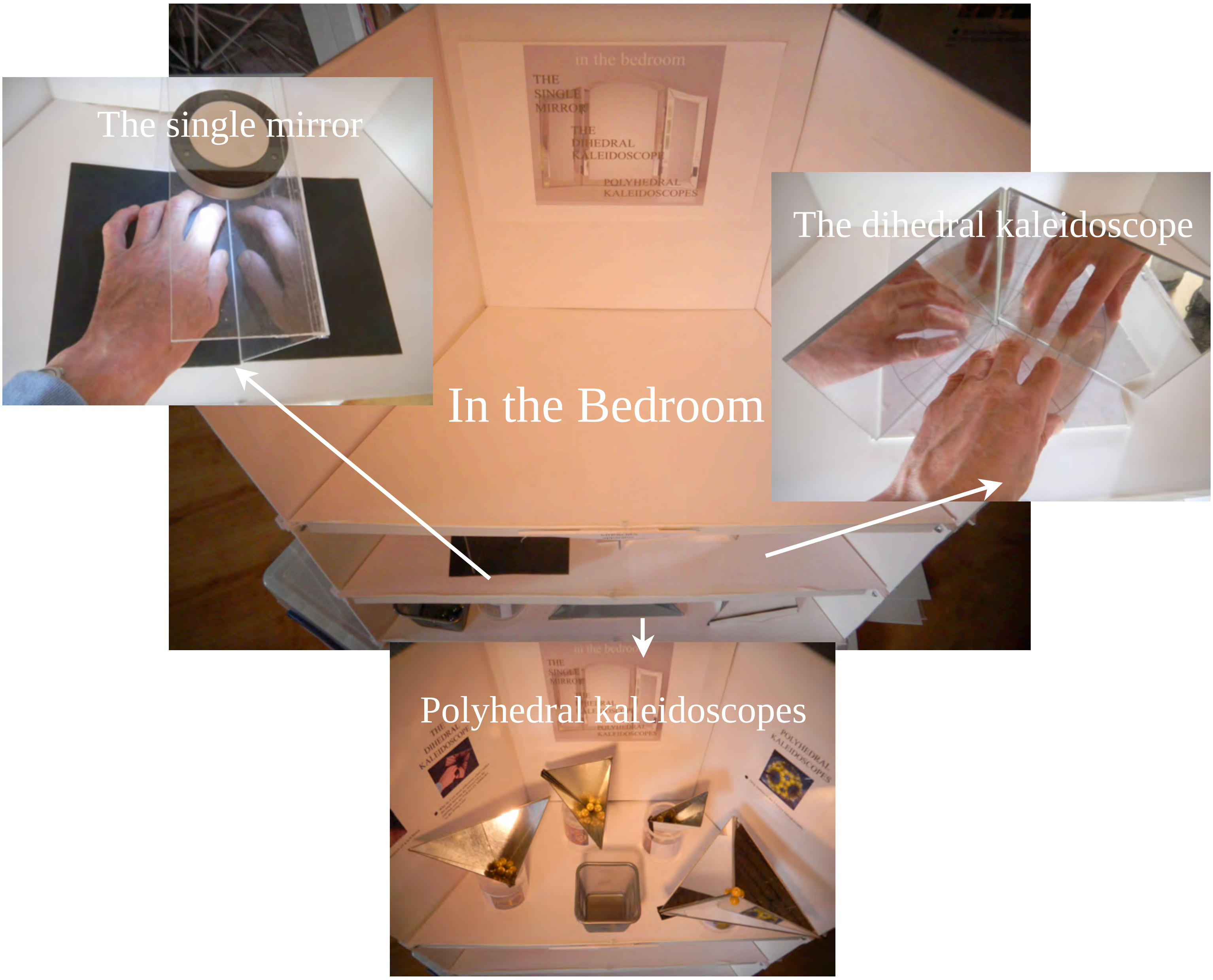
Notes

The phenomenon here is called a Moiré effect. Moiré effects are caused by phase differences between a pattern and the same pattern, shrunk or twisted, superimposed on the original. In our case the shrinking is achieved by parallax. The tiling of rhombuses in the back wall of the tidy subtends a smaller angle at the eye than that at the front. The upper profile below shows the case where *k* rhombuses at the back fit the space of *l* at the front. If *k* and l are coprime, counting in terms of back rhombuses, the two sets come into phase every *kl.* In the lower profile, *k* back rhombuses coincide with a greater number, *m*, of front rhombuses. *km* > *kl*, so the 'ghost' rhombuses in the second case are (fewer and) bigger.



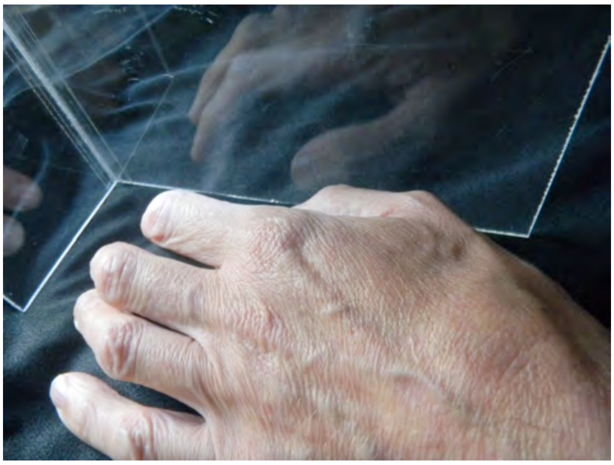


For the provisional draft of of a morning workshop on the subject for 13-14-year-olds, to be given first 27.9.14, e-mail me at the address below.



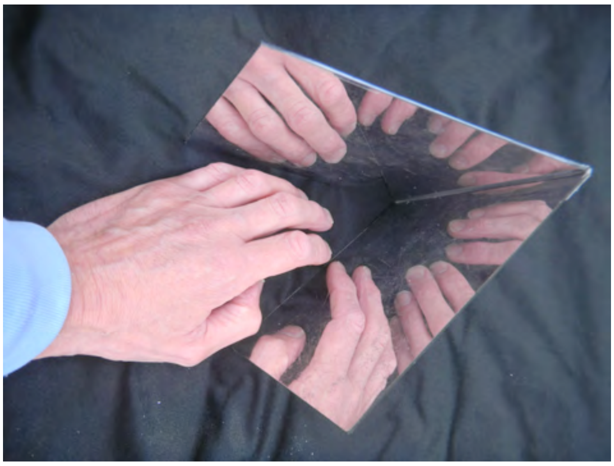
The single mirror
in the bedroom
THE
SINGLE
MIRROR
THE
DIHEDRAL
KALEIDOSCOPE
POLYHEDRAL
KALEIDOSCOPES
The dihedral kaleidoscope
In the Bedroom
Polyhedral kaleidoscopes

THE SINGLE MIRROR



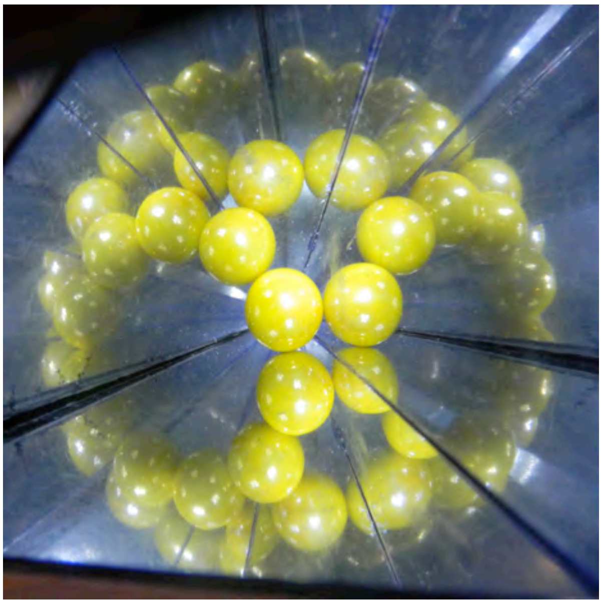
Why is the reflection of a left hand, a right hand?

THE DIHEDRAL KALEIDOSCOPE



Why do I see more reflections when I reduce the angle between the mirrors? (For the limiting case, try PARALLEL MIRRORS ‘in the living room’.)

POLYHEDRAL KALEIDOSCOPES



Drop marbles in the 3-mirror kaleidoscopes.

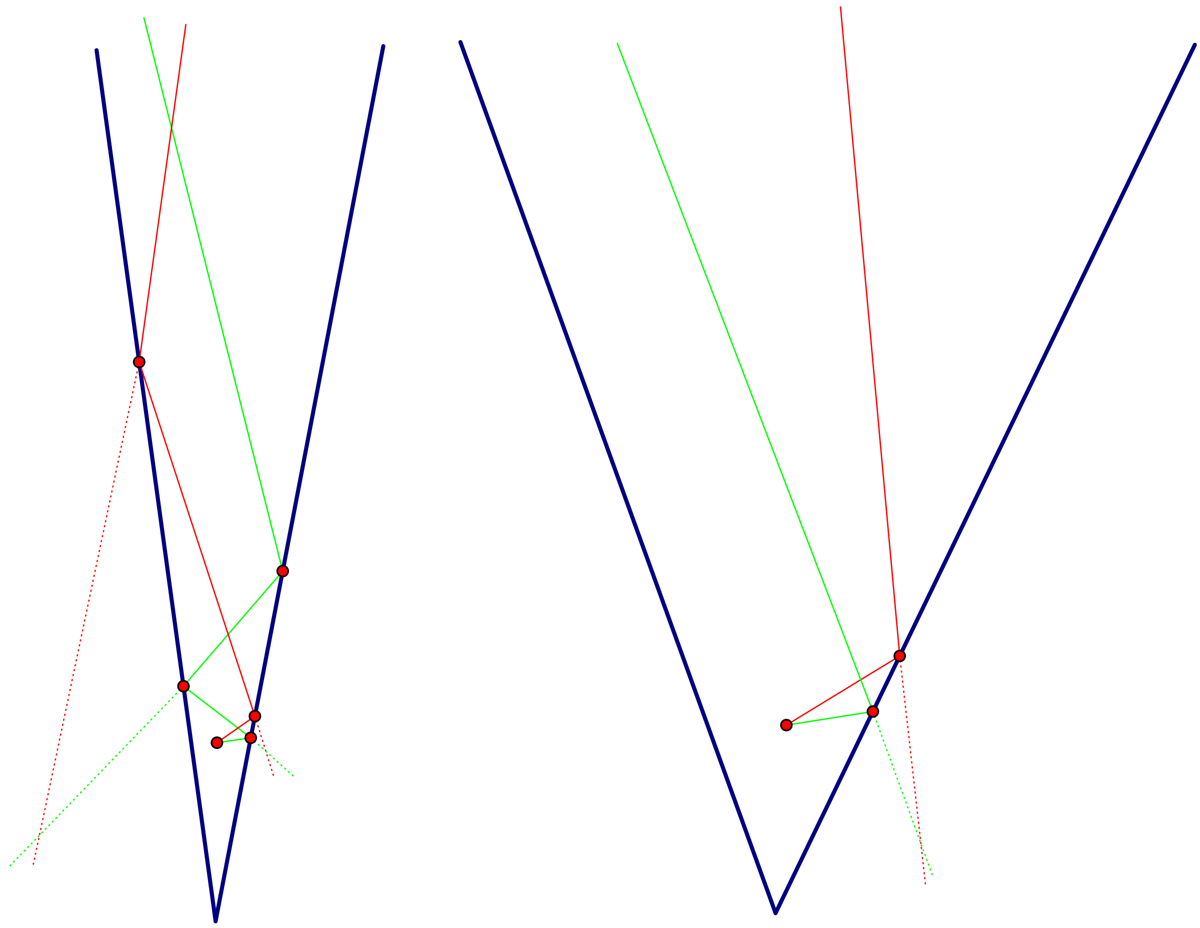
Notes

The single mirror

A plane mirror reverses front and back. This is equivalent to turning the hand - or a glove containing it - inside out. The left and right hands cannot be superimposed: they are the definitive *enantiomorphs*.

The dihedral kaleidoscope

Below we contrast the effect of setting the kaleidoscope at different angles. We have taken a 'red' and a 'green' ray leaving the object at the same angles in each case. On the left a number of subsequent collisions lead to more reflected images, represented by the intersecting pairs of dotted red and green lines.

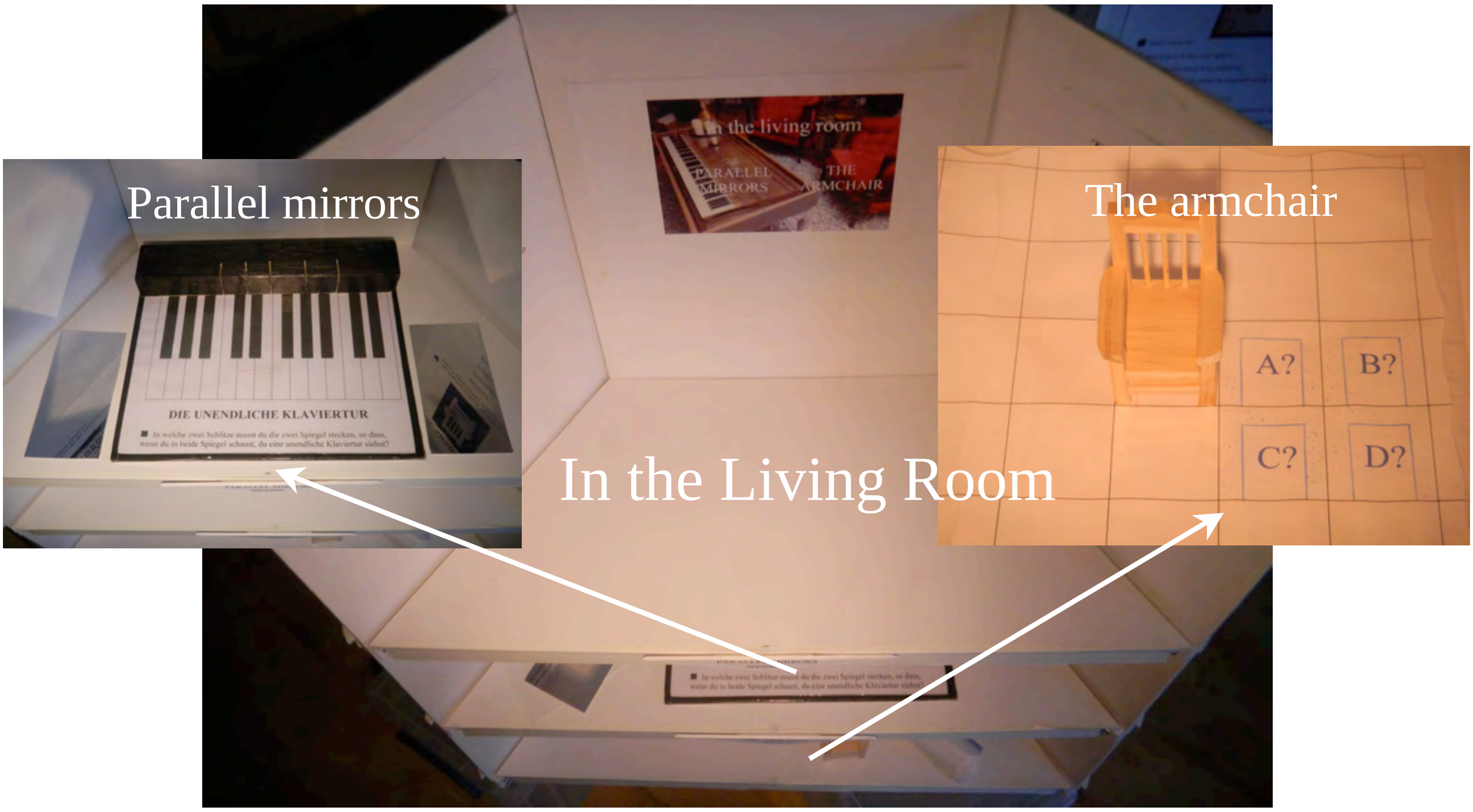


Polyhedral kaleidoscopes

These kaleidoscopes are to the prismatic kaleidoscopes used to construct tilings 'on the patio' as the dihedral kaleidoscope is to the pair of parallel mirrors 'in the living room'. The former construct something finite; the latter, something infinite. What is created here is a polyhedron. A tiling is a polyhedron with an infinite number of faces. In the case of a tiling the *angular defect* at each vertex which causes the polyhedron to close around a sphere, is zero.

We can imagine fitting together the solid angles defined by the 3 mirrors, to make the complete polyhedron. If the kaleidoscope has the smallest possible solid angle θ steradians to reproduce the whole solid, the full order of symmetry of the latter is $\frac{4\pi}{\theta}$.

Go to www.magicmathworks.org/masterclasses and select 'The Cube' as a .pdf . (Like the others cited in these notes, this wo0rkshop is for 13-14-year-old children.)

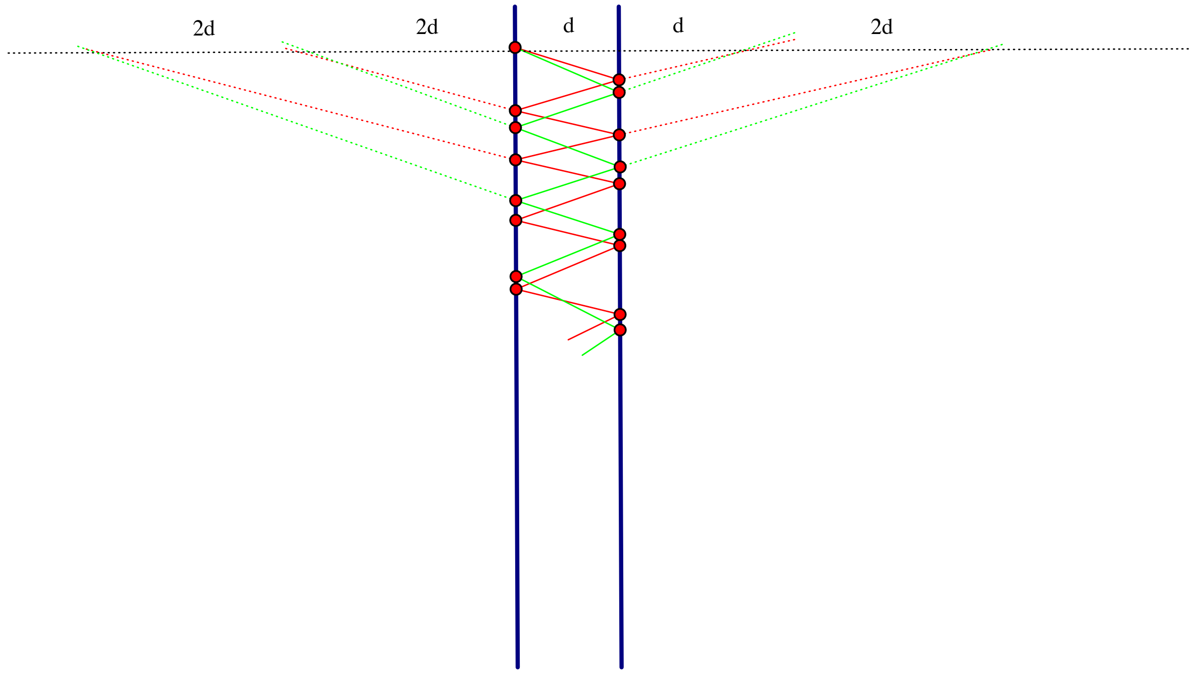


PARALLEL MIRRORS

Slide the two mirrors into the correct slots so that, when you look into one, you see an infinite keyboard.

Notes

Compare the figure below with those for the dihedral kaleidoscope 'in the bedroom'.



Were the rays perfectly parallel, a ray bouncing back and forth along a normal would make an infinite number of collisions, corresponding to an infinite number of reflections. You see, by similar triangles, that the successive image distances follow an arithmetic progression. The image positions fall on alternate sides of the diagram, at distances $2kd$ to the left, $(2k + 1)d$ to the right.

THE ARMCHAIR

The armchair is heavy. To move it, you must swing it about corners. Can you bring it to

position A?

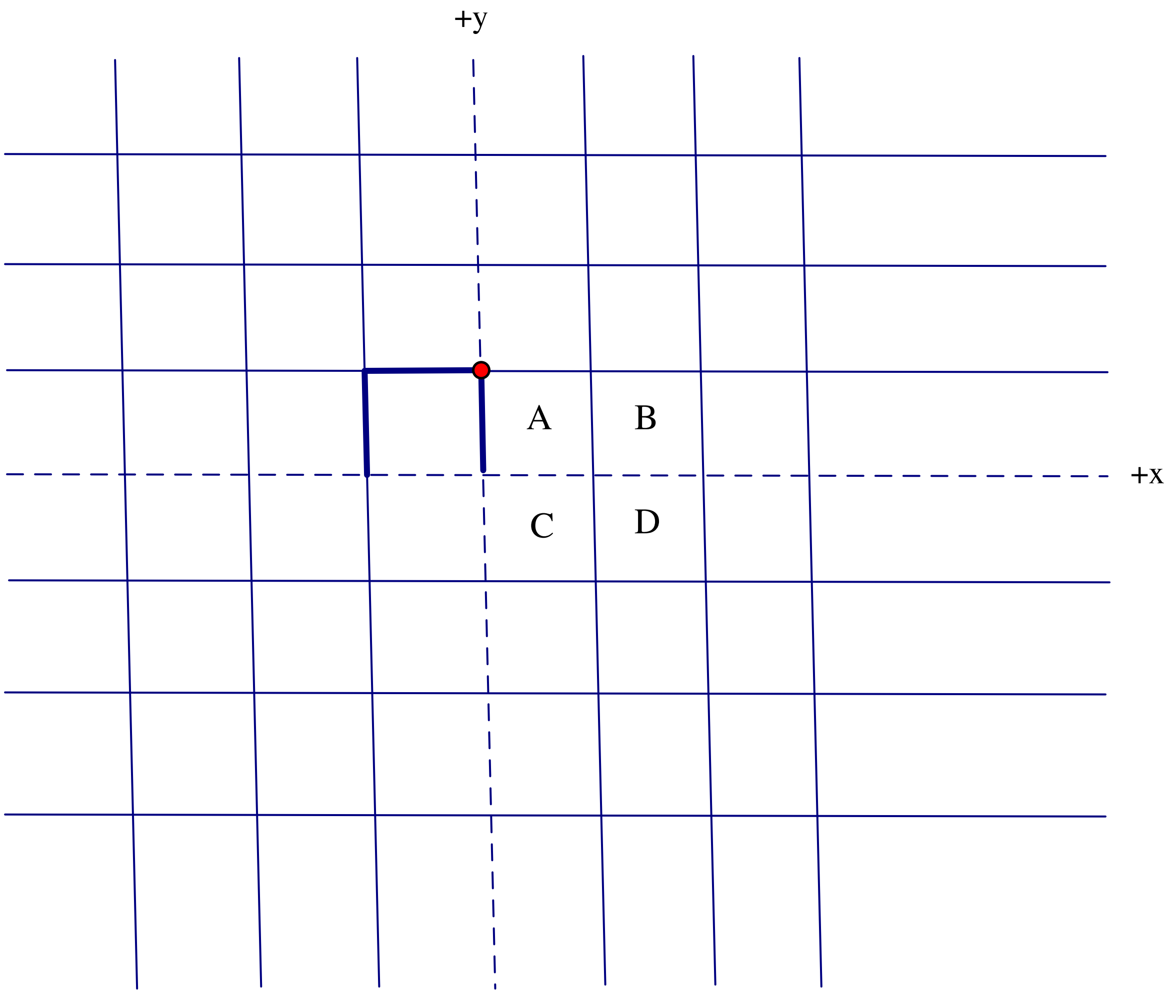
position B?

position C?

position D?

Notes

The turns required are multiples of a right angle. Here is the armchair shown in plan on a Cartesian grid, the left back corner marked with a dot.



If you track its progress as you move the toy chair around, and note the difference in the x-coordinate and the y-coordinate on each swing, you find that the sum changes by either 2 or 0. To get the chair into positions **A** and **D** would require a change in this sum of 1 and 3 respectively. Only positons **B** and **C** (and squares with the same checkerboard colouring) are possible.

This problem is taken from 'Thinking mathematically' by J. Mason, L. Burton & K. Stacey.

* * *

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