# Heuristics
## applied in
## The Magic Mathworks Travelling Circus

## INTRODUCTION

The Magic Mathworks Travelling Circus is a touring maths lab. These notes comprise a check-list for teachers who have visited the Circus and are interested in helping their students develop the ways of thinking a maths lab encourages.

The booklet is in two parts. **Part A** lists twenty-some **stations**. For each, the relevant heuristic methods are listed. **Part B** lists the **methods** themselves, taking as examples the stations keyed from **Part A**.

Edition: 10.11.05

| STATION \ METHOD | Seeking analogies | Splitting a problem into parts | Bounding the possibilities | Structuring and restructuring the problem | Generalising and specialising | Seeking symmetries | Applying the analysis-synthesis procedure | Applying the principle of similarity | Exhausting cases | Seeking a recursive structure | Devising a notation | Backtracking |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1.4 The Seesaw | ● | | | | | | | | | | | |
| 2.6 Perspective Drawing | ● | | | | | | | | | | | |
| 3.9.2 Motorway Networks | ● | | | | | ● | ● | | | | | |
| 4.2 Penrose Rhombuses | | | | ● | | | | | | | | |
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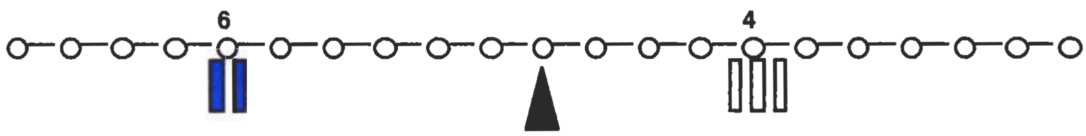
| 7.6.1 Sliding Sam | ● | | | | | ● | | | | ● | | |
| 7.6.2 The Ferry Problem | ● | ● | | | | | | | | | | |
| 7.7 Leapfrog | | | | ● | | ● | | | | ● | | |
| 7.8 The Hanoi Pagoda | | | | | | ● | | | | ● | | |
| 8.2 The Ladder | | | | | | ● | | | | | | |
| 10.1.3 The Pie Game | | | | ● | | | | | | | | |
| 10.1.4 Domino Block | | | | | | ● | | | | | | |
| 10.1.5 3-D Os & Xs | | | | ● | | ● | | | | | | |
| 10.2 Safe Queens | | | ● | | | | | | | | ● | ● |
| 10.3.1 Grandpa's Armchair | | | | | | | | | | | ● | |
| 10.5.3 Graeco-Latin Squares | | ● | | | | | | | | | | |
| 10.6 A Domino Rectangle | | | | ● | | | | | | | | |
| 11.3 The Magic Cone | | | | ● | | | | ● | | | | |

# A. STATIONS

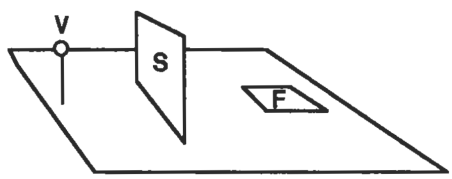
## 1.4 The Seesaw

'A' puts hangers on a chosen peg. 'B' must choose a different peg and the correct number of hangers in order to balance the beam.




## 2.6 Perspective Drawing

A geometrical figure **F** is laid on the table and traced on the screen **S** from viewpoint **V**.

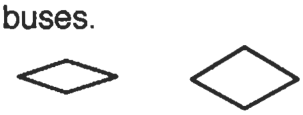



## 3.9.2 Motorway Networks

A motorway network is to connect (e.g.) 3 cities shown by dots on a board. The experimenters draw what they consider to be the network of smallest total length. 2 perspex plates joined by pins in the corresponding positions are dipped in soap solution and withdrawn to reveal the true configuration.

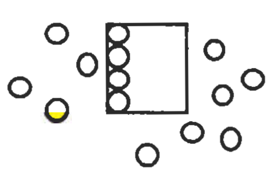
## 4.2 Penrose Rhombuses

The rhombuses required for an aperiodic tiling have angles of $36^{0}$ and $72^{0}$. As an exercise preliminary to making such a tiling, the students find how many vertex types are possible with the unmarked rhombuses.



## 4.5 14 People in a Lift

A rectangle is of such dimensions that it will just accommodate 14 disks without overlap. The experimenter must find the packing necessary.

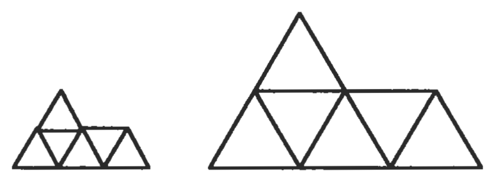


## 5.1.1 Tangram Polygons

Outlines of a selection of convex polygons are offered. These must be filled with those constituting the standard 7-piece tangram.

## 5.1.2 The Riddle of the Sphinx

The 'Sphinx' is a hexiamond (6 equilateral triangles joined edge-to-edge) and a 'rep-tile', a figure which aggregates into similar shapes. The experimenters must arrange unit sphinxes in similar outlines of successive orders of size.



## 5.2.1 The Soma Cube

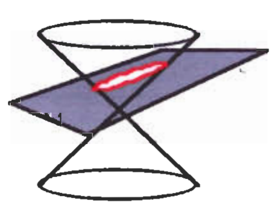
The Soma Cube is an example of a 3 x 3 x 3 cube dissected into 7 polycubes: 1 3-cube, 6 4-cubes. The precise significance of the forms is discussed in **Section B**.

## 6.2 Map-colouring Solids

The experimenter must build polyhedra from polygonal tiles in such a way that no two tiles of the same colour touch edge-to-edge.

## 6.5.1 Baravalle Cuts

Hermann von Baravalle pioneered this technique, which is to section a translucent solid by means of a plane of light. In the Circus the position and orientatior of the plane and the position and orientation of the model may all be altered independently.

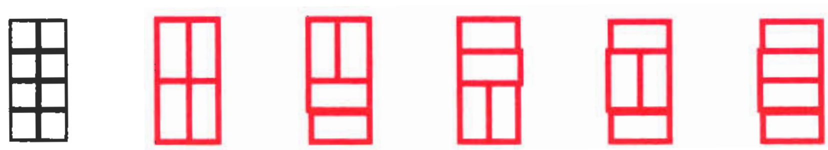


## 7.1.3 Handshakes

n people meet and all shake hands. How many handshakes will there be? Beginning with n = 2, 3, ... the experimenters predict, then model the people with pegs and the handshakes with rubber bands. Beyond n = 5, these become respectively dots and lines on a whiteboard.

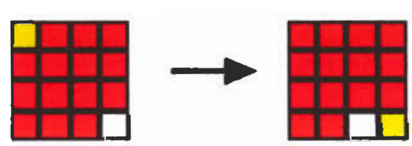
## 7.4.3 Path-paving

The paths are rectangles 2 squares wide; the paving slabs are dominoes. How many different patterns are possible for paths 1 square long? 2 squares long? 3 squares long? ... ?



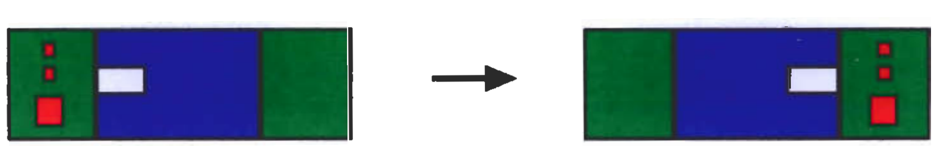
## 7.6.1 Sliding Sam

In a 'sliding-block' puzzle one constructs a picture by arranging square tiles in a frame with one missing tile. An adjacent tile can move into the space, allowing completion by a succession of such moves. At this station the object is to transfer one tile (Sam) from top left to bottom right in the fewest moves and in frames of increasing size.



## 7.6.2 The Ferry Problem

1 man and 2 boys must cross a river. All can row but the boat will accommodate only 1 man or 2 boys. How can it be done? The experimenters model the situation. Having achieved success, they extend the investigation to 2 men and 4 boys, 3 men and 6 boys, ...

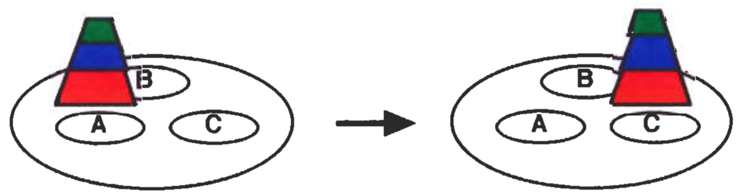


## 7.7 Leapfrog

A line of n red frogs must change places with an opposing line of n blue frogs. There is a single space between them. A frog may jump into a space over one of the other colour or slide into an adjacent space. The aim is to achieve the swap in the fewest moves. We model n = 1, 2, 3, ...

## 7.8 The Hanoi Pagoda

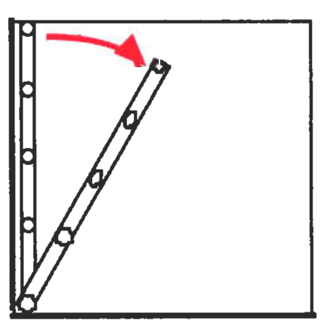
A tower of n inverted cups of decreasing size stands on site **A**. One must move it to site **C**, using site **B**, according to 3 rules: **1)** A cup may only be taken from on top. **2)** A larger cup may not be placed on a smaller. **3)** The cups must be moved one at a time. As at other stations, the experimenter is encouraged to begin with n = 1 and progress upwards.



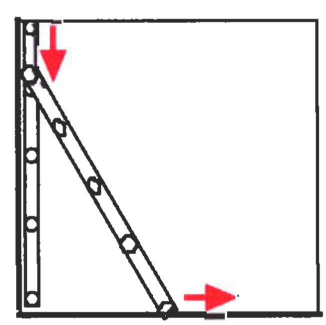

## 8.2 The Ladder

A ladder rests against a wall. The problem has 2 variants: **8.2.1**: the ladder falls backwards, pivoting about its foot; **8.2.2**: the foot of the ladder slides outwards and the head downwards correspondingly. The exercise is to predict the loci of significant points on the ladder. The apparatus models the situation in cross-section. Dry-wipe pens are located in holes in the model ladder and trace paths on a whiteboard.

**8.2.1**

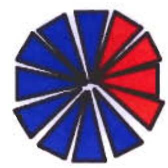


**8.2.2**



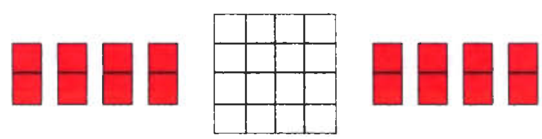
## 10.1.3 The Pie Game

On the plate are 8 slices of pie: 8 blueberry and 4 cherry. The 2 players take turns removing slices. The player who takes the last slice wins. A player has 3 options: **1)** to take a chosen number of blueberry slices, **2)** to take a chosen number of cherry slices, **3)** to take the same number of each.



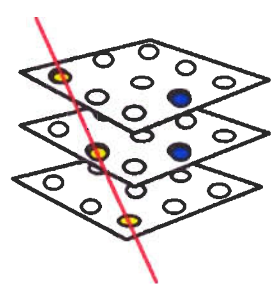
## 10.1.4 Domino Block

Each player has 4 dominoes.The players take turns placing their dominoes on a 4 x 4 grid. The last to do so, wins.



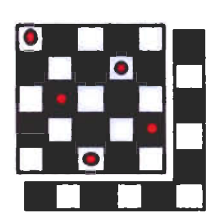
## 10.1.5 3D Os & Xs

The players take turns placing marbles in their colour on the 'board'. The board is a hollow 3-cube with a depression in each of its 27 cells. The first player to make a line of 3 marbles wins.



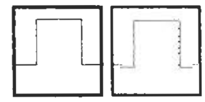
## 10.2 Safe Queens

The investigators work on chessboards of increasing size. The aim is to arrange as many queens as possible in such a way that no queen threatens another, that is to say, no 2 queens share the same vertical, horizontal or diagonal line.



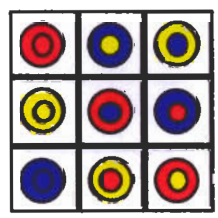
## 10.3.1 Grandpa's Armchair

A heavy square armchair can only be moved by swinging it through right-angles about its corners. Can it be moved to a new position directly alongside the old?



## 10.5.3 Graeco-Latin Squares

The investigator has cups and saucers, n each of n colours - initially 3 but later 4. They must be paired and arranged in an n x n array in such a way that, **1)** all n colours are represented in every row and every column in both cups and saucers, **2)** all $n^2$ pairings are present.



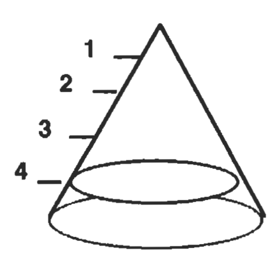
## 10.6 A Domino Rectangle

The investigator is faced with a rectangular grid 8 squares x 7. Each cell bears a number between 0 and 6. A full set of dominoes is to be arranged on this grid so that the number of spots on each half of the domino corresponds to the number written in the corresponding cell.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 4 | 6 | 6 | 6 | 6 | 0 | 4 |
| 0 | 3 | 5 | 5 | 5 | 5 | 2 | 4 |
| 5 | 2 | 4 | 4 | 4 | 4 | 0 | 5 |
| 1 | 1 | 3 | 3 | 3 | 3 | 6 | 5 |
| 4 | 6 | 2 | 2 | 2 | 2 | 6 | 3 |
| 1 | 6 | 1 | 1 | 1 | 1 | 3 | 0 |
| 5 | 2 | 0 | 0 | 0 | 0 | 3 | 2 |

## 11.3 The Magic Cone

A perspex cone is graduated linearly in 5 equal steps. Powder reaches the lowest mark. Which mark will the powder reach when the cone is inverted?



# B. METHODS

Soldiers distinguish 'strategy' from 'tactics'. The first is the general approach adopted; the second, the particular dispositions and movements of troops needed to implement it. The solution of a mathematical problem involves the same 2 stages. The study of problem-solving strategies is called 'heuristics'; the study of particular tactics, 'algorithmics'.

In this booklet we concentrate exclusively on the first.

In the Circus itself one section (**10**) is indeed called 'Heuristics'. It contains stations without specified content where the exercise is to find, as it were from first principles, the mathematics applicable. In this process the investigator must call on many of the strategies described here. However, this section contributes only 30% of the stations discussed. From the other 70% take for example the station 'Leapfrog' from the section 'Sequences'. It appears here under 2 headings: **Structuring and restructuring the problem** and **Seeking a recursive structure**. At each appearance we derive the relation between the number of moves and the number of frogs in a different way and each sheds light on what's going on. At the station itself the children proceed empirically. They record the number of moves for 1, 2, 3 and 4 frogs each side, compile a difference table, use it to predict the number of moves for 5 frogs, then test their prediction. Students with some algebra can apply the method of finite differences to the table and derive a general formula. The tasks on both levels are useful and satisfying but shed no light on the nature of the problem. Our slogan might be, 'No algorithmics without heuristics'.

Though a difference table itself is not helpful in all cases, the principle of starting with simple cases invariably is. It enables one to make a conjecture and see if it survives a change in the parameters - in the case of the frogs, an increase in their number (or indeed a change in their number*s* - see the introduction to **Seeking symmetries**.)

- Trying simple cases
- Seeking patterns
- Making conjectures

These heuristics are encouraged in the way the Circus is set up and presented. They do not appear among our headings only because they should precede those more specific strategies.

We have chosen 12 principles. Every author classifies heuristic methods in a different way or gives them different emphases so don't take our headings as definitive. For example, we have two headings: **Seeking analogies** and **Structuring and restructuring the problem**. Restructuring produces an analogy so the two are really one.

Concerning **Seeking symmetries**, a teacher who read a previous draft of these notes commented that the symmetries present in most of the examples I cite are likely to appear only *after* the initial problem has been cracked and can therefore hardly be enlisted as an aid in its solution. In a given case it may be that we only bring one of our 12 techniques into play when we have penetrated a certain depth beneath the first level.

# SEEKING ANALOGIES

The Circus is arranged according to the Multiple Embodiment Principle of Z.P.Dienes: each section takes a particular concept and realises it in physically disparate settings. The mathematics is precisely what the investigator abstracts from each group. In section **1** for example, the peg number and the number of hangers in station **1.4**, the allocation of the chime bar and the number of strikes in **1.5**, and the diameter of the wheel and the number of turns in **1.6** all model the same mathematical situation. The whole Circus exploits such analogies. Each station is indeed an analogue experiment - an analogue computer if you like. We pick out two such examples here. We also choose two examples where the analogy is more elusive and, by way of warning, three examples where the experimenter may be led to make false comparisons.

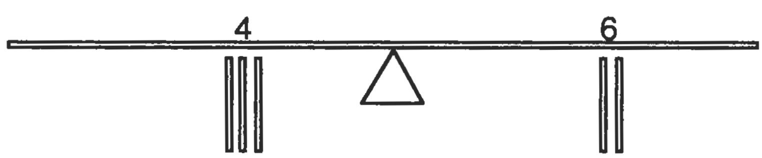
## 1.4 The Seesaw

The mathematics

The existence of a common multiple of 2 numbers means that we can form it as a product in 2 different ways. For example, 12 is a common multiple of 4 and 6; we can write 12 as 4 x 3 or as 6 x 2.

The physics

By representing the multiplicands in units of distance from a pivot, the multipliers in units of force applied normally at those points, we can model the products as torques. To equate products we find the point of balance for a centrally-pivoted beam with weights hung at graduations:




## 4.5 Can 14 People Fit in this Lift?

The mathematics

The problem of packing disks in the plane and spheres on a flat surface is the same: we need only consider cross-sections, viz. circles.

The physics

Therefore, if we want to pack disks as closely as possible, we can allow spheres to roll down a slope under gravity until they find a state of minimum potential energy.

## 7.6.1 Sliding Sam

## 7.6.2 The Ferry Problem

In **7.6.1** we are required to move a tile (Sam) from one corner of a square checkerboard to the one diagonally opposite. His progress is made possible by sliding bean bags in order to free squares. In **7.6.2** we are required to row men across a river. They can be transported only by using boys to return the boat to the original bank. **7.6.1** is extended by increasing the side of the checkerboard from k to (k + 1) units (and, correspondingly, the number of bean bags); **7.6.2**, by increasing the number of men from m to (m + 1) (and, correspondingly, the number of boys). We note this parallel:

**7.6.1**
When k increases by 1, the total number of moves increases by 8.

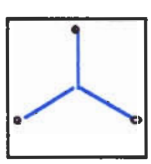
**7.6.2**
When m increases by 1, the total number of crossings increases by 8.

## 2.6 Perspective Drawing

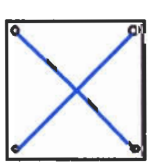
In the general case, a circle placed flat on the table will project as an ellipse on the screen. If at that point you substitute for the circle a square with an edge set parallel to the screen and ask, "What will you obtain *now* ?", the student will almost invariably reply, "A rectangle", presumably interpreting the ellipse as a squashed circle and inferring the equivalent result for a square. Mentally the investigator is performing a 1-way, fractional stretch, whereas the transformation involved is quite different. (Such students do not - at that stage - see the screen as a glass easel. If told the square was a trapdoor in the school stage and asked to draw it from the front, we suspect most would produce a correct view.)

## 3.9.2 Motorway Networks

Students are shown the result for 3 cities at the vertices of an equilateral triangle:



Challenged with the case of 4 cities at the comers of a square, almost all opt for:

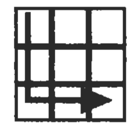


They have drawn a very reasonable analogy - but a false one. When we come to consider the use of symmetry as a heuristic guide, we shall try to find what's going wrong.

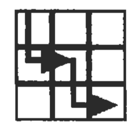
### 7.6.1 Sliding Sam

The best route for Sam on the 2 x 2 board is this:

Children often fix on the fact that his path follows the perimeter and adopt this inefficient procedure on bigger boards:



In fact what we are looking at is one ‘zig’ and one ‘zag’ of a zigzag path:



# Splitting a Problem into Parts

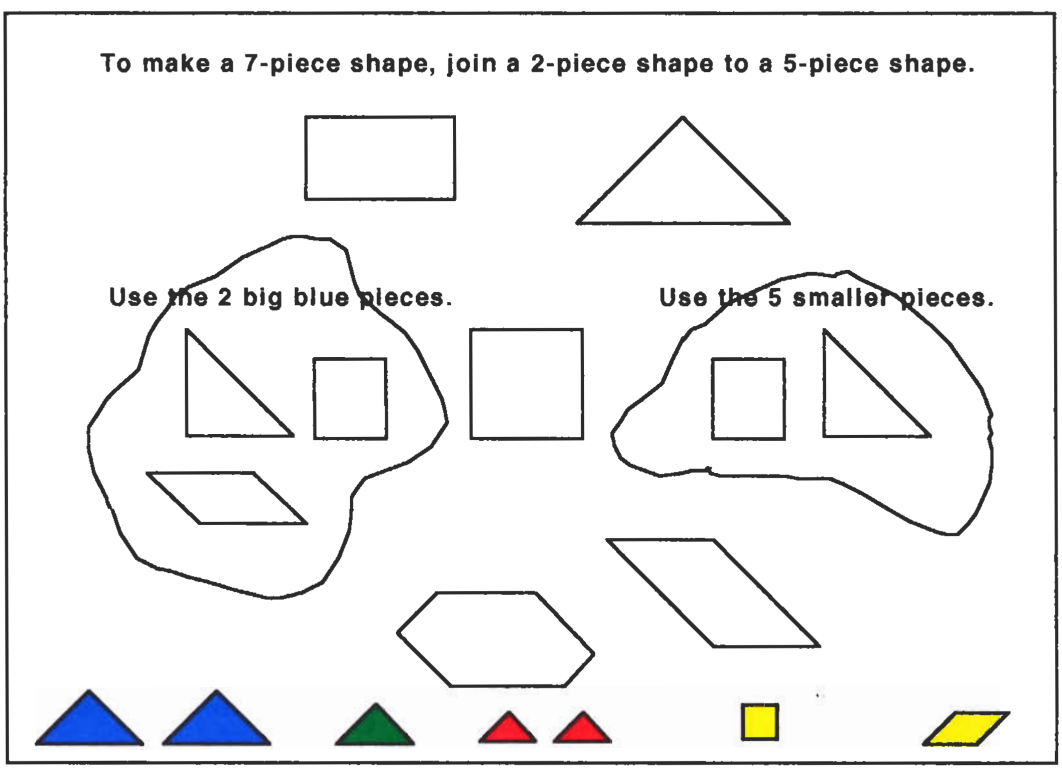
We look at 4 cases. In the first pair, the split is clear; in the second, not so.

## 5.1.1 Tangram Polygons

Though there is satisfaction in solving a dissection puzzle which is merely difficult, it is only of the kind derived from completing a big jigsaw. No dissection puzzle has a place in a maths lab unless it encourages ways of thinking which one can apply more widely.

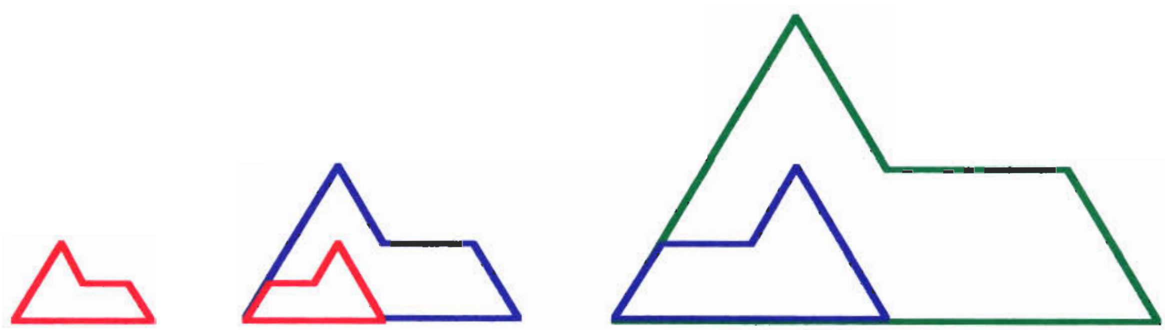
The 'Tangram' is based on the right-angled isosceles triangle. This is a 'rep-tile' of order 2, that is to say, a pair combine to form a similar shape - in this case they join symmetrically by a limb. This leads to a sequence of such triangles, each scaled $\sqrt{2}$ with respect to the last. 3 orders of size are represented in the classic 7-piece Tangram. 2 of the smallest, 1 of intermediate size, 2 of the largest account for 5 of the pieces. Of the remaining 2, one is a square formed by joining 2 of the smallest size by the hypotenuse; the other is a parallelogram, formed by joining 2 of the smallest by a limb but in the other possible relative orientation.

Because the relations between the parts are simple, it is possible to construct bigger polygons by prefabricating modules. This practice is encouraged at station **5.1.1** as shown:



### 5.1.2 The Riddle of the Sphinx

By the principle of similarity, if we can solve the order 2 puzzle, we can solve the order 4: the order 4 sphinx bears the same relation to the order 2 sphinx as the order 2 sphinx bears to the order 1:

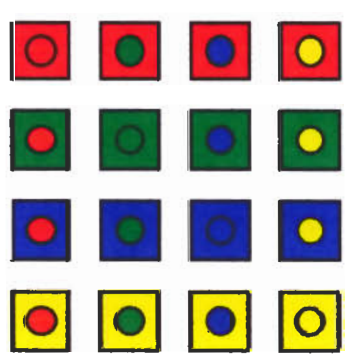


### 7.6.2 The Ferry Problem

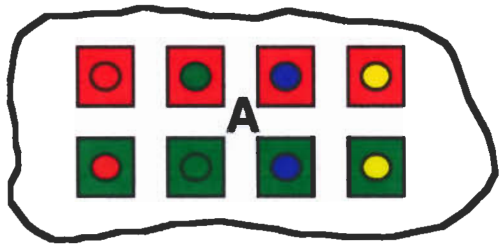
We can consider separately the problem of getting a boy across the river and that of getting a man across. But we then realise the movement of the boys is key to both.

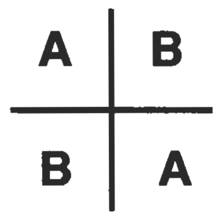
### 10.5.3 Graeco-Latin Squares

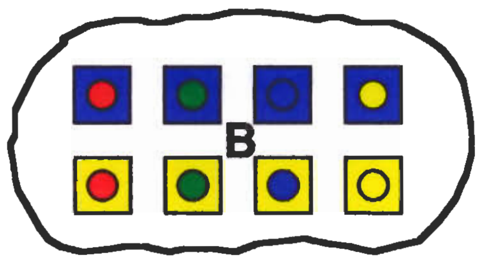
We must ensure the 'cup' square is Latin and the separate 'saucer' square is Latin before we try to combine them and fulfil the new condition, namely that we duplicare no cup-saucer pair. With a Graeco-Latin square of order 3 it is not difficult to check for repeats: we have only 9 pairs to check. With a square of order 4, however, it is easy to produce the separate order-4 Latin squares only to find they are not orthogonal: we have duplicated a cup-saucer pair. With this problem in mind we can choose instead to make up our 16 cup-saucer pairs first, then handle them complete. Here are our 16 pairs of (round) cups on (square) saucers:



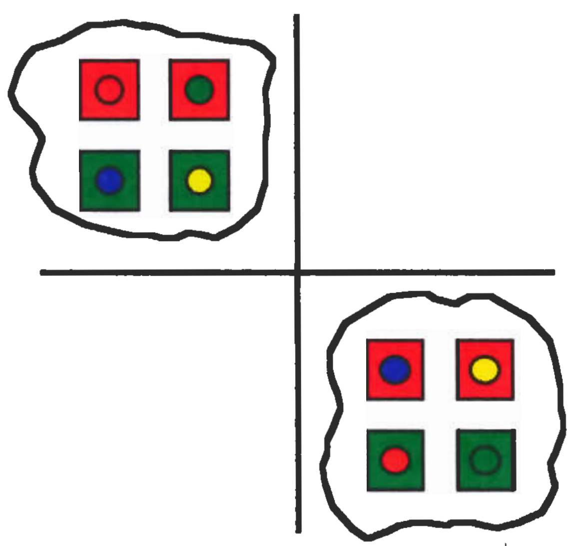
We split these into 2 sets of 8. We decide that all pairs with (say) red or green saucers will go in the top left and bottom right quadrants, all pairs with blue or yellow saucers will go bottom left or top right:




Each set of 8 we now break down into 2 sets of 4. We confine the red saucers bearing (say) red or green cups to the top left quadrant. This dictates not only which pairs with red saucers go bottom right but where the complementary pairs with green saucers go:



We must proceed with care as we converge on our goal. Nevertheless, we have now divided by 4 the number of possible positions a given pair can occupy.

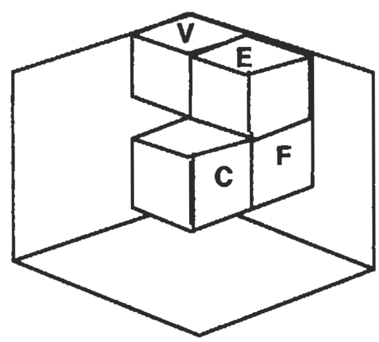
# Bounding the Possibilities

In general there are limits within which the parameters of a problem must lie. Determine these at the outset and you have simplified your task. We take three examples, including two from one particular station. An especially simple case of setting bounds to the range of possibilities is the application of the 'pigeonhole' principle. We cite one example.

## 5.2.1 The Soma Cube

In *Winning Ways* (1st edition: Academic Press, 1982, pp. 737-9) Berlekamp, Conway and Guy point out just how structured the 3 x 3 x 3 cube is, and we could well have included this example under the heading **Structuring and restructuring the problem**, but it is particularly appropriate here.

There are 27 cells but these are not equivalent for the purpose of accommodating the constituent Soma polycubes. On the contrary, the cells are of 4 types and there are different numbers of each: V (8), E (12), F (6), C (1):




Of the 7 pieces, 5 can occupy at most 1 V cell, the remaining 2 (the 'L' and 'T'), 2. That gives a total of 9, in excess by 1, so not all can occupy their maximum number of V cells. However, if The 'T' did not, it would occupy 0, giving a total deficit of 1, therefore it must. In any solution, therefore, the top of the 'T' must lie along an edge.

But we can say more. Colour the cube like a 3-D checkerboard, so that V and F cells receive one colour, E and C cells the other. Here are the cells occupied by the 7 pieces in one particular solution:

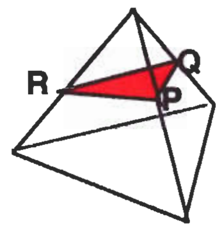
| 'T' | '2-D V' | '3-D V' | | the four others | | | |
|---|---|---|---|---|---|---|---|
| 3 | 2 | 1 | 2 | 2 | 2 | 2 | total: 14 FV cells |
| 1 | 1 | 3 | 2 | 2 | 2 | 2 | total: 13 EC cells |

The effect of translating a piece 1 cell is to reverse the order of these two numbers. In four cases, this makes no difference. We know that the order for the 'T' is fixed. To preserve the totals therefore, the FV sum and the EC sum for the two remaining pieces must stay the same. We cannot move either piece 1 cell without upsetting these. Therefore we may not do so. In other words, our table defines the FV, EC pair for every piece in every possible solution.

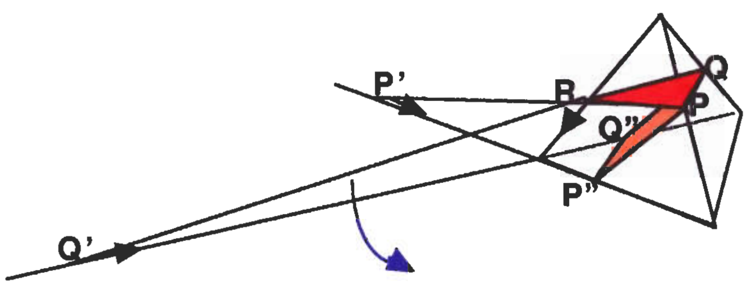
## 6.5.1 Baravalle Cuts

***A tetrahedron is sectioned with a plane. What polygons can result?***

A plane sections a convex polyhedron when it isolates at least one vertex from the rest. In a tetrahedron 3 edges meet in a vertex. Therefore the plane must cut at least 3 edges. Therefore the polygon must have at least 3 vertices:




Imagine that we have been lucky enough to hit on the points **P**, **Q** which will enable our plane to cut the greatest possible number of edges. We rotate the plane about **PQ** in search of the best orientation:




At the point where the plane passes through a second vertex of the tetrahedron, our polygon gains 2 vertices (**P"** and **Q"**) but loses 1 (**R**).

Only 2 polygons are therefore possible: the triangle and the quadrilateral.

***Every plane section of a sphere is a circle. What can you say about a circle of greatest size (a 'great' circle)?***

A great circle must lie in a symmetry plane of the sphere. Were it not so there would be 2 circles of maximum size for each such plane.

***What can you say about the shortest route between 2 points on the sphere?***

Because a great circle lies in a symmetry plane, the route must follow a great circle. Were it not so there would be 2 routes of minimum length.

## 10.2 Safe Queens

We know that a board of order n cannot accommodate (n+1) queens for there would then be 2 queens in 1 row.

# Structuring and Restructuring the Problem

A hard problem may become easy when looked at in a different way.

## 4.2 Penrose Rhombuses

***In how many ways can rhombuses of $36^0$ and $72^0$ be arranged around a vertex?***

The angles are of 4 sizes: 1/10, 2/10, 3/10, 4/10 of a whole angle. We can transform the combinatorial problem from a geometrical one to the following arithmetical one:

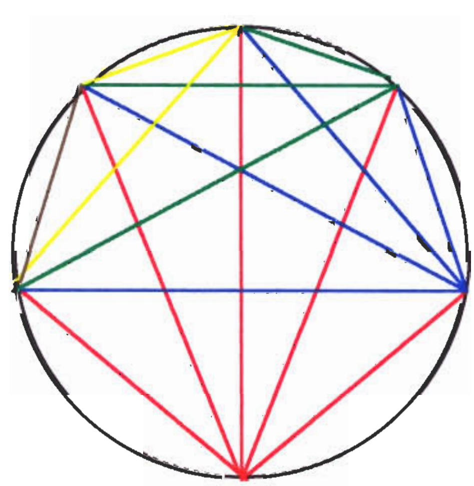
***How many ordered partitions of 10 into parts of size 1, 2, 3, 4 are there if cyclic permutations are counted as equivalent?***

(To produce an aperiodic ('Penrose') tiling we must apply certain rules about the edges and vertices we may place next to each other. These cut the number down to just 8. But we can apply our technique to that problem too.)

## 7.1.3 Handshakes

***How many handshakes are there when n people meet?***

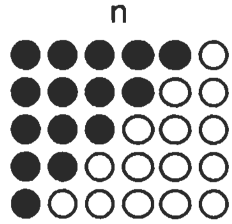
<u>First structure</u>
Instead of considering the group shaking hands all at once, consider the people in order: how many new handshakes are needed each time? We can draw a diagram and use a different colour for each person:



The sum is 1 + 2 + 3 + ... + (n - 1): a triangular number. We can arrange the 2 triangles to make a rectangle, then halve it again:

n

(n - 1)



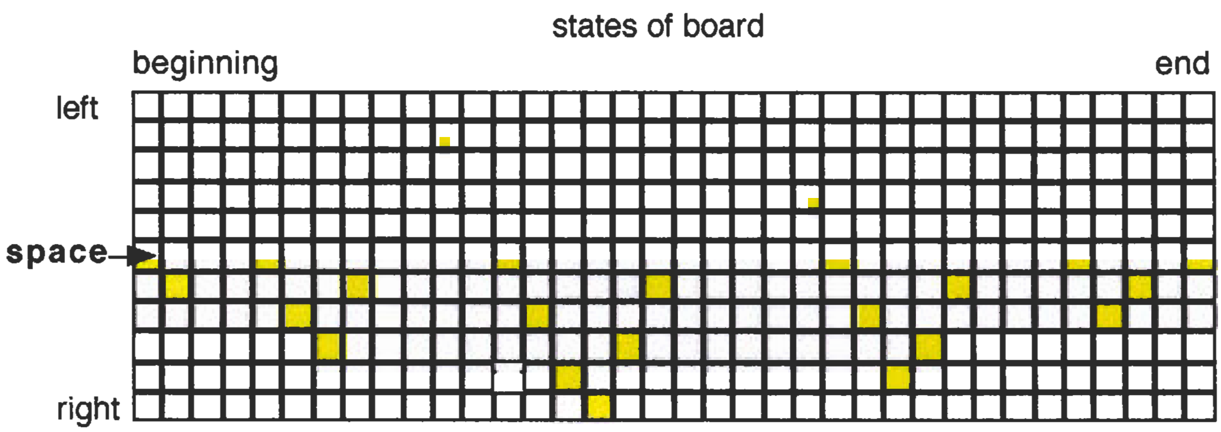
### Second structure

Each person must shake hands with (n - 1) others. We multiply by n for the total number of people, but divide by 2 because we've double-counted.

## 7.7 Leapfrog

### First structure

We can turn our attention from the movement of the frogs to the movement of the space:




This pattern helps us when we try to find an iterative formula for the number of moves. (See **Seeking a recursive structure**.)

### Second structure

To find a formula for the number of moves in terms of the number of frogs each side, we need look only at the initial state and the final state and ask what must have happened to produce the second from the first.

For the minimum number of moves, no frog must move backwards.
The frogs of one colour, $\mathbf{A_0}$, $\mathbf{B_0}$, ... , $\mathbf{F_0}$, end in the positions $\mathbf{A_1}$, $\mathbf{B_1}$, ... , $\mathbf{F_1}$:

$$\xrightarrow{\quad (f+1) \text{ shifts} \quad}$$
$$\mathbf{A_0\, B_0 \ldots \ldots \ldots F_0 \; \_ \; A_1\, B_1 \ldots \ldots \ldots F_1}$$
$$\longleftarrow f \text{ positions} \longrightarrow$$

Let the minimum number of moves be m,
the number of shifts in position, t,
the number of jumps, j,
the number of slides, s.

Out of interest, we give here on the right the corresponding expressions when there are not f frogs of each colour but a of one, b of the other.

m = j + s. (I)

Since 2 sets of f frogs each make (f+1) shifts in position (see above diagram):

t = 2f (f+1). (II) a(b+1) + b(a+1)

1 jump = 2 shifts; 1 slide = 1 shift. Therefore:

t = 2j + s. (III)

Each frog of one colour must jump all the frogs of the other colour. Therefore:

| | | |
|---|---|---|
| $j = f^2$. | (IV) | ab |

Substituting successively in (I), we have:

| | | |
|---|---|---|
| $m = j + s$, | (I) | |
| $m = f^2 + s$, | (from IV) | ab + s |
| $m = f^2 + (t - 2j)$, | (from III) | ab + (t-2j) |
| $m = f^2 + 2f(f+1) - 2f^2$, | (from II & IV) | ab + a(b+1) + b(a+1) - 2ab |
| $m = f(f+2)$. | | ab + a + b |

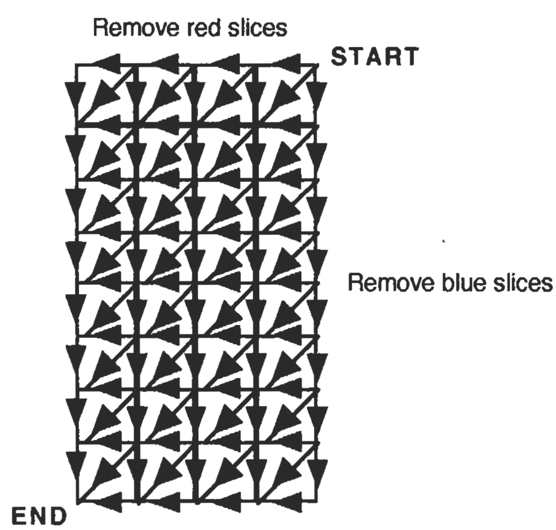
Notice that the number of board states, $b = m + 1 = (f+1)^2$.

### 10.1.3 The Pie Game

By showing each move as a vector we can record the progress of the game. Most importantly, we can spot which moves lead to positions which are safe for a player and which unsafe.

| <u>Move</u> | <u>Equivalent vector</u> |
|---|---|
| Take r red slices | Move r places west |
| Take b blue slices | Move b places south |
| Take k red, k blue slices | Move k places west, k south (i.e. south-west) |

This grid is the game 'space': the gridpoints represent all possible positions. The record of the game is a track from the northeast to the southwest corner. We can chart progress in real time by moving a counter, or mark each vertex or edge for a permanent record:

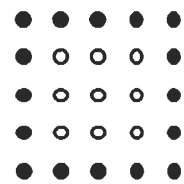


## 10.1.5 3-D Os & Xs

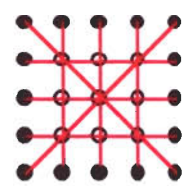
In the section **Seeking symmetries** we mention the task of counting how many lines-of-3 there are. The problem is to avoid double- and triple-counting. In an inspired piece of restructuring Leo Moser shows how it can be done.

Take the 2-D case as an example.

Embed the 3 x 3 grid in a 5 x 5:



Pair the outer points via lines-of-3 on the inner grid:

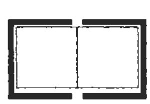


The number of lines-of-3 is then $(5^2 - 3^2)/2$.

The procedure generalises to higher dimensions. For our 3 x 3 x 3 case, the number is therefore $(5^3 - 3^3)/2 = 49$.

## 10.6 A Domino Rectangle

**1)** A domino has 2 cells and each has 3 borders:



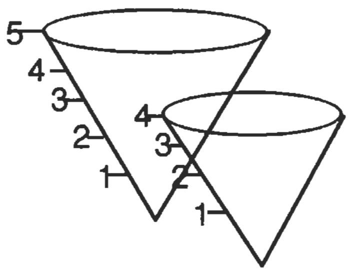
This means that each grid cell must likewise have 3 borders.
**2)**
**a)** Any cell in the body of the grid may belong to any of 4 adjacent cells.
**b)** Any cell at the edge of the grid may belong to any of 3 adjacent cells.
**c)** Any cell at the corner of the grid may belong to either of 2 adjacent cells.

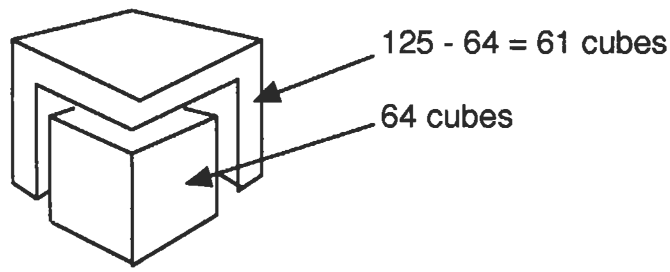
Interpreting **(2)** in the light of **(1)**:
**3)**
**a)** There are 4 ways of choosing the selection of borders to a cell in the body of the grid.
**b)** There are 3 ways of choosing the selection of borders to a cell at the edge of the grid.
**c)** There are 2 ways of choosing the selection of borders to a cell at the corner of the grid.

This suggests an approach to the puzzle: examine the border region, particularly the corners and see if you can place a piece with certainty. (It turns out that this is not possible. Nevertheless, inferences can be drawn which narrow the possibilities.)

## 11.3 The Magic Cone

The surprise is almost as great but the effect easier to appreciate if we invoke the principle of similarity and transform the solid from a cone to a cube:



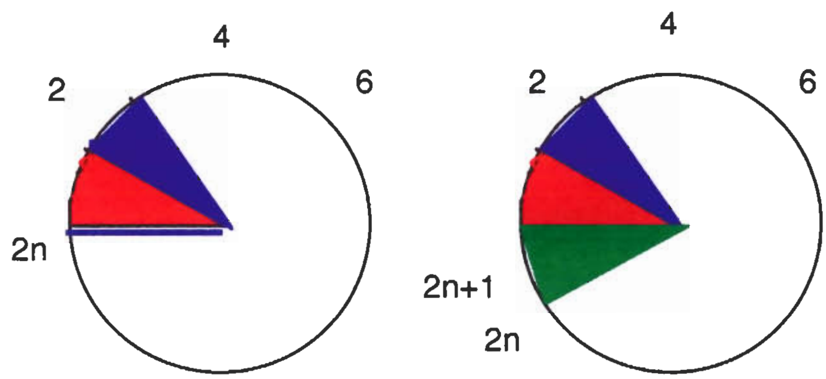



# Generalising and Specialising

We can examine *special* cases, but are they truly representative of the *general* case? They fulfil *necessary* conditions, but are these also *sufficient* ?

## 6.2 Map-colouring Solids

When an even number of faces meet in a vertex, 2 colours are needed; when an odd number meet, 3. These are *necessary* conditions:




But are they also *sufficient* ?

We must try individual cases: if we find we need more colours in any one case, we know that they are in general *not* sufficient.

# Seeking Symmetries

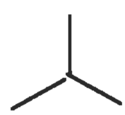
A whole section of the Circus (**3**) is devoted to spatial symmetry. We can widen the concept as when we say, "The expression bc + ca + ab is symmetrical in a, b, c", though all our examples happen to be geometric. But the point is that here we *use* symmetry to tell us something new about a situation.

Symmetry is a powerful weapon in the mathematician's armoury. Hence the many cases where we invoke its aid. But it's also a dangerous one. Station **3.9.2** is one case in point. Here we have to consider the phenomenon of symmetry-*breaking* before we can fully appreciate the role of symmetry. Another is **7.7**. The essence of the problem remains even when the numbers of red and blue frogs differ - note the entry under **Structuring and restructuring the problem** (second structure).

## 3.9.2 Motorway Networks

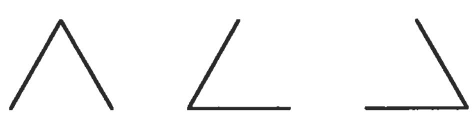
***Draw the network which connects n cities with the smallest total length of road.***

With cities at the vertices of an equilateral triangle, it turns out this is the solution:



A plausible rival is:

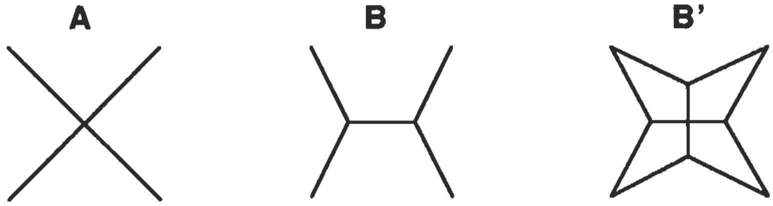
(Indeed, if you draw the model from the soapy water apex-last, this is the configuration you obtain - in energy terms a 'local' minimum.) Though this solution has less symmetry as it stands, it can be regarded as nature's choice of one of these 3:



- in other words, a selection of 2 out of the complete triangle's 3 sides.
If the principle of 'symmetry-breaking' is applied, therefore, the two solutions have the same symmetry and an equal claim on our attention:

The point is that, when it comes to 4 cities in a square, what is ostensibly the most symmetric tree is *not* the solution. **A** is the usual guess, **B** the correct solution, **B'** the same but allowing for symmetry-breaking: the full symmetry of the square is restored:

**A** **B** **B'**



## 5.2.1 The Soma Cube

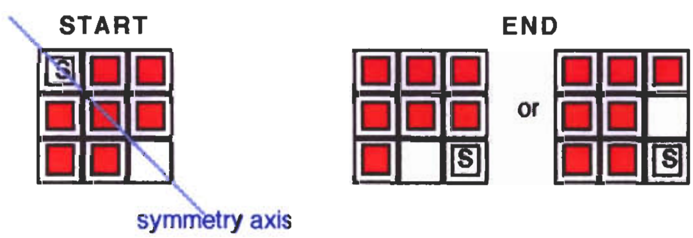
In exercise **5.2.1.2** the investigator builds a mirror image of the particular solution constructed in **5.2.1.1**. All the pieces have a plane of symmetry except 2 and those are mirror image forms ('enantiomorphs'). This means that every model made from the pieces - whether a cube or some other shape - has a mirror form, which in turn means that the number of solutions for the cube itself must be even.

## 6.5.1 Baravalle Cuts

In **Bounding the possibilities** we deduce an important consequence of the sphere's symmetry.

## 7.6.1 Sliding Sam

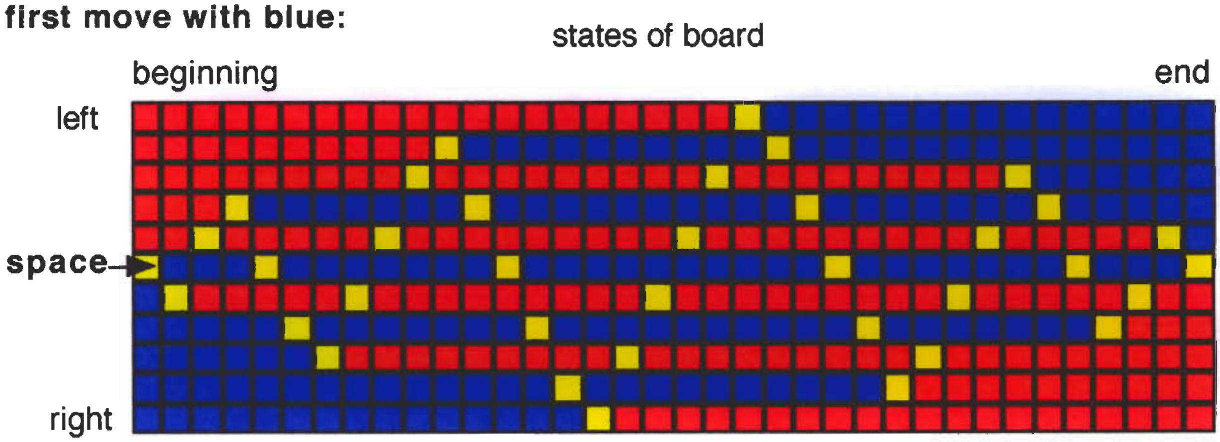
The board is symmetrical about the leading diagonal so it does not matter whether we move the space first left or first up, or whether it finally appears to the left of Sam or above him:



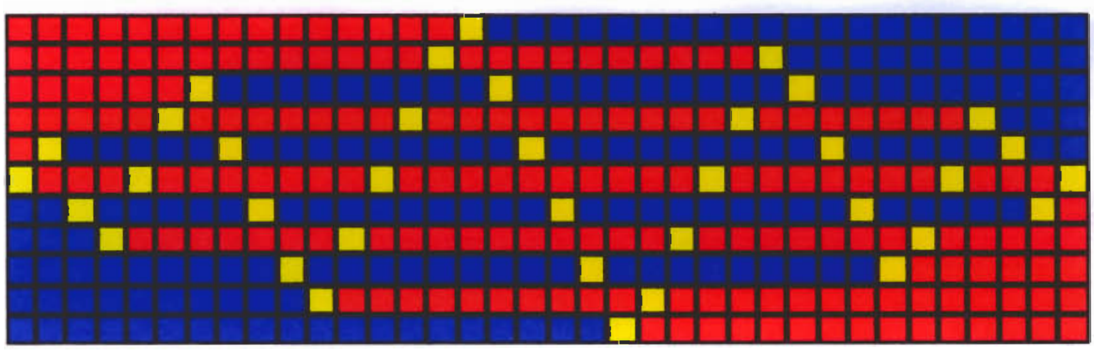
## 7.7 Leapfrog

The board is symmetrical about its perpendicular bisector. Therefore it doesn't matter whether we move a red frog first or a blue one. The effect on the 'space' track on our state diagram of changing that decision is to reflect it in the horizontal mid-line. The blue and red panels are reflected in the same line but swap colour. In either case, the whole diagram has point symmetry:

**first move with blue:**

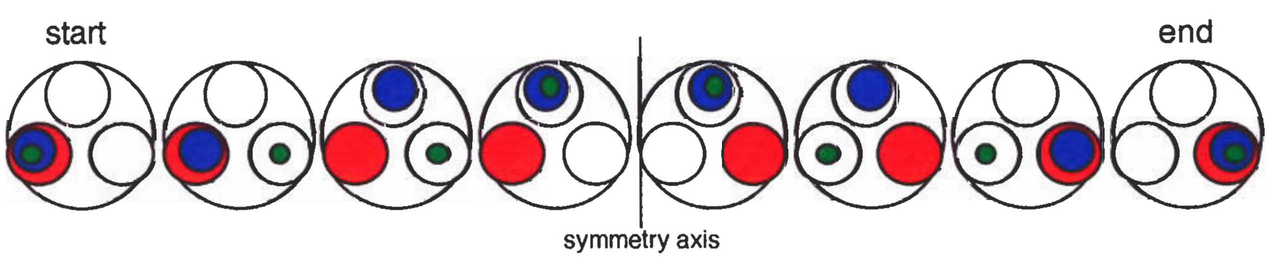



**first move with red:**



## 7.8 The Hanoi Pagoda

Here is a state diagram of the above kind for the Hanoi Pagoda:



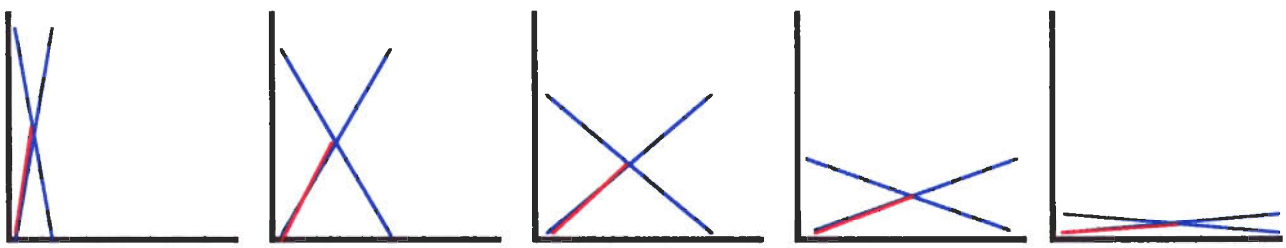

2 features emerge from the symmetry of the diagram:

**1.** The bottom cup moves anticlockwise round the board, the next clockwise, and so alternately.

**2.** Each cup makes twice as many moves as the one below.

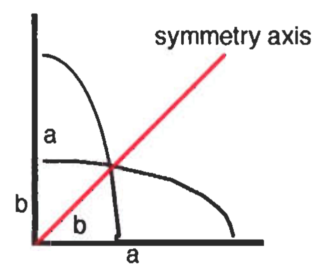
## 8.2 The Ladder

That the midpoint of the falling ladder (**8.2.1**) follows a circular arc comes as no surprise; that the same is true of the sliding ladder (**8.2.2**) often does. Most who perform the experiment have not yet learnt the necessary geometry to prove the result. However, intuition can be satisfied by thinking of the experiments as not two but one; the ladder, as not one but two, hinged at the centre. The motion is then as follows and the limb shown in red seen to be the radius of a circle:



By combining the two experiments we have obtained a figure of higher symmetry, and therefore one easier to interpret.

Symmetry also enters in another way. Say the students have not used the central hole but two which are equidistant from the centre. They will have obtained two equal 1/4-ellipses orientated as shown. Imagine a continuous transformation of the one into the other. Think of the axes of the first as 'a' and 'b' and of the corresponding axes of the second as 'b' and 'a'. In the intermediate position these values swap. At this point we have an ellipse with equal axes, i.e. a circle.



### 10.1.4 Domino Block

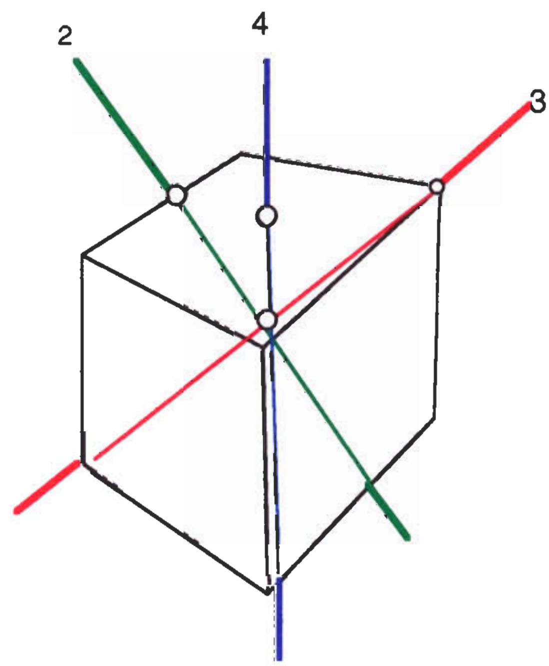
By matching the first player's positionings to preserve overall point symmetry (rotation symmetry of order 2), the second player guarantees victory.

### 10.1.5 3-D Os & Xs

There are only 4 kinds of site: corners, edge-centres, face-centres and the centre of the cube itself, each of which is repeated a certain number of times by the symmetry of the cube. This simplifies the analysis of possible strategies and the calculation of how many lines-of-3 there are.

o types of site

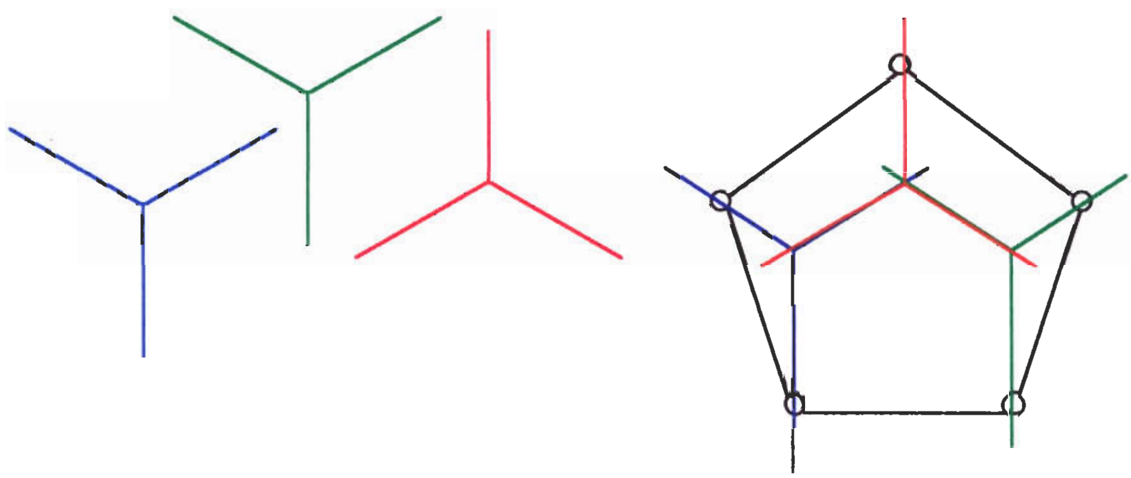
rotation
symmetry axes
and their orders



Time spent examining simple cases is always repaid when one comes to deal with the more complex.

### 3.9.2 Motorway Networks

We have found the solutions for 3 and 4 cities. We note that in each case 3 lines meet at $120^0$. Can we use this fact to construct the solution for 5 cities? Yes: we can draw 'trigons' on acetate and slide them around till they fit:



# Applying the Principle of Similarity

Similar figures are to be found thoughout the Circus, particularly in sections **2**, **3** and **11**, but the stations where similarity is employed as a problem-solving method are few.

### 5.1.2 The Riddle of the Sphinx

We cover this example under the heading **Splitting a problem into parts**, when we see that we can use the principle to give us outlines within the region of a large dissection puzzle, thus - literally - "splitting the problem into parts".

### 11.3 The Magic Cone

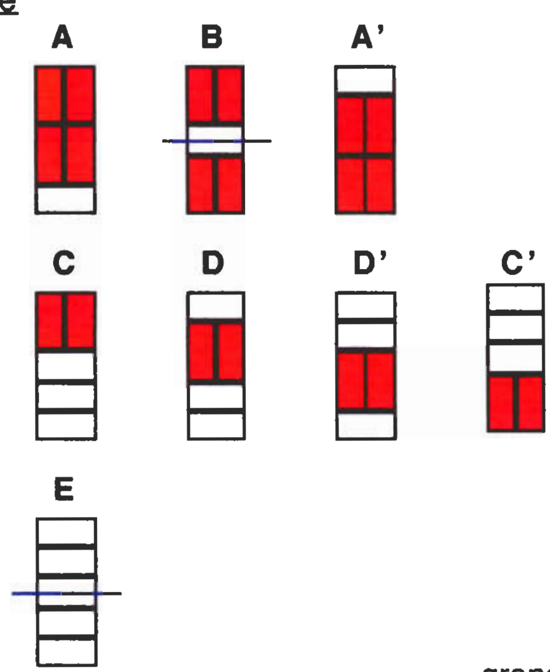
We discuss this station under the heading **Structuring and restructuring**, as a case where we can recast a problem in simpler terms.

# Exhausting Cases

In the method of exhaustion we simply ensure that we have counted all the cases. This is a poor option but it may be an essential first step in getting a handle on the problem. We must find a systematic way to make the count and the search for that may itself prove instructive.

## 7.4.3 Path-paving

Take the example of a path 5 units long. We can decide, for example, to sort the solutions in terms of how many sections of the path use slabs placed lengthways. Then, within each set so formed, we can move the lengthwise sections according to some scheme. We can also make an additional check: every solution not symmetrical about the horizontal mid-line must have a complement, viz. that pattern inverted. In the present case this gives us the pairs (**A**, **A'**), (**C**, **C'**), (**D**, **D'**):

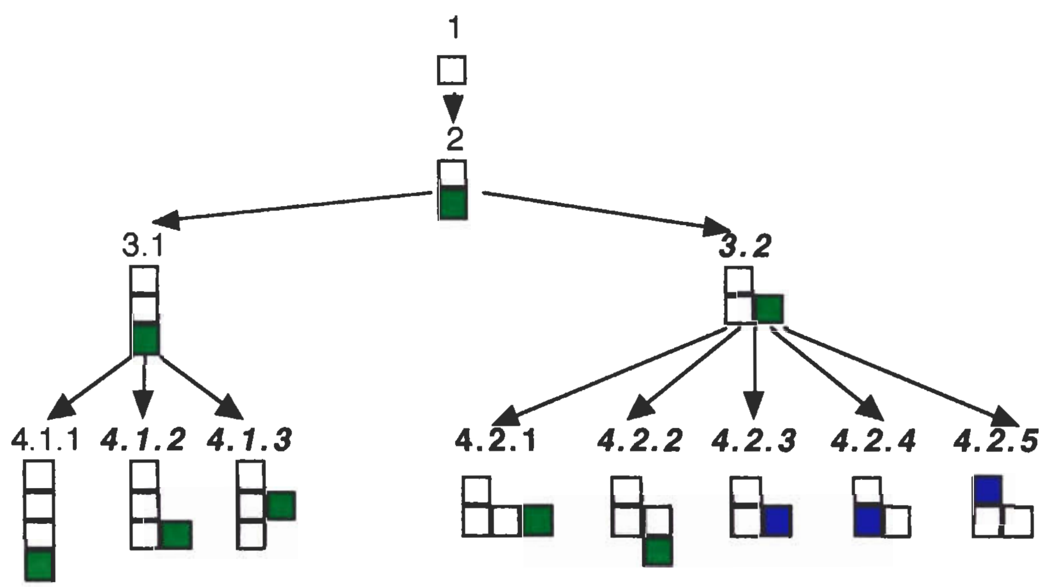



| pairs lengthwise | subtotal |
|---|---|
| 2 | 3 |
| 1 | 4 |
| 0 | 1 |
| grand total: | 8 |

### 5.2.1 The Soma Cube

This particular cube dissection consists of all n-cubes up to n = 4 which are not cub*oids*. Give students a set of interlocking cubes and set them to find all these. Their identity with the Soma Cube pieces comes as a pleasant surprise.

Here we grow a tree of forms, proceeding through the generations n = 1, n = 2, and so on. Each branch spreading from a node represents the addition of a cube in a different position. A number string identifies the family history of that particular form. We show the forms in plan. A green square shows a cube added in the same plane; a blue square, one added on the second level. In other words, a blue square indicates a tower of 2 cubes. The symbols for the n-cubes which are cuboids are shown in light type; the symbols for the non-cuboids, in **bold**. Notice that only the forms notated in ***bold italic*** are distinct: our scheme repeats one - a salutary example.

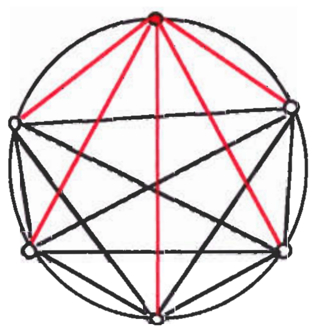


# Seeking a Recursive Structure

A recursive procedure derives the next case from the previous one. We choose 4 stations where such a structure helps to explain the growth of a sequence.

### 7.1.3 Handshakes

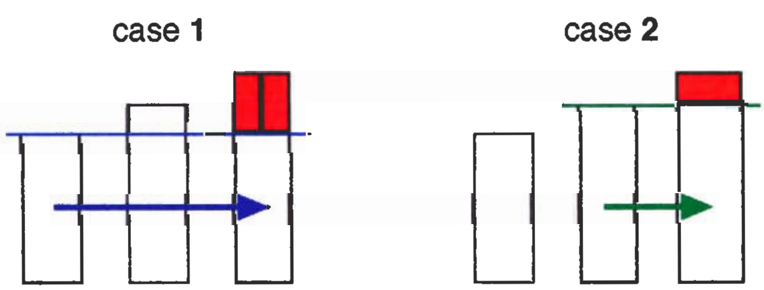
The $n^{th}$ person joining the group must shake hands with each of the (n - 1) people already present. Thus the number of handshakes $H_n = H_{n-1} + (n - 1)$:



### 7.4.3 Path-paving

Consider the $(k+2)^{th}$ path. We have two options:
**1.** We can take all the ways to pave the $k^{th}$ path and add a pair of slabs lengthways.
**2.** We can take all the ways to pave the $(k+1)^{th}$ path and add a single slab sideways.
Thus the number of ways in which we can pave the $(k+2)^{th}$ path is the sum of the totals for the previous two. So, if the number of ways to pave the $n^{th}$ path is $P_n$, we have:

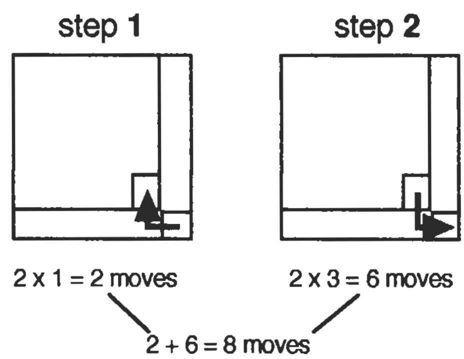
$$P_k = P_{k-1} + P_{k-2}.$$



## 7.6.1 Sliding Sam

**1.** Consider the $(k+1)^{th}$ board at the beginning. All we have to do is make 2 'space' moves to produce the $k^{th}$ board at the beginning.
**2.** Consider the $k^{th}$ board at the end. All we have to do is make 2 'Sam' moves to produce the $(k+1)^{th}$ board at the end. Each 'Sam' move entails 3 in total (2 moves by bean bags and 1 by Sam). The toal number of moves needed is therefore 2 x 3 = 6.

When we advance from the $k^{th}$ board to the $(k+1)^{th}$ we must perform both these steps. Therefore we must add 2 + 6 = 8 moves to the minimum total for the first to obtain the second.




## 7.7 Leapfrog

The diagram below shows board state charts for successive numbers of frogs. We see how the chart for n frogs is embedded in the chart for (n+1). Look at the part which is spliced in. Notice:
**1)** It is only here that the frogs added each side come into play.
**2)** The space executes a complete wave of the maximum amplitude.

Let the new number of frogs each side be f.
Let the number of shifts needed to execute a complete wave be w.
Since the amplitude = 2f + 1, we have:

$w = 2\,(2f+1).$ (I)

In this wave, all the moves are jumps except 2: a slide associated with the peak and a slide associated with the trough.
Let the number of jumps in the wave be p; the number of slides, q (=2).
Since a jump accounts for 2 shifts, a slide for 1, we have:

$w = 2p + 1q = 2p + 2 = 2\,(p+1).$ (II)

And, combining (I) & (II), we have:

$p = 2f.$

Thus the total number of moves in the wave, $v = p + q = 2f + 2$.

So the number of extra board states spliced in = $v - 1 = 2f + 1$.

Let the number of states for the f-frog board be $b_f$. We now have our recurrence relation:

$b_f = b_{f-1} + (2f+1)$. (III)

Writing: $b_f - b_{f-1} = [2f+1][1] = [(f+1) + (f)][(f+1) - (f)]$, we see that:

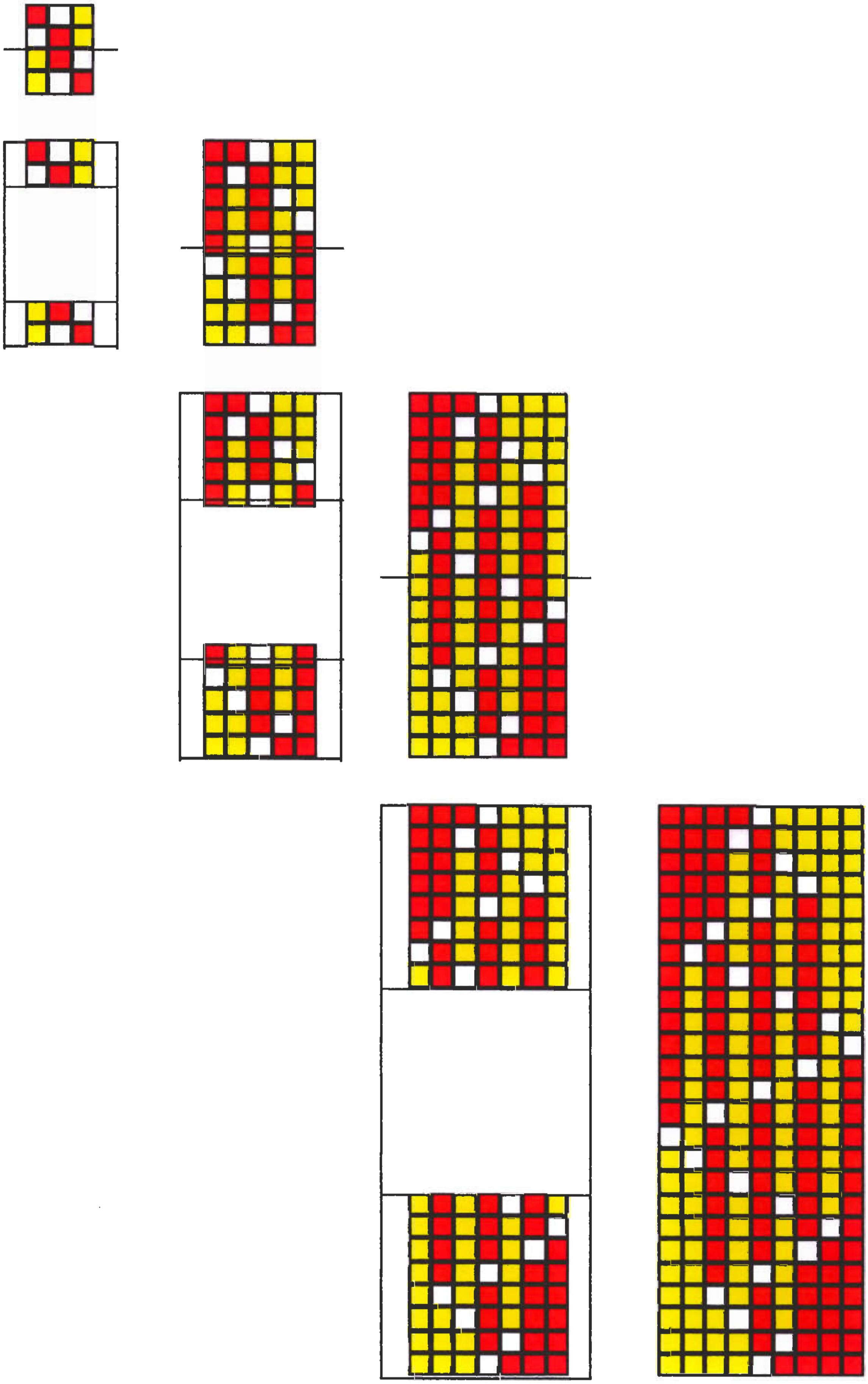
$b_f - b_{f-1} = (f+1)^2 - f^2$,

whence the explicit formula derived under the heading **Structuring and restructuring the problem**:

$b = (f+1)^2$.

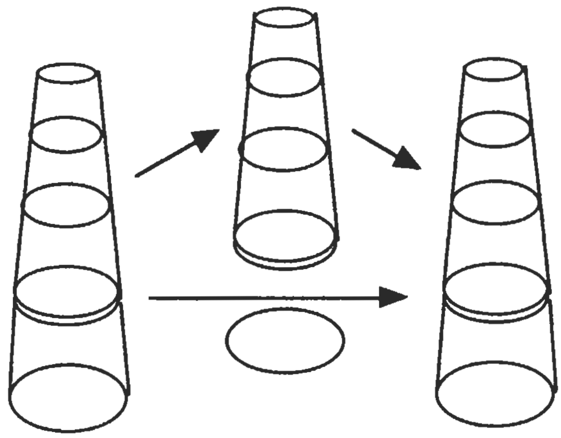
Note also that we can write (III):

$b_f = (\sqrt{b_{f-1}} + 1)^2$.



## 7.8 The Hanoi Pagoda

When we move a tower of height (k+1) from **A** to **C**, we move a tower of height k first to **B**, then to **C**, with an additional move to transfer the new base direct to **C**. Thus, to obtain the number of moves for a tower of height (k+1), we must double the number of moves needed for a tower of height k and add 1.

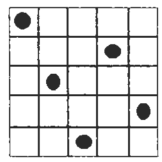


# Devising a Notation

When faced with a problem, typically we first enact the situation, then represent the elements with ikons and finally substitute symbols for the ikons, thus fully abstracting the problem from its particular setting. Spurious features are discarded and the underlying pattern - the mathematics - is revealed. However, it may not be clear at the outset what sort of notation will help. We choose two examples where a Cartesian plot proves a good choice.

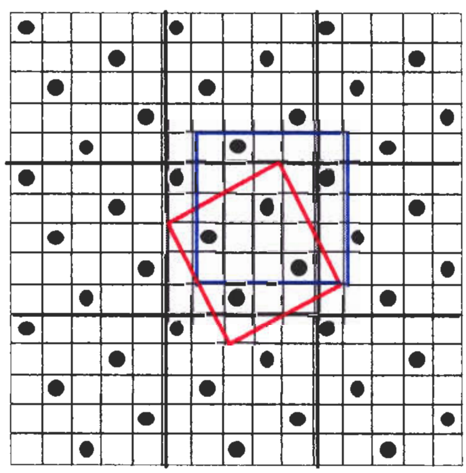
## 10.2 Safe Queens

Take the 5x5 case. We begin by working with cubes or counters on a checkerboard. We find a solution:



We draw it on an acetate square and find equivalent solutions by rotating the square through 1/4-turns or flipping it about symmetry axes.

Next we consider translations. Can we move the whole solution by the vector $\binom{k}{l}$ and use the 'wrap-around' property to obtain a valid solution?
We embed our solution in a regular tiling of identical squares:

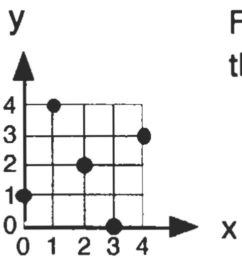


We see a simple pattern defined by the unit cell picked out in red. Notice how we can generate the whole tiling by means of knights' moves in two perpendicular directions, shown by the vectors $\binom{2}{1}$ and $\binom{-1}{2}$.

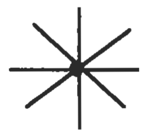
Notice also that we can translate our 5x5 frame to the 'blue' position to give a different solution, one which has a queen at the centre and rotation symmetry of order 4. Indeed we see that *wherever* we move the blue frame we obtain a valid solution.

Can we do the same with a 6x6 solution? We find we cannot. What's the difference?

Go back to the 5x5 case. Set the queens at points on a Cartesian grid and write equations for all 4 lines through each:



For $(x_k, y_k)$ we have these lines:



$y = x + (y_k - x_k)$

$y = y_k$

$y = -x + (y_k + x_k)$

$x = x_k$

$A = x_k$
$B = y_k$
$C = (y_k - x_k)$
$D = (y_k + x_k)$

This gives 5 x 4 = 20 equations and the 'safe' condition is that all are distinct. Looking at the equations, we realise we can extract the 4 numbers A, B, C and D from each set of 4. Numbering our points 1 to 5 from left to right, we can now consider 4 sets of 5 numbers separately: $A_k$ contains all 5 'A' numbers, $B_k$ all 5 'B' numbers, and so on. Tabulating these, we require that, in each column, the 5 be distinct:

| k | $A_k$ | $B_k$ | $C_k$ | $D_k$ |
|---|---|---|---|---|
| 1 | 0 | 1 | 1 | 1 |
| 2 | 1 | 4 | 3 | 5 |
| 3 | 2 | 2 | 0 | 4 |
| 4 | 3 | 0 | -3 | 3 |
| 5 | 4 | 3 | -1 | 7 |
| Totals: | 10 | 10 | 0 | 20 |

(We can check our column totals. Those for $A_k$ and $B_k$ must be the 4th triangular number; $C_k$ their difference, viz. 0; $D_k$ their sum, viz. 2 x same.)

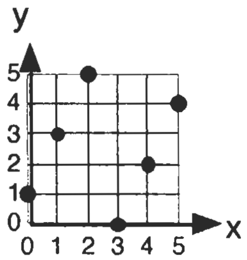
How does the 'wrap-around' property translate (pun intended) into arithmetic?

Take our solution and apply the vector $\binom{1}{0}$ . The y-coordinates are unchanged. The x-coordinates increase by 1 *modulo 5* :

| k | $A_k$ | $B_k$ | $C_k$ | $D_k$ |
|---|---|---|---|---|
| 1 | 1 | 1 | 0 | 2 |
| 2 | 2 | 4 | 2 | 6 |
| 3 | 3 | 2 | -1 | 5 |
| 4 | 4 | 0 | -4 | 4 |
| 5 | 0 | 3 | 3 | 3 |
| Totals: | 10 | 10 | 0 | 20 |

(Notice that the column totals remain as they were.)

In each case the entries remain distinct. Now try the same with a 6x6 solution. This time we must work modulo 6 in the '$A_k$' column (or in the '$A_k$' and '$B_k$' columns in the general case.)




Before:

| k | $A_k$ | $B_k$ | $C_k$ | $D_k$ |
|---|---|---|---|---|
| 1 | 0 | 1 | 1 | 1 |
| 2 | 1 | 3 | 2 | 4 |
| 3 | 2 | 5 | 3 | 7 |
| 4 | 3 | 0 | -3 | 3 |
| 5 | 4 | 2 | -2 | 6 |
| 6 | 5 | 4 | -1 | 9 |

After:

| k | $A_k$ | $B_k$ | $C_k$ | $D_k$ |
|---|---|---|---|---|
| 1 | 1 | 1 | 0 | 2 |
| 2 | 2 | 3 | 1 | 5 |
| 3 | 3 | 5 | 2 | 8 |
| 4 | 4 | 0 | -4 | 4 |
| 5 | 5 | 2 | -3 | 7 |
| 6 | 0 | 4 | 4 | 4 |

Failure!

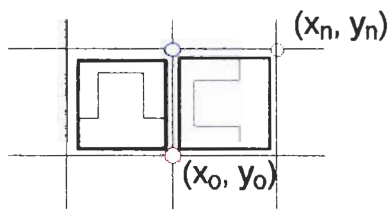
We have translated the problem from geometry to algebra to arithmetic. As we move from the 6x6 board to the 7x7 and the 8x8 we no longer have to draw diagrams to test properties but can work with tables of numbers. The properties of each board of order n are seen to be the properties of numbers modulo n themselves.

### 10.3.1 Grandpa's Armchair

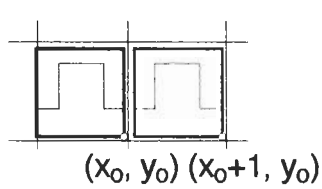
At the Circus station a Cartesian grid, a dry-wipe pen and a model chair - a square tile bearing an arrow - are provided. If the student moves the tile around the board, following the given rule, and records the orientation of the arrow in each position, a pattern emerges: 'left-right' and 'up-down' arrows are found to correspond to the black and white squares on a checkerboard.

This *suggests* the requirement of the problem cannot be fulfilled. But can we prove this?

Locate the front left corner at $(x_o, y_o)$ and rotate the chair (say) anticlockwise about (say) the back left corner. Note the new coordinates of the front left corner $(x_n, y_n)$:




Make a number of such trials. In every case the sums $(x_o + y_o)$ and $(x_n + y_n)$ differ by 0 or 2. You then realise the reason for this: every time you change the x-coordinate by 1, you change the y-coordinate by 1 - whether up or down is immaterial. The point is that the parity of the sum is unchanged. But we require our chair corner to move from $(x_o, y_o)$ to $(x_o+1, y_o)$:




The sum changes by 1. 1 is an odd number and changes the parity.

We can confirm indeed that parity is preserved whatever gridpoint we choose as our centre of rotation. With this as our origin and (a,b) our chosen point we apply the matrices $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ for an anticlockwise 1/4-turn, $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ for a clockwise 1/4-turn.

Consider the effect of the first:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -b \\ a \end{pmatrix}$$

$$(a,b) \longrightarrow (-b,a)$$

$$(a+b) \longrightarrow (a-b)$$

$$(a+b) - (a-b) = 2b.$$

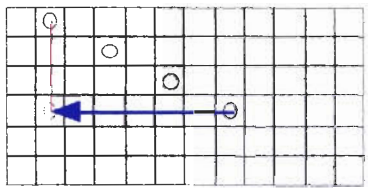
A 1/4-turn the other way gives 2a, an even number in both cases.

# Backtracking

Like exhaustion, backtracking is a method of last resort - unless we can devise an efficient algorithm a computer can apply. But, when a pattern eludes us, it may be our only option. The best way to explain the term is to show the process in use.

## 10.2 Safe Queens

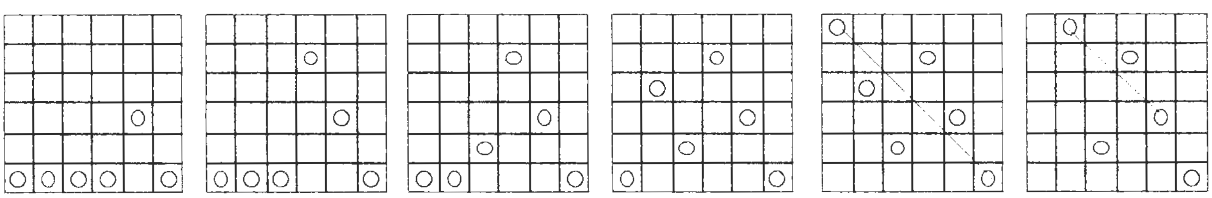
Though the puzzle is relatively easy to solve on boards of prime size, boards of composite order present increasing difficulty. if we apply a simple strategy, for example perform a knight's move, a composite board brings us back into phase with ourselves when we try to exploit the 'wrap-around' property:



On the order 6 board, shown here, it's not difficult to make an adjustment which will solve the puzzle. But this is not the case with the full-size board (order 8). Here we may resort to backtracking. We shall use the order 6 case to show the method in use:

**1.** Line up the 6 queens along the bottom row.
**2.** Advance the 2nd from right to a position where it doesn't threaten the 1st.
**3.** Advance the 3rd from right to a position where it doesn't threaten the 1st and 2nd.
.
.
.

Proceed in this way till no such position proves possible.
In our case, no safe position exists for the 6th.
Can we succeed by adjusting the position of the 5th?



No. In fact we find we must 'backtrack' right to the start, advance the 1st queen 1 square and begin again.