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Descartes' Analogy

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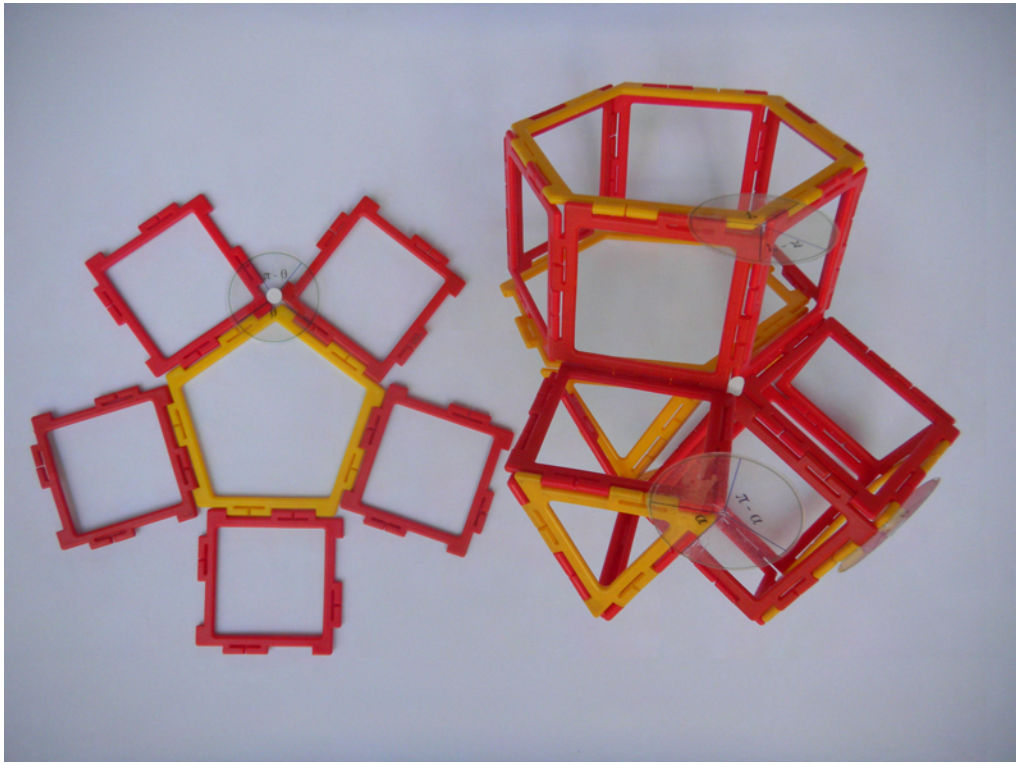
The total angle defect for a poly*gon* is 360°;
that for a poly*hedron*, 720°.

Use the notes and models to follow the analogy George Pólya thinks René Descartes may have drawn.

The geomlab file pursues the analogy to obtain Descartes' formula 2V - P + 2F = 4, where V is the number of vertices, F the number of faces and P the number of interior angles. Swap 2E for P and you have the Euler formula. Did Descartes see it? Opinion is divided.

Paraboloid
Hyperbola
Hyperboloid
Hyperbolic paraboloid
Line pair
Sine curve
Tractrix
Exponential curve
Catenary
Catenoid
Helix
Helicoid
Plane
Polygon
Polyhedron
Tiling
Archimedean spiral
Equiangular spiral
Loxodrome
Line family
Circle
Cylinder
Cone
Sphere
Ellipse
Line
Parabola

George Pólya's reconstruction of an analogy Descartes may have drawn



On the left:

On the right:

Build *rectangles* on the *sides* of the poly*gon*.
The *circle* sectors between them fit together to complete a *circle*.

A *plane* angle can be stated in *radians* by quoting the *length* of *arc* of the unit *circle* subtending it.
A whole *plane* angle is 2π radians.

Build *prisms* on the *faces* of the poly*hedron*.
The *sphere* sectors between them fit together to complete a *sphere*.

A *spherical* angle can be stated in *ste*radians by quoting the *area* of *surface* of the unit *sphere* subtending it.
A whole *solid* angle is 4π *ste*radians.

The vertex model shows that the spherical triangle representing the angle defect, the amount by which a vertex falls short of a ‘flat’ angle, has angles of $A = \pi - \alpha$, $B = \pi - \beta$, $C = \pi - \gamma$. (*A,B,C* are the dihedral angles between the faces, α, β, γ the face angles.)

The total angle defect is a whole angle, 4π *ste*radians.

We shall use the sphere model to determine the area of our spherical triangle, and hence the angle defect it represents.

The white dots mark the vertices of the spherical triangle whose area Δ we require. Completing the great circle for each side, we find we've produced a congruent triangle in the antipodal position, shown by yellow dots.

Corresponding vertices mark the ends of lunes. We have a blue lune and a congruent, vertically opposite, blank lune. Likewise for red and green. Take the angle *A* in our triangle. The area of the lune to which it belongs is that fraction of 2π x the surface area of the sphere, $\frac{A}{2\pi}\times 4\pi = 2A$. Adding the corresponding blank lune, the area is double that, $4A$. The lunes of *A*, *B* and *C* and their doubles together cover the whole sphere, and the triangle and its double each 3 times. Thus the total exceeds 4π by the area of 4 triangles:

$4A + 4B + 4C = 4\pi + 4\Delta$, whence $\Delta = A + B + C - \pi$.

Since $A = \pi - \alpha$, $B = \pi - \beta$, $C = \pi - \gamma$, the area of our spherical triangle is $(\pi-\alpha)+(\pi-\beta)+(\pi-\gamma)-\pi = 2\pi-(\alpha+\beta+\gamma)$. This gives us a representation of the angle defect in terms of plane angles, (the angles of the faces meeting in the vertex), and a total angle defect of 4π radians.

Returning to our analogy,

On the left:

The vertex model shows that the angle defect, the amount by which a vertex falls short of a *straight* angle, is $\pi - \theta$ radians.

The total angle defect is 2π radians.

On the right:

We now know that the angle defect, the amount by which a vertex falls short of a *flat* angle, is $2\pi-(\alpha+\beta+\gamma)$ radians.

The total angle defect is 4π radians.

How Descartes got close to the Euler polyhedral formula

Having established the total angle deficit expressed in plane angles for the polyhedron, Descartes is able (conjecturally) to reason as follows.

Let there be V vertices $v_1,...,v_i,...,v_V$, F faces $f_1,...,f_i,...,f_F$, P interior angles.
Let the face f_i have n_i interior angles.

At each vertex v_i
the angle deficit $d_i = 2\pi -$ (the sum of the interior angles meeting there).

So the total angle deficit $D = \sum_1^V d_i = 2V\pi - S$,

where S is the sum of all the interior angles meeting in all V vertices.
But S is also the sum of all the interior angles of all F faces

$= \sum_1^F (n_i - 2)\pi = \sum_1^F n_i\pi - 2F\pi = P\pi - 2F\pi$.

Thus $D = 4\pi = 2V\pi - S = 2V\pi - (P\pi - 2F\pi)$.
Whence $4 = 2V - P + 2F$.

What interests commentators is that he is at this point just two steps from the Euler formula $V - E + F = 2$, where E is the number of edges. It is necessary to realise first that P is also the total number of sides of all the faces, second that this is twice the number of edges of the polyhedron. But the first observation is of little interest on its own and the second requires one to identify ‘edge’ as a descriptor in the first place. Euler did so (his term was ‘acies’) in the course of drawing a different analogy.

Descartes had been interested in a general, metric property of all polyhedra. Euler wished to find a way to classify different ones. What descriptors are key? The affinity between any two convex 5-sided polygons is clear; on the other hand a square-based pyramid and a triangular prism both have 5 faces but seem to be qualitatively different. Euler realised that, whereas in two dimensions, it is sufficient to consider only [vertices] (0 dimensions) and sides (1 dimension), in three dimensions “… three kinds of bounds are to be considered …” (*Elementa doctrinae solidorum*, 1758): vertices (0 dimensions), edges (1 dimension), faces (2 dimensions). (We put the second ‘vertices’ in brackets because Euler’s Latin would translate as ‘solid angles’ but the context makes his meaning clear: in the two-dimensional case, a point where 2 sides meet; in the three-dimensional case, a point where more than 2 edges meet.)

We do not know how Euler arrived at his polyhedral formula. But he now had the 3 relevant parameters and could make the 3 counts on a whole range of tabulated examples. Pólya conjectures that Euler may have made the observation that, when F and V match, so does E, leading to the hypothesis that E is a function of F and V. Pólya is careful to reconstruct the heuristics in a way that excludes hindsight. When we present such a table to contemporary children, we cannot avoid steering them towards ‘the answer’. The activity is fun, and indeed satisfying, for the children, but it is not history.