THE TIMES TABLES FOR TEACHERS

Graded activities from KS1 to KS5
on the structure and properties of
the multiplication square

A handbook for teachers, and instructors
running preservice and inservice courses

Collected by
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Introduction

When we look about us, we do not see numbers, but we may see a set of square tiles or a staircase. The teacher can always use geometry in the service of arithmetic. We arrange square tiles in an *a* by *b* rectangle and count them to find the product. The 'times tables' (plural) make up the operation table (singular) for multiplication of the natural numbers, usually called *the multiplication square*. This tabulation is already a visual aid. We look for the product *a* times *b* on the multiplication square and we find it labelling the corner of our rectangle of tiles. The idea of this book is to start from such simple beginnings and see how far we get.

In 1983 I was lucky enough to attend a course run by the late Edith Biggs, devoted to enriching the classroom experiences of upper primary children, including their experience of the 'times tables'. Inspired by this, I went through old copies of the journals and one-off publications of The Mathematical Association and The Association of Teachers of Mathematics. I found, for example, an account of a weekend brain-storming conference by a group of teachers devoted entirely to the multiplication square. What is its structure? What properties arise from this? As a long-time member of both assocations, I continue to trawl through the journals, to which I occasionally contribute our experiences with the touring maths lab I launched in 1989, and continue to run helped by colleagues and students, The Magic Mathworks Travelling Circus (www.magicmathworks.org). Though the lab tours internationally, my reading of the teaching literature is still largely confined to these Isles. I therefore invite readers in other countries to bring to my attention for future editions ideas not found here (pstephenson1@me.com).

Intended readership

British teachers elect to train either for primary or for secondary work. It is valuable for the primary teacher to learn how her work might be developed later in the child's schooling, and for the secondary teacher to realise what mental constructs the child brings to his secondary school. The emphasis in this book is not on a command of products as number facts but on the structure of the table which summarises them. Familiarity with the algebraic laws which lie behind this structure gives the child alternative ways to access these facts, indeed alternative ways to do and check any calculation. Those running courses for teachers have the opportunity to convey the freedom this understanding provides and, though the book is addressed to teachers directly, it is among their tutors that it is likely to find its widest readership.

Pedagogy

There is a judgement the teacher must make, which for simplicity I'll personalise: whether to follow Gattegno or Dienes, twentieth century mathematics educators with different attitudes to classroom aids in the service of arithmetic and algebra.

The case for Dienes

We learn through all our senses. Initially the brain receives an unsorted input of sensory data, which is only separated according to specific sense modalities at a later stage. If we wish to identify birds visually, we are more likely to retain the images if, simultaneously, we hear recordings of their songs and calls. (If, however, the different senses receive conflicting inputs, the effect, as you expect, is counterproductive.)

Before the era of brain research Z. P. Dienes suggested that we come to understand a concept by meeting it in a range of perceptually different embodiments then abstracting the features which are common to them. (The topic suites exhibited in The Magic Mathworks Travelling Circus follow this scheme.) The Dienesian programme requires the classroom to be a sensorily rich environment, in which the child moves between number base blocks, the mathematical balance, and other pieces of apparatus.

<u>The case for Gattegno</u>

Caleb Gattegno championed the use of wooden rods, colour-coded according to their integral lengths, devised by the Belgian, Georges Cuisenaire. Gattegno showed how all the arithmetical operations could be performed by their manipulation. He called his scheme with deliberate provocation ‘qualitative arithmetic’. Their modern champion, Ian Benson, uses the slogan ‘algebra before arithmetic’. Take for example the red and (light) green rods. Though the child may learn that ‘red = 2’, ‘green = 3’, he can simply write ‘R’ for the red, ‘G’ for the green, and thereafter ignore the particular value they represent, building equal trains and writing, for example, ‘RRRRRR’ = ‘GGGG’ = ‘RRRGG’, and so on, fully symbolic statements. As the child moves the blocks around he is carrying out algebraic operations in an enactive way. With the colour and feel of the rods, his experience is indeed multisensory, but confined to one particular piece of apparatus.

<u>Comment</u>

In this book I’ve attempted to do for the muliplication square what Gattegno did for Cuisenaire rods (except with a Dienesian tendency to use a variety of equipment in addition). This is not because I think this particular tabulation is indispensable but because it is already there on classroom walls and teachers may be interested to see how much they can do with it.

Conventions

In keeping with standard practice, the child is ‘he’; the teacher, ‘she’. ‘We’ will mean the teacher and her class.

On our multiplication squares, the axes run left to right along the top and top to bottom down the left side.

When we consider the algebraic laws governing the binary operation ‘multiplication’, it will be important to distinguish *multiplicand* and *multiplier*. We shall write the former first and the latter second. Thus in ‘3 x 2’, ‘3’ is the multiplicand – the number being multiplied; ‘2’, the multiplier – the number doing the multiplying.

In parallel with this, we shall find the multiplicand along the top of the square, the multiplier down the side.

The same will apply to addition and the addition square. (I wonder why the operation table for *addition* is not seen on classroom walls? Children could make them in the course of mastering their ‘number bonds’.)

The development

As we advance through the key stages we shall tackle about fifty tasks. In accord with Bruner's *spiral curriculum*, we shall revisit topics, but treat them in more depth each time.
This progression follows the sort of numbers we shall deal with, starting with the *natural numbers* and finishing with all the *reals.*

On the left is the pedagogy; inset in italics are the tasks, sometimes prompted by a question. I have on occasion suggested how an investigation might be presented for dramatic effect. *But hese are notes for you the teacher. Having selected a task, you will know best how to present it to your children, and how many times an exercise should be repeated.* I have been very free with the allocation to Key Stages. You know your children and can move up and down my list till you find a task you think they'll enjoy, whatever Key Stage label I've given it.

In tasks **1** to **11** we treat a ‘times table’ (singular).
We model the *binary operation* ‘multiply’ in several different ways.

Lower KS1

We begin with the *equal addition* model. In more advanced terms, a set of multiples is seen as an *arithmetic progression* in which the first term and common difference are equal.

1. *We look at a real staircase and talk about it. What is special about the way the steps go? If the school is on a single level, you will have to project a picture.*

2. *We take paper sheets with numbers written on them and place them on the correct steps. If the school is on a single level, this must be a homework task shared with the parents.*

As long advocated by Ruth Merttens, and supported by recent research, this sharing is essential to the early development of the child.

3. *With Multilink cubes we build staircases whose risers have unit height, and ...*

in accordance with Dienes’ principle of perceptual variability

... do the same horizontally with Cuisenaire rods.

Upper KS1

We model the product of whole numbers as a *Cartesian product*. The name *Cartesian product* comes from the *x* and *y* Cartesian axes. In a Cartesian product each member of one set $\{x_1, x_2, x_3, \ldots, x_m\}$ is paired with each member of another $\{y_1, y_2, y_3, \ldots, y_n\}$ to form the set of all *mn* pairs $\{(x_1, y_1), (x_2, y_1), (x_3, y_1), \ldots\}$.

I owe the following task to Julie Anghileri.

4. *With 3 types of top and 4 colours of shorts or skirts, how many different outfits can we make? Working in groups, the children make their own collection of clothes from coloured and patterned papers – and may wish to extend the task to more shorts/skirts. When they arrange their models on a big sheet of paper and wish to check that they’ve got all the possibilities, it is important not to steer them towards a tabular arrangement. However, the children should discuss their displays and you may call upon a group to offer reasons for their choice.*

5. *We build staircases whose risers have heights other than 1 unit, and again repeat the task horizontally with Cuisenaire rods.*

Here we’ve built 4 staircases, labelled the steps and set them alongside each other, anticipating **34**:

We explore analogous situations involving *rate.* An *analogy* takes the form:

a in domain *A* is like *b* in domain *B* (in respect of property *X*.)

In the view of some neuroscientists, the drawing of *analogies* is the brain's main way of incorporating new information into an existing schema. *X* is not a given; the brain has to supply it.

In a *rate* the two quantities compared are of different kinds.

6. *The 20 of us dance in 5 rings of 4. Over in the field are 5 cows. How many legs have they got? There are 5 cars. How many roadwheels have they got?*

Lower KS2

We remodel the staircase as a right triangle to realise multiplication as a *scaling* operation. Implicit is the idea of *ratio*. In a *ratio* the two quantities compared are of the same kind. Accordingly a ratio has dimension zero. It is a ratio*nal* number.

7. *On the table are right triangles of different shapes and sizes. (They are in fact restricted to those which have height-to-base ratios of 1, 2, 3, 4). Our job is to sort them into sets of similar triangles. We do this by overlaying one on another.*

The *commutative* law states that the products *ab* and *ba* are identical. It provides an opportunity to do some 'people maths'.

8. *Five children stand in line in front of the class. Ashok holds a card with '3' on it; Jill holds card 'X'; Tracy, '4'; Ben, '='; Mira, '12'. The class are asked 'Who can swap places?'*

9. *With Cuisenaire rods we make rectangles with 3 rows of 4, and 4 rows of 3. If we draw round the rectangles and fill them with Cuisenaire 1s, what will we find?*

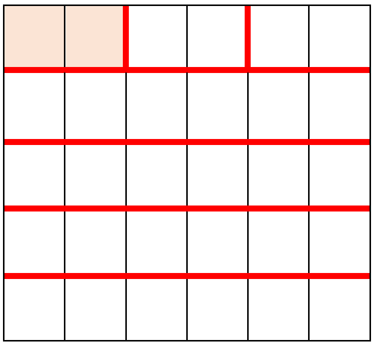
10. *On a mathematical balance we put a single hanger on peg '6' on the left. We balance the beam by putting 2 hangers on peg '3' on the right. If we're not allowed just to put a single hanger on peg '6' on the right to match the one on the left, how else can we balance the beam?*

For further activities with the mathematical balance go to www.magicmathworks.org , then 'Virtual Circus', then 'Multiplication', then 'Seesaw'.

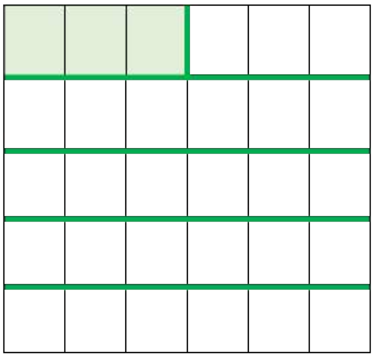
11. *How can we do* '$2 \times 3 \times 5$' *? We ask the children to draw blank 5 x 6 rectangles on squared paper and divide them into 2s, 3s and 5s in as many ways as they can. They can use shading or colours.*

Here is one way:

2 x 3 x 5



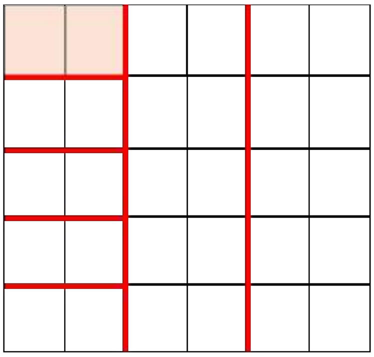
3 x 2 x 5



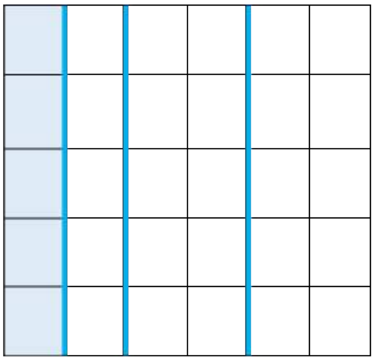
With Cuisenaire rods:

Swap 3 2s for 2 3s.

2 x 5 x 3

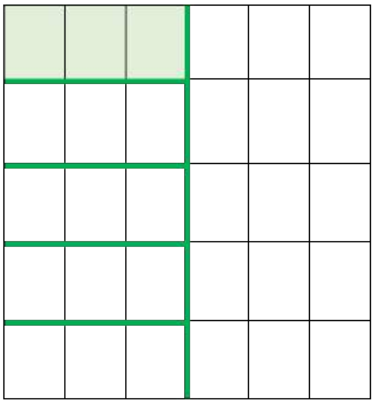


5 x 2 x 3

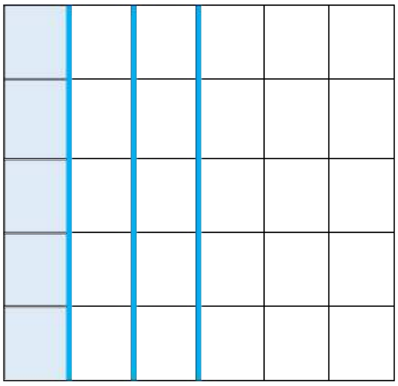


Swap 5 2s for 2 5s.

3 x 5 x 2



5 x 3 x 2



Swap 5 3s for 3 5s.

Upper KS2

At task **12** we finally put the individual 'tables' together as the multiplication square.

Now that we know the *commutative* law, we shall feel free to use the rows and columns of our table interchangeably.

Tasks **13** and **14** introduce the two other laws which govern the operation 'multiply': the *associative* and *distributive* laws. Task **15** concerns the *inverse* operation. Tasks **16** and **17**, **18** show two ways in which two numbers can be compared: *highest common factor* (h.c.f.)/*greatest common divisor* (g.c.d) and *lowest common multiple* (l.c.m.). These quantities in fact depend on the prime factorisation of the two numbers, whose study we meet in task **19**.

12. *We fill in an empty 10 x 10 table, then use it as if we did not know the products. That is to say, to find the product of of 8 and 7, we run one finger along the top to '8', another finger down the side to '7', and see where the '8' column and the '7' row intersect.*

In these notes we are not concerned with how the children learn the multiplication facts. However, a good exercise, having made the 10 x 10 squares, is for the children to mount them on card, cut them up and offer them to other groups to solve as jigsaw puzzles (as long advocated by Afzal Ahmed). Go to www.magicmathworks.org , then 'Virtual Circus', then 'Multiplication', then 'Tables jigsaws'.

13. *We use the table to show that* $(3\times2)\times5 = 3\times(2\times5)$,

demonstrating that the result of a multiplication does not depend on how we group the terms, i.e. that the operation is *associative*:

In the two squares below we perform the same calculation in two different ways. On the left we multiply 3 by 2, and the product by 5; on the right we multiply 3 by the product of 2 and 5. We can do this because multiplication is *associative*.

$(3\times2)\times5$

X	1	2	3	4	5	6	7	8	9	10
1	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
2	**2**	**4**	**6**	**8**	**10**	**12**	**14**	**16**	**18**	**20**
3	**3**	**6**	**9**	**12**	**15**	**18**	**21**	**24**	**27**	**30**
4	**4**	**8**	**12**	**16**	**20**	**24**	**28**	**32**	**36**	**40**
5	**5**	**10**	**15**	**20**	**25**	**30**	**35**	**40**	**45**	**50**
6	**6**	**12**	**18**	**24**	**30**	**36**	**42**	**48**	**54**	**60**
7	**7**	**14**	**21**	**28**	**35**	**42**	**49**	**56**	**63**	**70**
8	**8**	**16**	**24**	**32**	**40**	**48**	**56**	**64**	**72**	**80**
9	**9**	**18**	**27**	**36**	**45**	**54**	**63**	**72**	**81**	**90**
10	**10**	**20**	**30**	**40**	**50**	**60**	**70**	**80**	**90**	**100**

$3\times(2\times5)$

X	1	2	3	4	5	6	7	8	9	10
1	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
2	**2**	**4**	**6**	**8**	**10**	**12**	**14**	**16**	**18**	**20**
3	**3**	**6**	**9**	**12**	**15**	**18**	**21**	**24**	**27**	**30**
4	**4**	**8**	**12**	**16**	**20**	**24**	**28**	**32**	**36**	**40**
5	**5**	**10**	**15**	**20**	**25**	**30**	**35**	**40**	**45**	**50**
6	**6**	**12**	**18**	**24**	**30**	**36**	**42**	**48**	**54**	**60**
7	**7**	**14**	**21**	**28**	**35**	**42**	**49**	**56**	**63**	**70**
8	**8**	**16**	**24**	**32**	**40**	**48**	**56**	**64**	**72**	**80**
9	**9**	**18**	**27**	**36**	**45**	**54**	**63**	**72**	**81**	**90**
10	**10**	**20**	**30**	**40**	**50**	**60**	**70**	**80**	**90**	**100**

14. *We complete a blank addition square to 10 + 10, and use it in conjunction with our multiplication square to show that* $(4+3)\times 2 = 4\times 2 + 3\times 2$,

demonstrating that multiplication is *distributive over addition*.

On the left below is an addition square with a multiplication square beneath; on the right is a multiplication square with an addition square beneath. Again we perform the same calculation in two different ways. One the left we add 3 to 4, then multiply the sum by 2; on the right we multiply 4 by 2, and 3 by 2, then add the two products. We can do this because multiplication is *distributive* over addition.

$(4+3)\times 2$

+	1	2	3	4	5	6	7	8	9	10
1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
2	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
3	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**
4	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**
5	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
6	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**
7	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**
8	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**
9	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**
10	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**

X	1	2	3	4	5	6	7	8	9	10
1	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
2	**2**	**4**	**6**	**8**	**10**	**12**	**14**	**16**	**18**	**20**
3	**3**	**6**	**9**	**12**	**15**	**18**	**21**	**24**	**27**	**30**
4	**4**	**8**	**12**	**16**	**20**	**24**	**28**	**32**	**36**	**40**
5	**5**	**10**	**15**	**20**	**25**	**30**	**35**	**40**	**45**	**50**
6	**6**	**12**	**18**	**24**	**30**	**36**	**42**	**48**	**54**	**60**
7	**7**	**14**	**21**	**28**	**35**	**42**	**49**	**56**	**63**	**70**
8	**8**	**16**	**24**	**32**	**40**	**48**	**56**	**64**	**72**	**80**
9	**9**	**18**	**27**	**36**	**45**	**54**	**63**	**72**	**81**	**90**
10	**10**	**20**	**30**	**40**	**50**	**60**	**70**	**80**	**90**	**100**

$4\times 2 + 3\times 2$

X	1	2	3	4	5	6	7	8	9	10
1	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
2	**2**	**4**	**6**	**8**	**10**	**12**	**14**	**16**	**18**	**20**
3	**3**	**6**	**9**	**12**	**15**	**18**	**21**	**24**	**27**	**30**
4	**4**	**8**	**12**	**16**	**20**	**24**	**28**	**32**	**36**	**40**
5	**5**	**10**	**15**	**20**	**25**	**30**	**35**	**40**	**45**	**50**
6	**6**	**12**	**18**	**24**	**30**	**36**	**42**	**48**	**54**	**60**
7	**7**	**14**	**21**	**28**	**35**	**42**	**49**	**56**	**63**	**70**
8	**8**	**16**	**24**	**32**	**40**	**48**	**56**	**64**	**72**	**80**
9	**9**	**18**	**27**	**36**	**45**	**54**	**63**	**72**	**81**	**90**
10	**10**	**20**	**30**	**40**	**50**	**60**	**70**	**80**	**90**	**100**

+	1	2	3	4	5	6	7	8	9	10
1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
2	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
3	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**
4	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**
5	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
6	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**
7	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**
8	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**
9	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**
10	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**

15. *We use our table to show that* $21 \div 7 = 3$. *We move a finger down the '7' column to '21', then across to identify the 'row' heading.*

We illustrate the operation and its inverse with some other examples on the table below. Look at the orange arrows and the orange box.

What is 7 times 12?

We move down from '7' and across from '12'. Their product lies at the intersection of the row and the column.

What is 84 divided by 7?

To perform the *inverse* operation to multiplication, we find 84 in the '7' column and identify the row.

16. *What is the h.c.f. of 9 and 15? We move right from '1' till we find the heading of the column in which 9 and 15 are adjacent. That heading gives the h.c.f. .*

On the same table we'll illustrate another example. Look at the blue box and the red box and arrow.

What is the highest common factor (h.c.f.) of 8 and 12?

We find 8 and 12 in the column headed '2'. We know therefore that 2 is a factor common to 8 and 12. But it is not the *highest* such factor. We need to find the column in which the two numbers are *adjacent*. In the column headed '4' we find 8 directly above 12. We therefore know that no multiple of 4 lies between 8 and 12 and 4 must therefore be the highest factor common to both.

17. *What is the l.c.m. of 6 and 10? Simultaneously, we go down from '6' and across from '10' until we hit the same number.*

We'll use the table for another example. Look at the green boxes and arrows:

What is the lowest l.c.m. of 8 and 12?

We begin at the top of the '12' column, and the lefthand end of the '8' row. All the numbers in the first are multiples of 12, and all the numbers in the second are multiples of 8. However, only certain numbers in each are multiples of both. We require the *lowest* such. So we move down from '12' and across from '8' till we find the same number. This is the lowest multiple common to the two numbers.

For activities on common multiples, go to www.magicmathworks.org , then 'Virtual Circus', then 'Multiplication', then any or all of the following: 'Times chimes', 'Gear ratios', 'Magic masks'.

X	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
2	**2**	**4**	**6**	**8**	**10**	**12**	**14**	**16**	**18**	**20**	**22**	**24**	**26**	**28**	**30**
3	**3**	**6**	**9**	**12**	**15**	**18**	**21**	**24**	**27**	**30**	**33**	**36**	**39**	**42**	**45**
4	**4**	**8**	**12**	**16**	**20**	**24**	**28**	**32**	**36**	**40**	**44**	**48**	**52**	**56**	**60**
5	**5**	**10**	**15**	**20**	**25**	**30**	**35**	**40**	**45**	**50**	**55**	**60**	**65**	**70**	**75**
6	**6**	**12**	**18**	**24**	**30**	**36**	**42**	**48**	**54**	**60**	**66**	**72**	**78**	**84**	**90**
7	**7**	**14**	**21**	**28**	**35**	**42**	**49**	**56**	**63**	**70**	**77**	**84**	**91**	**98**	**105**
8	**8**	**16**	**24**	**32**	**40**	**48**	**56**	**64**	**72**	**80**	**88**	**96**	**104**	**112**	**120**
9	**9**	**18**	**27**	**36**	**45**	**54**	**63**	**72**	**81**	**90**	**99**	**108**	**117**	**126**	**135**
10	**10**	**20**	**30**	**40**	**50**	**60**	**70**	**80**	**90**	**100**	**110**	**120**	**130**	**140**	**150**
11	**11**	**22**	**33**	**44**	**55**	**66**	**77**	**88**	**99**	**110**	**121**	**132**	**143**	**154**	**165**
12	**12**	**24**	**36**	**48**	**60**	**72**	**84**	**96**	**108**	**120**	**132**	**144**	**156**	**168**	**180**
13	**13**	**26**	**39**	**52**	**65**	**78**	**91**	**104**	**117**	**130**	**143**	**156**	**169**	**182**	**195**
14	**14**	**28**	**42**	**56**	**70**	**84**	**98**	**112**	**126**	**140**	**154**	**168**	**182**	**196**	**210**
15	**15**	**30**	**45**	**60**	**75**	**90**	**105**	**120**	**135**	**150**	**165**	**180**	**195**	**210**	**225**

18. *On the mathematical balance we add hangers to the '6' peg on one side and the '10' peg on the other till we achieve balance. On either side the product 'peg number' times 'hanger number' is the l.c.m.*

Lower KS3

Task **11** treated under another heading what was in fact the *prime factorisation* of 30. Tasks **19** to **23** address this topic. It's best here to use a bigger square, say 15 x 15.

19. *Which numbers occur only in row 1, 2, 3, 5, 7, 11, 13? What can we say of these numbers?*

They have no divisors apart from themselves and 1: they are prime numbers.

20. *How can we break down 60 into its prime factors?*

We find 60 somewhere in the body of the table and note its row and column numbers:

$60 = 12 \times 5.$

We know 5 is prime, 12 isn't, so we repeat the process for 12.

$12 = 2 \times 6$. Finally, $6 = 2 \times 3$.

So we have $60 = 12 \times 5 = (2 \times 6) \times 5 = 2 \times (2 \times 3) \times 5 = 2 \times 2 \times 3 \times 5 = 2^2.3.5.$

Our 3 laws ensure that, wherever we start, we shall end with the same product. It's worth asking the children if they find this reasonable. If they do, they're accepting the truth of the *fundamental theorem of arithmetic*, which asserts that a number has a unique prime factorisation.

We can interpret tasks **16** to **18** in the light of this.

21. *How can we find the h.c.f. and l.c.m. of two numbers by comparing their prime factorisations?*

9: 3×3
15: $\quad 3 \times 5$

h.c.f.: $\quad 3$
l.c.m.: $3 \times 3 \times 5 = 3^2.5$

Though the prime factorisation of a number is unique, we can group these factors in different ways. Hence the many divisors of 60.

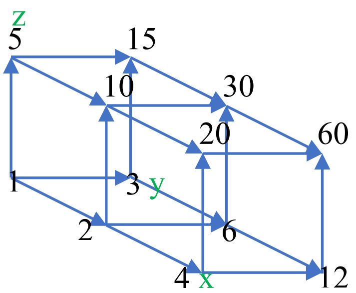
22. *In how many ways can we show 60 as the product of two factors. (The children will have to imagine the products which are off the scale of their 15 x 15 squares):*

$(1 \times 60), (2 \times 30), (3 \times 20), (4 \times 15), (5 \times 12), (6 \times 10).$

There are 12 divisors here, including 1 and 60. We obtain the above products by pairing them like this:

1 2 3 4 5 6 10 12 15 20 30 60

We can show the 12 divisors on this 3-dimensional *Hasse diagram*:



Each step in the x direction raises the power of 2 by one; likewise in the y direction for the power of 3, and in the z direction for the power of 5.

We see from this that we can find the number of divisors by adding 1 to each index and multiplying the new index numbers together:

$60 = 2^2.3^1.5^1$. Number of divisors = $(2 + 1)\times(1 + 1)\times(1 + 1) = 3\times2\times2 = 12$.

Such discussions will prepare the children for the next group of three tasks, **23**, **24** and **25**.

23. *Which numbers have exactly 1, 2, 3, 4 prime factors?*

See the table below. Because the heading numbers are prime, we see that the numbers in the blue boxes have exactly two distinct prime factors. So 15 is one. But so in the same row is, for example, 24, since 8 is a power of 2. In fact only 23 products on the whole table have 3 distinct prime factors. On the table we've taken 15 and multiplied it by 2 to produce a number, 30, with this number of prime factors. Just one number has more: we've taken 14 and multiplied it by 15 to get the number in the red box, 210, with 4 prime factors.

X	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
2	**2**	**4**	**6**	**8**	**10**	**12**	**14**	**16**	**18**	**20**	**22**	**24**	**26**	**28**	**30**
3	**3**	**6**	**9**	**12**	**15**	**18**	**21**	**24**	**27**	**30**	**33**	**36**	**39**	**42**	**45**
4	**4**	**8**	**12**	**16**	**20**	**24**	**28**	**32**	**36**	**40**	**44**	**48**	**52**	**56**	**60**
5	**5**	**10**	**15**	**20**	**25**	**30**	**35**	**40**	**45**	**50**	**55**	**60**	**65**	**70**	**75**
6	**6**	**12**	**18**	**24**	**30**	**36**	**42**	**48**	**54**	**60**	**66**	**72**	**78**	**84**	**90**
7	**7**	**14**	**21**	**28**	**35**	**42**	**49**	**56**	**63**	**70**	**77**	**84**	**91**	**98**	**105**
8	**8**	**16**	**24**	**32**	**40**	**48**	**56**	**64**	**72**	**80**	**88**	**96**	**104**	**112**	**120**
9	**9**	**18**	**27**	**36**	**45**	**54**	**63**	**72**	**81**	**90**	**99**	**108**	**117**	**126**	**135**
10	**10**	**20**	**30**	**40**	**50**	**60**	**70**	**80**	**90**	**100**	**110**	**120**	**130**	**140**	**150**
11	**11**	**22**	**33**	**44**	**55**	**66**	**77**	**88**	**99**	**110**	**121**	**132**	**143**	**154**	**165**
12	**12**	**24**	**36**	**48**	**60**	**72**	**84**	**96**	**108**	**120**	**132**	**144**	**156**	**168**	**180**
13	**13**	**26**	**39**	**52**	**65**	**78**	**91**	**104**	**117**	**130**	**143**	**156**	**169**	**182**	**195**
14	**14**	**28**	**42**	**56**	**70**	**84**	**98**	**112**	**126**	**140**	**154**	**168**	**182**	**196**	**210**
15	**15**	**30**	**45**	**60**	**75**	**90**	**105**	**120**	**135**	**150**	**165**	**180**	**195**	**210**	**225**

Tasks **24** and **25** concern the fact established in activities **19** to **23** that a number may have alternative factorisations.

24. *Where do all the instances of the same number lie?*

On the table below we've colour-coded the numbers up to 12. We see that each colour picks out a particular curve. Take 12 itself, for example. Call the column factor x, the row factor y. Then the equation of the curve through the points at the centres of the dark blue cells is $xy = 12$. Such a curve is called a *rectangular hyperbola*.

X	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
2	**2**	**4**	**6**	**8**	**10**	**12**	**14**	**16**	**18**	**20**	**22**	**24**	**26**	**28**	**30**
3	**3**	**6**	**9**	**12**	**15**	**18**	**21**	**24**	**27**	**30**	**33**	**36**	**39**	**42**	**45**
4	**4**	**8**	**12**	**16**	**20**	**24**	**28**	**32**	**36**	**40**	**44**	**48**	**52**	**56**	**60**
5	**5**	**10**	**15**	**20**	**25**	**30**	**35**	**40**	**45**	**50**	**55**	**60**	**65**	**70**	**75**
6	**6**	**12**	**18**	**24**	**30**	**36**	**42**	**48**	**54**	**60**	**66**	**72**	**78**	**84**	**90**
7	**7**	**14**	**21**	**28**	**35**	**42**	**49**	**56**	**63**	**70**	**77**	**84**	**91**	**98**	**105**
8	**8**	**16**	**24**	**32**	**40**	**48**	**56**	**64**	**72**	**80**	**88**	**96**	**104**	**112**	**120**
9	**9**	**18**	**27**	**36**	**45**	**54**	**63**	**72**	**81**	**90**	**99**	**108**	**117**	**126**	**135**
10	**10**	**20**	**30**	**40**	**50**	**60**	**70**	**80**	**90**	**100**	**110**	**120**	**130**	**140**	**150**
11	**11**	**22**	**33**	**44**	**55**	**66**	**77**	**88**	**99**	**110**	**121**	**132**	**143**	**154**	**165**
12	**12**	**24**	**36**	**48**	**60**	**72**	**84**	**96**	**108**	**120**	**132**	**144**	**156**	**168**	**180**
13	**13**	**26**	**39**	**52**	**65**	**78**	**91**	**104**	**117**	**130**	**143**	**156**	**169**	**182**	**195**
14	**14**	**28**	**42**	**56**	**70**	**84**	**98**	**112**	**126**	**140**	**154**	**168**	**182**	**196**	**210**
15	**15**	**30**	**45**	**60**	**75**	**90**	**105**	**120**	**135**	**150**	**165**	**180**	**195**	**210**	**225**

25. *How many products are there on a 1x1, 2x2, 3x3, … times square?*

Look at the table below. If we look in the 4 x 4 square in the top left corner, we find there are only 9 products, even though there are 16 cells. The dark blue cells are counted twice because they are duplicated by the axis of symmetry. But, even within the remaining patch, the number 4 occurs twice: once as 4 x 1, once as 2 x 2. By the time we get to the 9 x 9 square, the number of distinct products has fallen to 36, well under half the number of cells, 81. Jon Millington found that you can write these 36 numbers on a set of 9 tetrahedra, one on each face, in such a way that the blocks display each of the times tables from 1 to 9. Go to www.magicmathworks.org , then ‘Virtual Circus’, then ‘Multiplication’, then ‘Tables race’.

X	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
2	**2**	**4**	**6**	**8**	**10**	**12**	**14**	**16**	**18**	**20**	**22**	**24**	**26**	**28**	**30**
3	**3**	**6**	**9**	**12**	**15**	**18**	**21**	**24**	**27**	**30**	**33**	**36**	**39**	**42**	**45**
4	**4**	**8**	**12**	**16**	**20**	**24**	**28**	**32**	**36**	**40**	**44**	**48**	**52**	**56**	**60**
5	**5**	**10**	**15**	**20**	**25**	**30**	**35**	**40**	**45**	**50**	**55**	**60**	**65**	**70**	**75**
6	**6**	**12**	**18**	**24**	**30**	**36**	**42**	**48**	**54**	**60**	**66**	**72**	**78**	**84**	**90**
7	**7**	**14**	**21**	**28**	**35**	**42**	**49**	**56**	**63**	**70**	**77**	**84**	**91**	**98**	**105**
8	**8**	**16**	**24**	**32**	**40**	**48**	**56**	**64**	**72**	**80**	**88**	**96**	**104**	**112**	**120**
9	**9**	**18**	**27**	**36**	**45**	**54**	**63**	**72**	**81**	**90**	**99**	**108**	**117**	**126**	**135**
10	**10**	**20**	**30**	**40**	**50**	**60**	**70**	**80**	**90**	**100**	**110**	**120**	**130**	**140**	**150**
11	**11**	**22**	**33**	**44**	**55**	**66**	**77**	**88**	**99**	**110**	**121**	**132**	**143**	**154**	**165**
12	**12**	**24**	**36**	**48**	**60**	**72**	**84**	**96**	**108**	**120**	**132**	**144**	**156**	**168**	**180**
13	**13**	**26**	**39**	**52**	**65**	**78**	**91**	**104**	**117**	**130**	**143**	**156**	**169**	**182**	**195**
14	**14**	**28**	**42**	**56**	**70**	**84**	**98**	**112**	**126**	**140**	**154**	**168**	**182**	**196**	**210**
15	**15**	**30**	**45**	**60**	**75**	**90**	**105**	**120**	**135**	**150**	**165**	**180**	**195**	**210**	**225**

As the size of the table increases, the proportion of distinct products continues to *trend* downwards. However, because of the effect of primes, the function is not *monotone decreasing.*

Here is a 20 x 20 square. We've proceeded downwards row by row, entering only new products and giving their frequency at the end in red:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	20
										22	24	26	28	30	32	34	36	38	40	10
						21		27		33		39	42	45	48	51	54	57	60	11
										44		52	56		64	68	72	76	80	8
				25		35			50	55		65	70	75		85	90	95	100	11
										66		78	84		96	102	108	114	120	8
						49		63		77		91	98	105	112	119	126	133	140	11
										88		104			128	136	144	152	160	7
								81		99		117		135		153	162	171	180	8
										110		130		150		170		190	200	6
										121	132	143	154	165	176	187	198	209	220	10
												156	168		192	204	216	226	240	7
												169	182	195	208	221	234	247	260	8
													196	210	224	238	252	266	280	7
														225		255	270	285	300	5
															256	272	288	304	320	5
																289	306	323	340	4
																	364	382	360	3
																		361	380	2
																			400	1

Notice that the prime columns are complete where the prime exceeds half of 20. Otherwise multiples will already have been taken care of. By contrast the ‘12’ column is almost empty. 12 is a ‘highly divisible’ number so we will have met many entries in other columns. (One might have expected the ‘18’ column to be less full, and so it would be if our square extended to ‘27’.)

Tasks **26** to **34** explore how and where *figurate* numbers occur on the multiplication square.
26 to **30** treat two-dimensional figures; **30** to **34**, three-dimensional ones. Though the topic rightly belongs to recreational mathematics, the patterns are pleasing.

Rectangular numbers are so called because they count the number of dots in an array of that shape. *Square* numbers, *triangular* numbers likewise. Go to www.magicmathworks.org , then ‘Virtual circus’, then ‘Number patterns’, then ‘2-D number shapes’.

26. *All the numbers which lie beyond the first row and column are by definition rectangular numbers. Where do the square numbers lie?*

The commutative law means the square has a symmetry axis running top left to bottom right.
The numbers which lie on this are the squares.

27. *Where do the triangular numbers lie?*

See the table below. As we've seen in **26**, the numbers in the chrome yellow boxes are the squares. The numbers in the blue boxes are the triangular numbers. Two consecutive triangular numbers make a square:

We've chosen two such pairs and shown with blue arrows the squares they sum to.

X	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
2	**2**	**4**	**6**	**8**	**10**	**12**	**14**	**16**	**18**	**20**	**22**	**24**	**26**	**28**	**30**
3	**3**	**6**	**9**	**12**	**15**	**18**	**21**	**24**	**27**	**30**	**33**	**36**	**39**	**42**	**45**
4	**4**	**8**	**12**	**16**	**20**	**24**	**28**	**32**	**36**	**40**	**44**	**48**	**52**	**56**	**60**
5	**5**	**10**	**15**	**20**	**25**	**30**	**35**	**40**	**45**	**50**	**55**	**60**	**65**	**70**	**75**
6	**6**	**12**	**18**	**24**	**30**	**36**	**42**	**48**	**54**	**60**	**66**	**72**	**78**	**84**	**90**
7	**7**	**14**	**21**	**28**	**35**	**42**	**49**	**56**	**63**	**70**	**77**	**84**	**91**	**98**	**105**
8	**8**	**16**	**24**	**32**	**40**	**48**	**56**	**64**	**72**	**80**	**88**	**96**	**104**	**112**	**120**
9	**9**	**18**	**27**	**36**	**45**	**54**	**63**	**72**	**81**	**90**	**99**	**108**	**117**	**126**	**135**
10	**10**	**20**	**30**	**40**	**50**	**60**	**70**	**80**	**90**	**100**	**110**	**120**	**130**	**140**	**150**
11	**11**	**22**	**33**	**44**	**55**	**66**	**77**	**88**	**99**	**110**	**121**	**132**	**143**	**154**	**165**
12	**12**	**24**	**36**	**48**	**60**	**72**	**84**	**96**	**108**	**120**	**132**	**144**	**156**	**168**	**180**
13	**13**	**26**	**39**	**52**	**65**	**78**	**91**	**104**	**117**	**130**	**143**	**156**	**169**	**182**	**195**
14	**14**	**28**	**42**	**56**	**70**	**84**	**98**	**112**	**126**	**140**	**154**	**168**	**182**	**196**	**210**
15	**15**	**30**	**45**	**60**	**75**	**90**	**105**	**120**	**135**	**150**	**165**	**180**	**195**	**210**	**225**

28. *What happens when we add the first few numbers in a row?*

29. *What happens when we add the numbers in a rectangular block?*

30. *What happens when the block is square?*

We'll take **28**, **29**, **30** one at a time:

X	1	2	3	4	5	6	7	8	9	10
1	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
2	**2**	**4**	**6**	**8**	**10**	**12**	**14**	**16**	**18**	**20**
3	**3**	**6**	**9**	**12**	**15**	**18**	**21**	**24**	**27**	**30**
4	**4**	**8**	**12**	**16**	**20**	**24**	**28**	**32**	**36**	**40**
5	**5**	**10**	**15**	**20**	**25**	**30**	**35**	**40**	**45**	**50**
6	**6**	**12**	**18**	**24**	**30**	**36**	**42**	**48**	**54**	**60**
7	**7**	**14**	**21**	**28**	**35**	**42**	**49**	**56**	**63**	**70**
8	**8**	**16**	**24**	**32**	**40**	**48**	**56**	**64**	**72**	**80**
9	**9**	**18**	**27**	**36**	**45**	**54**	**63**	**72**	**81**	**90**
10	**10**	**20**	**30**	**40**	**50**	**60**	**70**	**80**	**90**	**100**

The sum of the palest green row is the 4th triangular number, T_4. The sum of the next, twice this; of the next, three times this. So the grand total is $T_4 \times T_3$.

If we add a 4th row, the total is ${T_4}^2$.

The figure below shows that ${T_4}^2 = 1\times1^2 + 2\times2^2 + 3\times3^2 + 4\times4^2$, i.e. ${T_4}^2 = 1^3 + 2^3 + 3^3 + 4^3$.

Here is one way to introduce **29** to a class who know the first four triangular numbers:

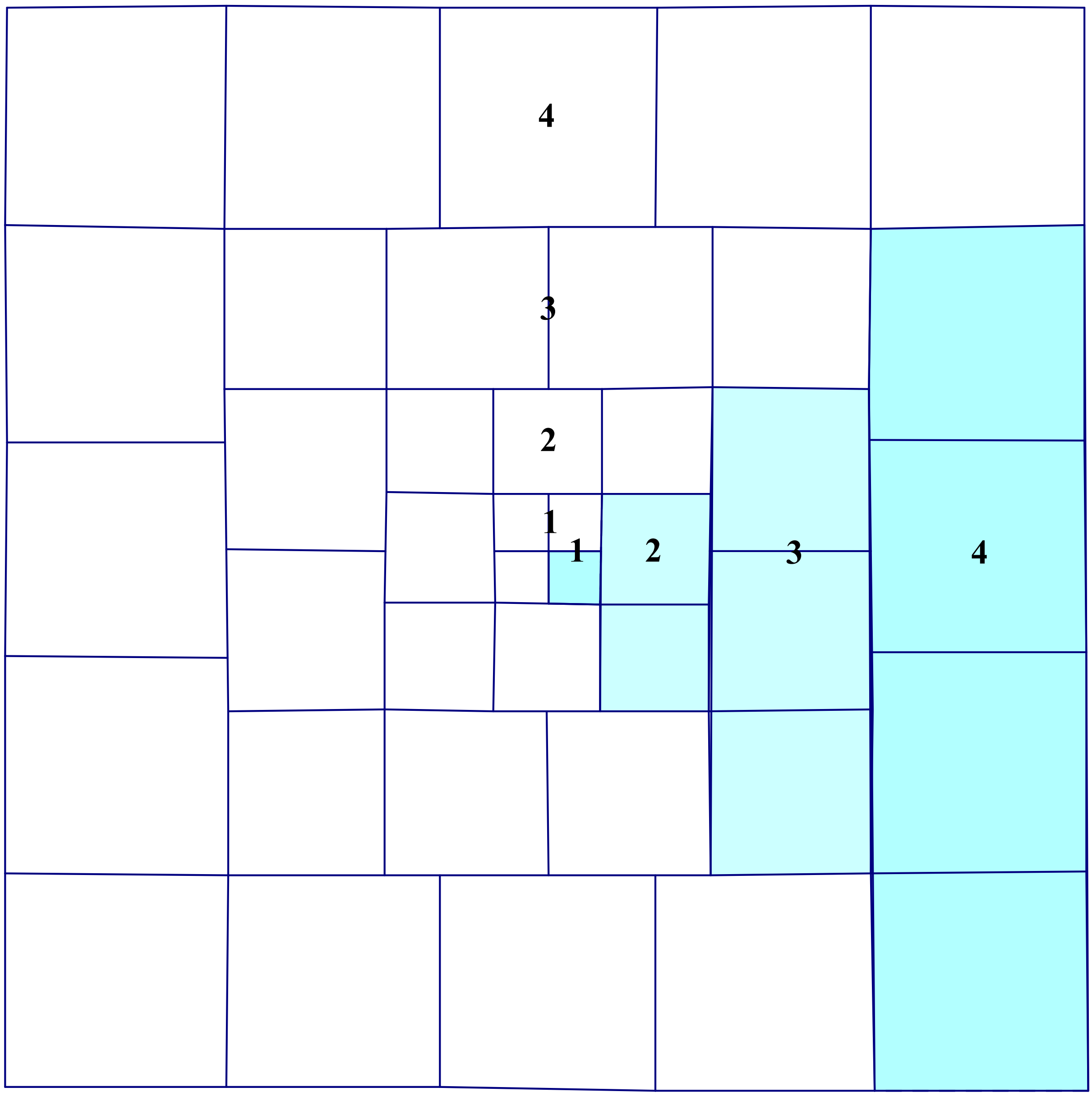
"*Tanya's table, choose a small number across and a small number down. Add all the numbers in the rectangle underneath but keep your answer secret."*

"Right, now tell me your 'across' and 'down' numbers."

"4 and 3."

Make a show of going across to the 4th triangular number and down to the 3rd triangular number And locating the intersection.

"I think your answer's 60. Am I right?"



For the 3-dimensional figurate numbers which follow, go to www.magicmathworks.org then 'Virtual circus', then 'Number patterns', then '2-D number shapes'.
We sum the entries along diagonals running top right to bottom left.

31. *What happens when the diagonal lies at 45°?*

32. *What happens when we total two such adjacent diagonals? … and then sum the entries vertically?*

We get a steeper diagonal.

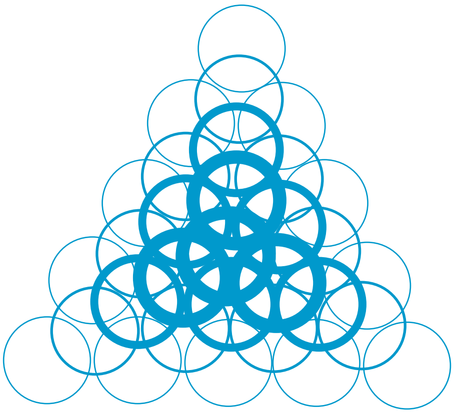
33. *What happens when we total two adjacent steep diagonals horizontally?*

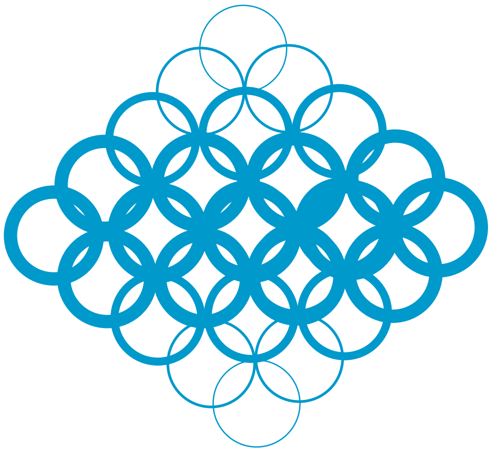
We'll take **31**, **32**, **33** one at a time:

The 5^{th} tetrahedral number is the sum of the first 5 triangular numbers.

On the left we show the tetrahedron standing on a base, so that the horizontal layers are the constituent triangular numbers: 15 + 10 + 6 + 3 + 1.

On the right we show it standing on an edge. The layers are rectangles:
(1 x 5), (2 x 4), (3 x 3), (4 x 2), (5 x 1).





These numbers lie here on the multiplication square:

X	1	2	3	4	5	6	7	8	9
1	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
2	**2**	**4**	**6**	**8**	**10**	**12**	**14**	**16**	**18**
3	**3**	**6**	**9**	**12**	**15**	**18**	**21**	**24**	**27**
4	**4**	**8**	**12**	**16**	**20**	**24**	**28**	**32**	**36**
5	**5**	**10**	**15**	**20**	**25**	**30**	**35**	**40**	**45**
6	**6**	**12**	**18**	**24**	**30**	**36**	**42**	**48**	**54**
7	**7**	**14**	**21**	**28**	**35**	**42**	**49**	**56**	**63**
8	**8**	**16**	**24**	**32**	**40**	**48**	**56**	**64**	**72**
9	**9**	**18**	**27**	**36**	**45**	**54**	**63**	**72**	**81**

We've seen that two consecutive triangular numbers make a square number.

In three dimensions, comparing two consecutive *tetrahedra* slice by horizontal slice, we see that they make a *pyramid*.

Below we show the diagonals which sum to the 4^{th} and 5^{th} tetrahedral numbers.

Adding cells vertically, we have a steeper diagonal whose cells sum to the 5^{th} pyramidal number.

If we stick two consecutive pyramids base to base, we have an octahedron.

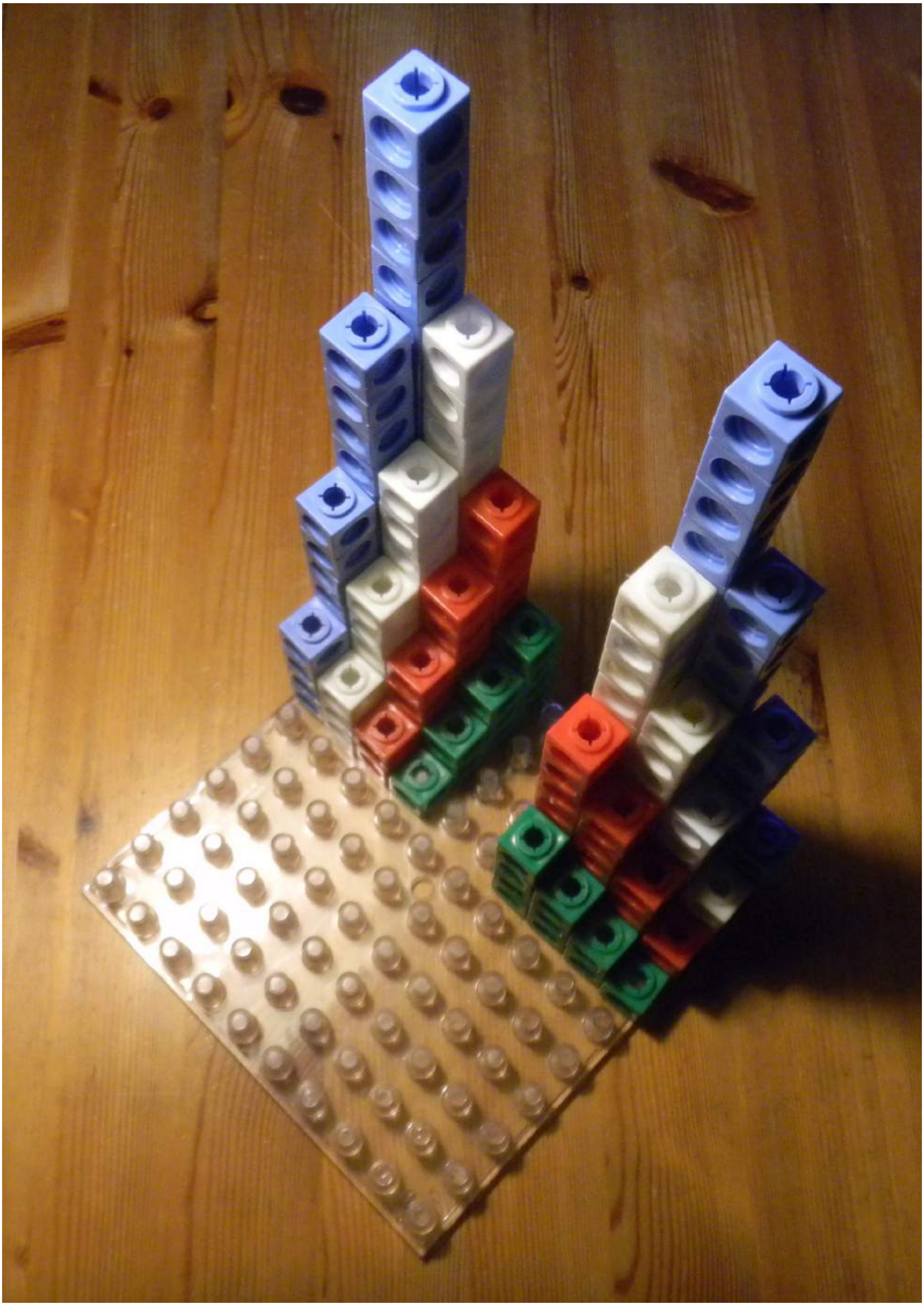
On the square below we've shown the 5^{th} and 6^{th} pyramidal numbers. This time we've added the cells horizontally. The result is again a 45° diagonal, but the entries are spaced out.

X	1	2	3	4	5	6	7	8	9
1	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
2	**2**	**4**	**6**	**8**	**10**	**12**	**14**	**16**	**18**
3	**3**	**6**	**9**	**12**	**15**	**18**	**21**	**24**	**27**
4	**4**	**8**	**12**	**16**	**20**	**24**	**28**	**32**	**36**
5	**5**	**10**	**15**	**20**	**25**	**30**	**35**	**40**	**45**
6	**6**	**12**	**18**	**24**	**30**	**36**	**42**	**48**	**54**
7	**7**	**14**	**21**	**28**	**35**	**42**	**49**	**56**	**63**
8	**8**	**16**	**24**	**32**	**40**	**48**	**56**	**64**	**72**
9	**9**	**18**	**27**	**36**	**45**	**54**	**63**	**72**	**81**

X	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
2	**2**	**4**	**6**	**8**	**10**	**12**	**14**	**16**	**18**	**20**	**22**	**24**	**26**	**28**	**30**
3	**3**	**6**	**9**	**12**	**15**	**18**	**21**	**24**	**27**	**30**	**33**	**36**	**39**	**42**	**45**
4	**4**	**8**	**12**	**16**	**20**	**24**	**28**	**32**	**36**	**40**	**44**	**48**	**52**	**56**	**60**
5	**5**	**10**	**15**	**20**	**25**	**30**	**35**	**40**	**45**	**50**	**55**	**60**	**65**	**70**	**75**
6	**6**	**12**	**18**	**24**	**30**	**36**	**42**	**48**	**54**	**60**	**66**	**72**	**78**	**84**	**90**
7	**7**	**14**	**21**	**28**	**35**	**42**	**49**	**56**	**63**	**70**	**77**	**84**	**91**	**98**	**105**
8	**8**	**16**	**24**	**32**	**40**	**48**	**56**	**64**	**72**	**80**	**88**	**96**	**104**	**112**	**120**
9	**9**	**18**	**27**	**36**	**45**	**54**	**63**	**72**	**81**	**90**	**99**	**108**	**117**	**126**	**135**
10	**10**	**20**	**30**	**40**	**50**	**60**	**70**	**80**	**90**	**100**	**110**	**120**	**130**	**140**	**150**
11	**11**	**22**	**33**	**44**	**55**	**66**	**77**	**88**	**99**	**110**	**121**	**132**	**143**	**154**	**165**
12	**12**	**24**	**36**	**48**	**60**	**72**	**84**	**96**	**108**	**120**	**132**	**144**	**156**	**168**	**180**
13	**13**	**26**	**39**	**52**	**65**	**78**	**91**	**104**	**117**	**130**	**143**	**156**	**169**	**182**	**195**
14	**14**	**28**	**42**	**56**	**70**	**84**	**98**	**112**	**126**	**140**	**154**	**168**	**182**	**196**	**210**
15	**15**	**30**	**45**	**60**	**75**	**90**	**105**	**120**	**135**	**150**	**165**	**180**	**195**	**210**	**225**

34. *We build our '1', 2', '3' and '4' staircases alongside each other on a Multilink base. What we find we've done is to graph the 4 x 4 multiplication square vertically.*

Though we now take the commutative law for granted, it comes as a surprise that, if we colour-code the individual 'times tables' as below, we can trace identical staircases going the other way:



Upper KS3

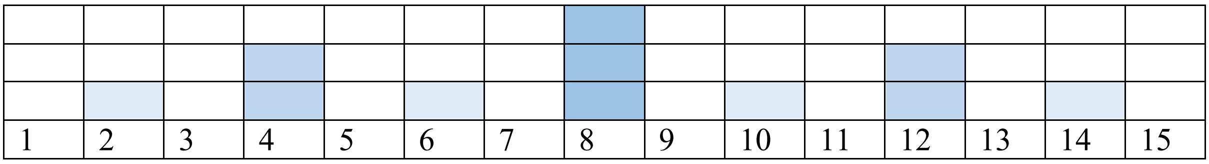
In activities **35** to **37** we investigate how the prime factorisations of products show themselves on the multiplication square when we select particular primes and products of primes.

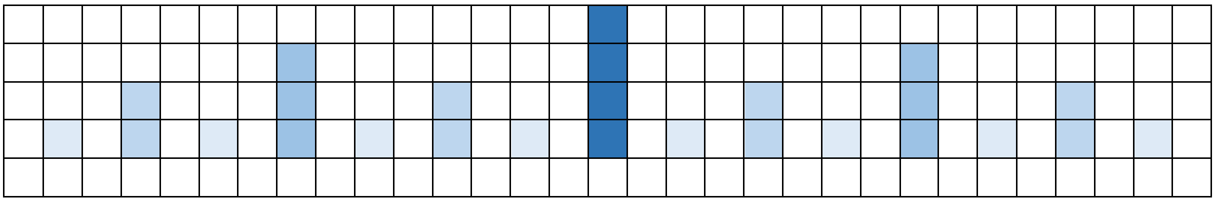
35. *How do the powers of 2, 3, 5, ... appear on the number line?*

36. *What patterns does this structure give rise to on the multiplication square?*

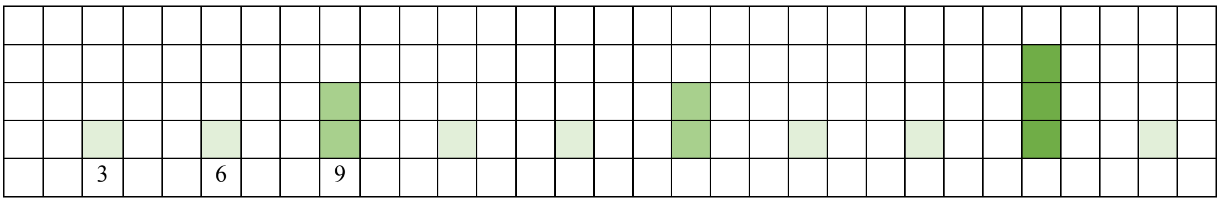
We'll take questions **35** and **36** in order:

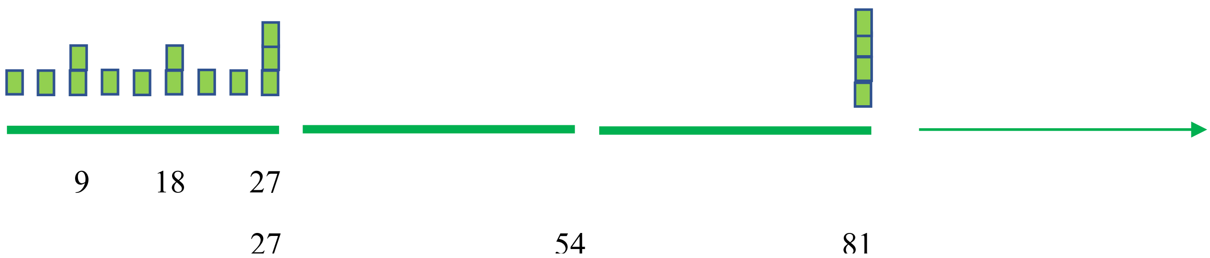
These figures are block graphs. The height of each block in the first two rows is the power of 2 the number contains. Note how the whole of the first row nests in the two halves of the second: the pattern has a fractal structure.





The height of each block below is the power of 3 the number contains. The pattern is different: we have a new power at the end of a cycle of three instead of two, but it has
the same fractal structure.





In each case, a tower of a new height contains the symmetry axis for the block containing all lower powers to left and right – we can think of a baron in his castle commanding all the lands around whose rulers are less powerful.

On the next pages, where products are shown in a multiplication square, this pattern appears in two dimensions. We can imagine we're looking down on a landscape of three-dimensional towers.

We've marked in black the squares which have the vertical and diagonal symmetry axes of the geometrical square.

First, powers of 2:

X	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
2	**2**	**4**	**6**	**8**	**10**	**12**	**14**	**16**	**18**	**20**	**22**	**24**	**26**	**28**	**30**
3	**3**	**6**	**9**	**12**	**15**	**18**	**21**	**24**	**27**	**30**	**33**	**36**	**39**	**42**	**45**
4	**4**	**8**	**12**	**16**	**20**	**24**	**28**	**32**	**36**	**40**	**44**	**48**	**52**	**56**	**60**
5	**5**	**10**	**15**	**20**	**25**	**30**	**35**	**40**	**45**	**50**	**55**	**60**	**65**	**70**	**75**
6	**6**	**12**	**18**	**24**	**30**	**36**	**42**	**48**	**54**	**60**	**66**	**72**	**78**	**84**	**90**
7	**7**	**14**	**21**	**28**	**35**	**42**	**49**	**56**	**63**	**70**	**77**	**84**	**91**	**98**	**105**
8	**8**	**16**	**24**	**32**	**40**	**48**	**56**	**64**	**72**	**80**	**88**	**96**	**104**	**112**	**120**
9	**9**	**18**	**27**	**36**	**45**	**54**	**63**	**72**	**81**	**90**	**99**	**108**	**117**	**126**	**135**
10	**10**	**20**	**30**	**40**	**50**	**60**	**70**	**80**	**90**	**100**	**110**	**120**	**130**	**140**	**150**
11	**11**	**22**	**33**	**44**	**55**	**66**	**77**	**88**	**99**	**110**	**121**	**132**	**143**	**154**	**165**
12	**12**	**24**	**36**	**48**	**60**	**72**	**84**	**96**	**108**	**120**	**132**	**144**	**156**	**168**	**180**
13	**13**	**26**	**39**	**52**	**65**	**78**	**91**	**104**	**117**	**130**	**143**	**156**	**169**	**182**	**195**
14	**14**	**28**	**42**	**56**	**70**	**84**	**98**	**112**	**126**	**140**	**154**	**168**	**182**	**196**	**210**
15	**15**	**30**	**45**	**60**	**75**	**90**	**105**	**120**	**135**	**150**	**165**	**180**	**195**	**210**	**225**

Now powers of 3:

X	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
2	**2**	**4**	**6**	**8**	**10**	**12**	**14**	**16**	**18**	**20**	**22**	**24**	**26**	**28**	**30**
3	**3**	**6**	**9**	**12**	**15**	**18**	**21**	**24**	**27**	**30**	**33**	**36**	**39**	**42**	**45**
4	**4**	**8**	**12**	**16**	**20**	**24**	**28**	**32**	**36**	**40**	**44**	**48**	**52**	**56**	**60**
5	**5**	**10**	**15**	**20**	**25**	**30**	**35**	**40**	**45**	**50**	**55**	**60**	**65**	**70**	**75**
6	**6**	**12**	**18**	**24**	**30**	**36**	**42**	**48**	**54**	**60**	**66**	**72**	**78**	**84**	**90**
7	**7**	**14**	**21**	**28**	**35**	**42**	**49**	**56**	**63**	**70**	**77**	**84**	**91**	**98**	**105**
8	**8**	**16**	**24**	**32**	**40**	**48**	**56**	**64**	**72**	**80**	**88**	**96**	**104**	**112**	**120**
9	**9**	**18**	**27**	**36**	**45**	**54**	**63**	**72**	**81**	**90**	**99**	**108**	**117**	**126**	**135**
10	**10**	**20**	**30**	**40**	**50**	**60**	**70**	**80**	**90**	**100**	**110**	**120**	**130**	**140**	**150**
11	**11**	**22**	**33**	**44**	**55**	**66**	**77**	**88**	**99**	**110**	**121**	**132**	**143**	**154**	**165**
12	**12**	**24**	**36**	**48**	**60**	**72**	**84**	**96**	**108**	**120**	**132**	**144**	**156**	**168**	**180**
13	**13**	**26**	**39**	**52**	**65**	**78**	**91**	**104**	**117**	**130**	**143**	**156**	**169**	**182**	**195**
14	**14**	**28**	**42**	**56**	**70**	**84**	**98**	**112**	**126**	**140**	**154**	**168**	**182**	**196**	**210**
15	**15**	**30**	**45**	**60**	**75**	**90**	**105**	**120**	**135**	**150**	**165**	**180**	**195**	**210**	**225**

Now powers of 5:

X	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
2	**2**	**4**	**6**	**8**	**10**	**12**	**14**	**16**	**18**	**20**	**22**	**24**	**26**	**28**	**30**
3	**3**	**6**	**9**	**12**	**15**	**18**	**21**	**24**	**27**	**30**	**33**	**36**	**39**	**42**	**45**
4	**4**	**8**	**12**	**16**	**20**	**24**	**28**	**32**	**36**	**40**	**44**	**48**	**52**	**56**	**60**
5	**5**	**10**	**15**	**20**	**25**	**30**	**35**	**40**	**45**	**50**	**55**	**60**	**65**	**70**	**75**
6	**6**	**12**	**18**	**24**	**30**	**36**	**42**	**48**	**54**	**60**	**66**	**72**	**78**	**84**	**90**
7	**7**	**14**	**21**	**28**	**35**	**42**	**49**	**56**	**63**	**70**	**77**	**84**	**91**	**98**	**105**
8	**8**	**16**	**24**	**32**	**40**	**48**	**56**	**64**	**72**	**80**	**88**	**96**	**104**	**112**	**120**
9	**9**	**18**	**27**	**36**	**45**	**54**	**63**	**72**	**81**	**90**	**99**	**108**	**117**	**126**	**135**
10	**10**	**20**	**30**	**40**	**50**	**60**	**70**	**80**	**90**	**100**	**110**	**120**	**130**	**140**	**150**
11	**11**	**22**	**33**	**44**	**55**	**66**	**77**	**88**	**99**	**110**	**121**	**132**	**143**	**154**	**165**
12	**12**	**24**	**36**	**48**	**60**	**72**	**84**	**96**	**108**	**120**	**132**	**144**	**156**	**168**	**180**
13	**13**	**26**	**39**	**52**	**65**	**78**	**91**	**104**	**117**	**130**	**143**	**156**	**169**	**182**	**195**
14	**14**	**28**	**42**	**56**	**70**	**84**	**98**	**112**	**126**	**140**	**154**	**168**	**182**	**196**	**210**
15	**15**	**30**	**45**	**60**	**75**	**90**	**105**	**120**	**135**	**150**	**165**	**180**	**195**	**210**	**225**

37. *Where do common multiples of* $2\times3 = 6, 2\times5 = 10, 3\times5 = 15$ *and other composite numbers lie?*

First, 6:

X	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
2	**2**	**4**	**6**	**8**	**10**	**12**	**14**	**16**	**18**	**20**	**22**	**24**	**26**	**28**	**30**
3	**3**	**6**	**9**	**12**	**15**	**18**	**21**	**24**	**27**	**30**	**33**	**36**	**39**	**42**	**45**
4	**4**	**8**	**12**	**16**	**20**	**24**	**28**	**32**	**36**	**40**	**44**	**48**	**52**	**56**	**60**
5	**5**	**10**	**15**	**20**	**25**	**30**	**35**	**40**	**45**	**50**	**55**	**60**	**65**	**70**	**75**
6	**6**	**12**	**18**	**24**	**30**	**36**	**42**	**48**	**54**	**60**	**66**	**72**	**78**	**84**	**90**
7	**7**	**14**	**21**	**28**	**35**	**42**	**49**	**56**	**63**	**70**	**77**	**84**	**91**	**98**	**105**
8	**8**	**16**	**24**	**32**	**40**	**48**	**56**	**64**	**72**	**80**	**88**	**96**	**104**	**112**	**120**
9	**9**	**18**	**27**	**36**	**45**	**54**	**63**	**72**	**81**	**90**	**99**	**108**	**117**	**126**	**135**
10	**10**	**20**	**30**	**40**	**50**	**60**	**70**	**80**	**90**	**100**	**110**	**120**	**130**	**140**	**150**
11	**11**	**22**	**33**	**44**	**55**	**66**	**77**	**88**	**99**	**110**	**121**	**132**	**143**	**154**	**165**
12	**12**	**24**	**36**	**48**	**60**	**72**	**84**	**96**	**108**	**120**	**132**	**144**	**156**	**168**	**180**
13	**13**	**26**	**39**	**52**	**65**	**78**	**91**	**104**	**117**	**130**	**143**	**156**	**169**	**182**	**195**
14	**14**	**28**	**42**	**56**	**70**	**84**	**98**	**112**	**126**	**140**	**154**	**168**	**182**	**196**	**210**
15	**15**	**30**	**45**	**60**	**75**	**90**	**105**	**120**	**135**	**150**	**165**	**180**	**195**	**210**	**225**

Now 10:

X	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
2	**2**	**4**	**6**	**8**	**10**	**12**	**14**	**16**	**18**	**20**	**22**	**24**	**26**	**28**	**30**
3	**3**	**6**	**9**	**12**	**15**	**18**	**21**	**24**	**27**	**30**	**33**	**36**	**39**	**42**	**45**
4	**4**	**8**	**12**	**16**	**20**	**24**	**28**	**32**	**36**	**40**	**44**	**48**	**52**	**56**	**60**
5	**5**	**10**	**15**	**20**	**25**	**30**	**35**	**40**	**45**	**50**	**55**	**60**	**65**	**70**	**75**
6	**6**	**12**	**18**	**24**	**30**	**36**	**42**	**48**	**54**	**60**	**66**	**72**	**78**	**84**	**90**
7	**7**	**14**	**21**	**28**	**35**	**42**	**49**	**56**	**63**	**70**	**77**	**84**	**91**	**98**	**105**
8	**8**	**16**	**24**	**32**	**40**	**48**	**56**	**64**	**72**	**80**	**88**	**96**	**104**	**112**	**120**
9	**9**	**18**	**27**	**36**	**45**	**54**	**63**	**72**	**81**	**90**	**99**	**108**	**117**	**126**	**135**
10	**10**	**20**	**30**	**40**	**50**	**60**	**70**	**80**	**90**	**100**	**110**	**120**	**130**	**140**	**150**
11	**11**	**22**	**33**	**44**	**55**	**66**	**77**	**88**	**99**	**110**	**121**	**132**	**143**	**154**	**165**
12	**12**	**24**	**36**	**48**	**60**	**72**	**84**	**96**	**108**	**120**	**132**	**144**	**156**	**168**	**180**
13	**13**	**26**	**39**	**52**	**65**	**78**	**91**	**104**	**117**	**130**	**143**	**156**	**169**	**182**	**195**
14	**14**	**28**	**42**	**56**	**70**	**84**	**98**	**112**	**126**	**140**	**154**	**168**	**182**	**196**	**210**
15	**15**	**30**	**45**	**60**	**75**	**90**	**105**	**120**	**135**	**150**	**165**	**180**	**195**	**210**	**225**

Now 15:

X	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30
3	3	6	9	12	15	18	21	24	27	30	33	36	39	42	45
4	4	8	12	16	20	24	28	32	36	40	44	48	52	56	60
5	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75
6	6	12	18	24	30	36	42	48	54	60	66	72	78	84	90
7	7	14	21	28	35	42	49	56	63	70	77	84	91	98	105
8	8	16	24	32	40	48	56	64	72	80	88	96	104	112	120
9	9	18	27	36	45	54	63	72	81	90	99	108	117	126	135
10	10	20	30	40	50	60	70	80	90	100	110	120	130	140	150
11	11	22	33	44	55	66	77	88	99	110	121	132	143	154	165
12	12	24	36	48	60	72	84	96	108	120	132	144	156	168	180
13	13	26	39	52	65	78	91	104	117	130	143	156	169	182	195
14	14	28	42	56	70	84	98	112	126	140	154	168	182	196	210
15	15	30	45	60	75	90	105	120	135	150	165	180	195	210	225

Now 30:

X	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
2	**2**	**4**	**6**	**8**	**10**	**12**	**14**	**16**	**18**	**20**	**22**	**24**	**26**	**28**	**30**
3	**3**	**6**	**9**	**12**	**15**	**18**	**21**	**24**	**27**	**30**	**33**	**36**	**39**	**42**	**45**
4	**4**	**8**	**12**	**16**	**20**	**24**	**28**	**32**	**36**	**40**	**44**	**48**	**52**	**56**	**60**
5	**5**	**10**	**15**	**20**	**25**	**30**	**35**	**40**	**45**	**50**	**55**	**60**	**65**	**70**	**75**
6	**6**	**12**	**18**	**24**	**30**	**36**	**42**	**48**	**54**	**60**	**66**	**72**	**78**	**84**	**90**
7	**7**	**14**	**21**	**28**	**35**	**42**	**49**	**56**	**63**	**70**	**77**	**84**	**91**	**98**	**105**
8	**8**	**16**	**24**	**32**	**40**	**48**	**56**	**64**	**72**	**80**	**88**	**96**	**104**	**112**	**120**
9	**9**	**18**	**27**	**36**	**45**	**54**	**63**	**72**	**81**	**90**	**99**	**108**	**117**	**126**	**135**
10	**10**	**20**	**30**	**40**	**50**	**60**	**70**	**80**	**90**	**100**	**110**	**120**	**130**	**140**	**150**
11	**11**	**22**	**33**	**44**	**55**	**66**	**77**	**88**	**99**	**110**	**121**	**132**	**143**	**154**	**165**
12	**12**	**24**	**36**	**48**	**60**	**72**	**84**	**96**	**108**	**120**	**132**	**144**	**156**	**168**	**180**
13	**13**	**26**	**39**	**52**	**65**	**78**	**91**	**104**	**117**	**130**	**143**	**156**	**169**	**182**	**195**
14	**14**	**28**	**42**	**56**	**70**	**84**	**98**	**112**	**126**	**140**	**154**	**168**	**182**	**196**	**210**
15	**15**	**30**	**45**	**60**	**75**	**90**	**105**	**120**	**135**	**150**	**165**	**180**	**195**	**210**	**225**

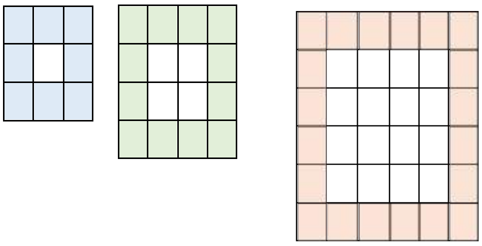
Now 12, where the prime factor 2 is repeated:

X	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
2	**2**	**4**	**6**	**8**	**10**	**12**	**14**	**16**	**18**	**20**	**22**	**24**	**26**	**28**	**30**
3	**3**	**6**	**9**	**12**	**15**	**18**	**21**	**24**	**27**	**30**	**33**	**36**	**39**	**42**	**45**
4	**4**	**8**	**12**	**16**	**20**	**24**	**28**	**32**	**36**	**40**	**44**	**48**	**52**	**56**	**60**
5	**5**	**10**	**15**	**20**	**25**	**30**	**35**	**40**	**45**	**50**	**55**	**60**	**65**	**70**	**75**
6	**6**	**12**	**18**	**24**	**30**	**36**	**42**	**48**	**54**	**60**	**66**	**72**	**78**	**84**	**90**
7	**7**	**14**	**21**	**28**	**35**	**42**	**49**	**56**	**63**	**70**	**77**	**84**	**91**	**98**	**105**
8	**8**	**16**	**24**	**32**	**40**	**48**	**56**	**64**	**72**	**80**	**88**	**96**	**104**	**112**	**120**
9	**9**	**18**	**27**	**36**	**45**	**54**	**63**	**72**	**81**	**90**	**99**	**108**	**117**	**126**	**135**
10	**10**	**20**	**30**	**40**	**50**	**60**	**70**	**80**	**90**	**100**	**110**	**120**	**130**	**140**	**150**
11	**11**	**22**	**33**	**44**	**55**	**66**	**77**	**88**	**99**	**110**	**121**	**132**	**143**	**154**	**165**
12	**12**	**24**	**36**	**48**	**60**	**72**	**84**	**96**	**108**	**120**	**132**	**144**	**156**	**168**	**180**
13	**13**	**26**	**39**	**52**	**65**	**78**	**91**	**104**	**117**	**130**	**143**	**156**	**169**	**182**	**195**
14	**14**	**28**	**42**	**56**	**70**	**84**	**98**	**112**	**126**	**140**	**154**	**168**	**182**	**196**	**210**
15	**15**	**30**	**45**	**60**	**75**	**90**	**105**	**120**	**135**	**150**	**165**	**180**	**195**	**210**	**225**

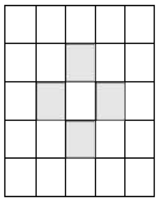
Let’s review the patterns we’ve observed by posing a little puzzle:

We see that multiples of a prime p form a gridiron pattern, in which the gaps are squares $(p-1)$ x $(p-1)$:

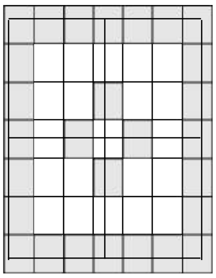
2 3 5



Say we are given this part of a multiplication square and told that a certain number divides the shaded cells but not the others: *What is the number?*



What we are seeing is a window in the '2' grid overlying a junction in the gridiron of a prime p. So the required number is $2p$. We would need to see more to determine p …



We now see that p is 3. Like the cells in the central cross, the outer grid contains even multiples of 3, i.e. multiples of 6.

We also see from the '3 x 5' square that multiples of two odd primes form a different pattern.

In tasks **38** and **39** we represent the products to a particular modulus.

38. *What patterns appear if we make a square (n − 1) by (n − 1) and write the products modulo n, e.g. an 8 x 8 square modulo 9?*

Here's an activity for Y6s or 7s. Write the numbers 1 to 8 on blank cards. Shuffle them and hand one each to a group so that nobody except that group knows what number they've been allocated. Their task is to work out the digital roots of the multiples, up to and including the 8th. With manuscript paper they make their answers the degrees of the musical scale, I suggest choosing C major as the key and making the B below middle C the 0th degree. One of the group then plays their phrase on a recorder or keyboard. The other groups must guess their number. Once all are known, the groups should be asked to play their phrases again and people should comment on what their ears notice. If they detect a mirror pair (or fail to), display the (or a) notated pair on a visualiser. Once all the pairs have been identified, the question is: What is special about the pairs?

39. *What if n is prime, e.g. a 10 by 10 square, modulo 11?*

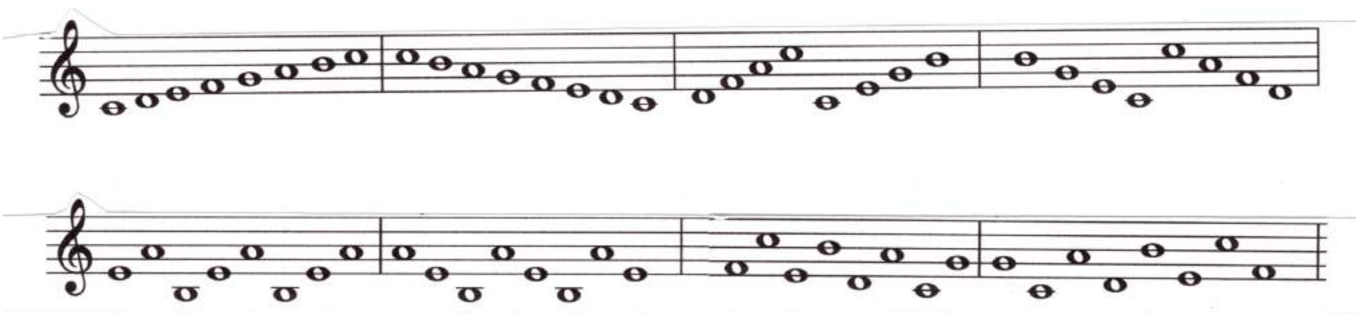
We'll compare the results in cases **38** and **39**:

In this 8 x 8 square the numbers are written mod 9. We find the two symmetry axes in red, which imply point symmetry (rotation symmetry of order 2).

If we reflect a number in a blue axis we find the additive complement with respect to 9.

X	1	2	3	4	5	6	7	8
1	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
2	**2**	**4**	**6**	**8**	**1**	**3**	**5**	**7**
3	**3**	**6**	**0**	**3**	**6**	**0**	**3**	**6**
4	**4**	**8**	**3**	**7**	**2**	**6**	**1**	**5**
5	**5**	**1**	**6**	**2**	**7**	**3**	**8**	**4**
6	**6**	**3**	**0**	**6**	**3**	**0**	**6**	**3**
7	**7**	**5**	**3**	**1**	**8**	**6**	**4**	**2**
8	**8**	**7**	**6**	**5**	**4**	**3**	**2**	**1**

Here are the 8 melodies for **38**:



In the next 10 x 10 square the numbers are written mod 11, a prime. Again we have two symmetry axes. This time the interesting feature is that all the numbers 1 to 10 appear in each row and each column: we have a *Latin* square. In fact the numbers 1 to (p – 1), where p is a prime, form a *group* under multiplication.

X	1	2	3	4	5	6	7	8	9	10
1	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
2	**2**	**4**	**6**	**8**	**10**	**1**	**3**	**5**	**7**	**9**
3	**3**	**6**	**9**	**1**	**4**	**7**	**10**	**2**	**5**	**8**
4	**4**	**8**	**1**	**5**	**9**	**2**	**6**	**10**	**3**	**7**
5	**5**	**10**	**4**	**9**	**3**	**8**	**2**	**7**	**1**	**6**
6	**6**	**1**	**7**	**2**	**8**	**3**	**9**	**4**	**10**	**5**
7	**7**	**3**	**10**	**6**	**2**	**9**	**5**	**1**	**8**	**4**
8	**8**	**5**	**2**	**10**	**7**	**4**	**1**	**9**	**6**	**3**
9	**9**	**7**	**5**	**3**	**1**	**10**	**8**	**6**	**4**	**2**
10	**10**	**9**	**8**	**7**	**6**	**5**	**4**	**3**	**2**	**1**

Lower KS4

In tasks **40** to **44** we study the properties of particular configurations of cells.

40. *What special relation appears when we isolate squares or rectangles within the multiplication square?*

I recommend that you use a multiplication square with a 2 cm grid and build rectangular frames with Multilink cubes which the children can slide around on it.

See the table below. The products of the blue numbers and the green numbers are equal.

Take a rectangle $(k + 1)$ cells wide and $(l + 1)$ cells high.

Take the top left product to be ab.

Then the blue product = top left x bottom right = $[(ab)][(a+k)(b+l)]$, the green product = bottom left x top right = $[a(b+l)]\ [(a+k)b]$ = the blue product.

X	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
2	**2**	**4**	**6**	**8**	**10**	**12**	**14**	**16**	**18**	**20**	**22**	**24**	**26**	**28**	**30**
3	**3**	**6**	**9**	**12**	**15**	**18**	**21**	**24**	**27**	**30**	**33**	**36**	**39**	**42**	**45**
4	**4**	**8**	**12**	**16**	**20**	**24**	**28**	**32**	**36**	**40**	**44**	**48**	**52**	**56**	**60**
5	**5**	**10**	**15**	**20**	**25**	**30**	**35**	**40**	**45**	**50**	**55**	**60**	**65**	**70**	**75**
6	**6**	**12**	**18**	**24**	**30**	**36**	**42**	**48**	**54**	**60**	**66**	**72**	**78**	**84**	**90**
7	**7**	**14**	**21**	**28**	**35**	**42**	**49**	**56**	**63**	**70**	**77**	**84**	**91**	**98**	**105**
8	**8**	**16**	**24**	**32**	**40**	**48**	**56**	**64**	**72**	**80**	**88**	**96**	**104**	**112**	**120**
9	**9**	**18**	**27**	**36**	**45**	**54**	**63**	**72**	**81**	**90**	**99**	**108**	**117**	**126**	**135**
10	**10**	**20**	**30**	**40**	**50**	**60**	**70**	**80**	**90**	**100**	**110**	**120**	**130**	**140**	**150**
11	**11**	**22**	**33**	**44**	**55**	**66**	**77**	**88**	**99**	**110**	**121**	**132**	**143**	**154**	**165**
12	**12**	**24**	**36**	**48**	**60**	**72**	**84**	**96**	**108**	**120**	**132**	**144**	**156**	**168**	**180**
13	**13**	**26**	**39**	**52**	**65**	**78**	**91**	**104**	**117**	**130**	**143**	**156**	**169**	**182**	**195**
14	**14**	**28**	**42**	**56**	**70**	**84**	**98**	**112**	**126**	**140**	**154**	**168**	**182**	**196**	**210**
15	**15**	**30**	**45**	**60**	**75**	**90**	**105**	**120**	**135**	**150**	**165**	**180**	**195**	**210**	**225**

41. *We investigate why numbers at the centres of rows, columns and any configuration with 4-fold rotation symmetry are the mean of those shapes.*

See the table below.

X	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
2	**2**	**4**	**6**	**8**	**10**	**12**	**14**	**16**	**18**	**20**	**22**	**24**	**26**	**28**	**30**
3	**3**	**6**	**9**	**12**	**15**	**18**	**21**	**24**	**27**	**30**	**33**	**36**	**39**	**42**	**45**
4	**4**	**8**	**12**	**16**	**20**	**24**	**28**	**32**	**36**	**40**	**44**	**48**	**52**	**56**	**60**
5	**5**	**10**	**15**	**20**	**25**	**30**	**35**	**40**	**45**	**50**	**55**	**60**	**65**	**70**	**75**
6	**6**	**12**	**18**	**24**	**30**	**36**	**42**	**48**	**54**	**60**	**66**	**72**	**78**	**84**	**90**
7	**7**	**14**	**21**	**28**	**35**	**42**	**49**	**56**	**63**	**70**	**77**	**84**	**91**	**98**	**105**
8	**8**	**16**	**24**	**32**	**40**	**48**	**56**	**64**	**72**	**80**	**88**	**96**	**104**	**112**	**120**
9	**9**	**18**	**27**	**36**	**45**	**54**	**63**	**72**	**81**	**90**	**99**	**108**	**117**	**126**	**135**
10	**10**	**20**	**30**	**40**	**50**	**60**	**70**	**80**	**90**	**100**	**110**	**120**	**130**	**140**	**150**
11	**11**	**22**	**33**	**44**	**55**	**66**	**77**	**88**	**99**	**110**	**121**	**132**	**143**	**154**	**165**
12	**12**	**24**	**36**	**48**	**60**	**72**	**84**	**96**	**108**	**120**	**132**	**144**	**156**	**168**	**180**
13	**13**	**26**	**39**	**52**	**65**	**78**	**91**	**104**	**117**	**130**	**143**	**156**	**169**	**182**	**195**
14	**14**	**28**	**42**	**56**	**70**	**84**	**98**	**112**	**126**	**140**	**154**	**168**	**182**	**196**	**210**
15	**15**	**30**	**45**	**60**	**75**	**90**	**105**	**120**	**135**	**150**	**165**	**180**	**195**	**210**	**225**

Look at the orange row.

It is an arithmetic progression. The mean is the centre number and the total 5 times this.
Each number to the right of the ‘12’ exceeds 12 by the amount the matching number on the left falls short of it.

The same argument applies to the green column.

Now look at the blue cross.

Make the centre number the product *ab*. Working round the outside from the top left cell, we have:

$(a-1)(b-1) = ab - a - b + 1$,
$(a+1)(b-1) = ab - a + b - 1$,
$(a+1)(b+1) = ab + a + b + 1$,
$(a-1)(b+1) = ab + a - b - 1$.

Adding, we have $4ab$. We can confirm that any design with rotation symmetry of order 4, like the magenta catherine wheel, will sum to the centre number times the number of cells.
And this applies to the cyan square (in which we've embedded the magenta wheel), and the whole multiplication square. Thus if our square is $(2n + 1)\times(2n+1)$, the total is $(2n+1)^2\times$ the centre number $= (2n+1)^2\times(n+1)^2$ or
$[(n+1)(2n+1)]^2$.

But we also know from **29** that the total is ${T_{2n+1}}^2$.
$T_k = \frac{k(k+1)}{2}$. So $T_{2n+1} = \frac{(2n+1)(2n+2)}{2} = (n+1)(2n+1)$.
So ${T_{2n+1}}^2 = [(n+1)(2n+1)]^2$, matching our first expression.

All this applies when the number of cells is even, except in that case the mean will be an integer + 1/4.
The total for a square $2n$ by $2n = {T_{2n}}^2 = \left[\frac{2n(2n+1)}{2}\right]^2 = [n(2n+1)]^2$.
Since there are $[2n]^2$ cells in the square,
the mean = $\left[\frac{n(2n+1)}{2n}\right]^2 = \left[\frac{2n+1}{2}\right]^2 = \frac{4n^2+4n+1}{4} = \frac{4n(n+1)+1}{4} = n(n+1) + \frac{1}{4}$.

If the square is *p* cells by *p*, whether *p* is odd or even, the total is ${T_k}^2$.
And from **30** we have the result
${T_k}^2 = 1^3 + 2^3 + 3^3 + \ldots + k^3$.

42. *We identify tetrahedral numbers as differences of diagonal sums, and show how it works in the top left corner of the table, where the leading diagonal is a sum of squares, i.e. a pyramidal number.*

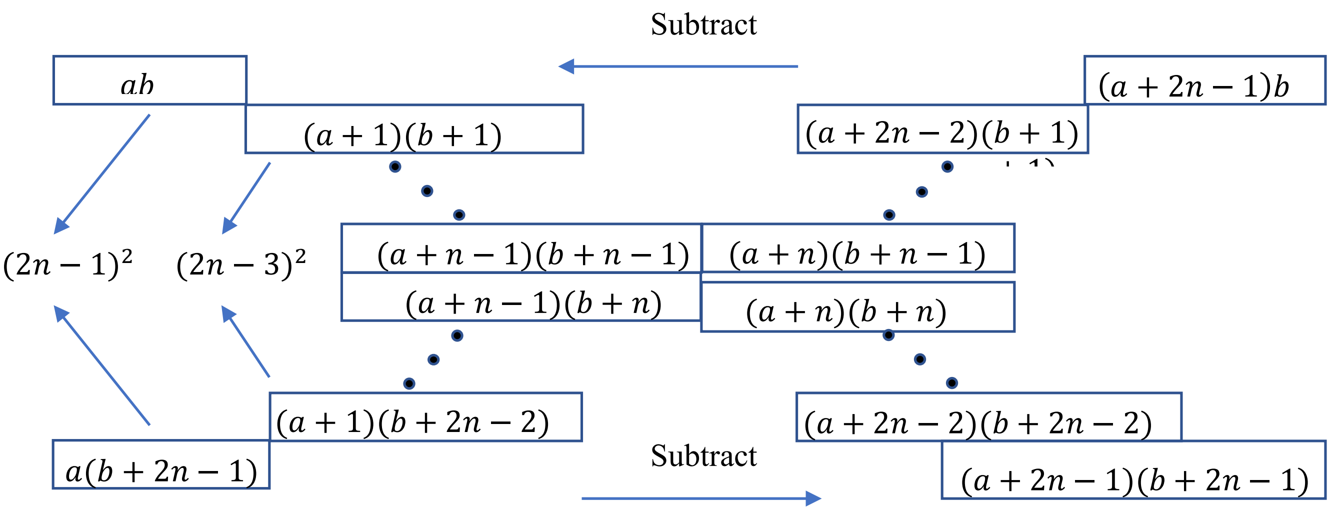
For this, difficult, investigation, begin with nrich.maths.org/2821.

X	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
2	**2**	**4**	**6**	**8**	**10**	**12**	**14**	**16**	**18**	**20**	**22**	**24**	**26**	**28**	**30**
3	**3**	**6**	**9**	**12**	**15**	**18**	**21**	**24**	**27**	**30**	**33**	**36**	**39**	**42**	**45**
4	**4**	**8**	**12**	**16**	**20**	**24**	**28**	**32**	**36**	**40**	**44**	**48**	**52**	**56**	**60**
5	**5**	**10**	**15**	**20**	**25**	**30**	**35**	**40**	**45**	**50**	**55**	**60**	**65**	**70**	**75**
6	**6**	**12**	**18**	**24**	**30**	**36**	**42**	**48**	**54**	**60**	**66**	**72**	**78**	**84**	**90**
7	**7**	**14**	**21**	**28**	**35**	**42**	**49**	**56**	**63**	**70**	**77**	**84**	**91**	**98**	**105**
8	**8**	**16**	**24**	**32**	**40**	**48**	**56**	**64**	**72**	**80**	**88**	**96**	**104**	**112**	**120**
9	**9**	**18**	**27**	**36**	**45**	**54**	**63**	**72**	**81**	**90**	**99**	**108**	**117**	**126**	**135**
10	**10**	**20**	**30**	**40**	**50**	**60**	**70**	**80**	**90**	**100**	**110**	**120**	**130**	**140**	**150**
11	**11**	**22**	**33**	**44**	**55**	**66**	**77**	**88**	**99**	**110**	**121**	**132**	**143**	**154**	**165**
12	**12**	**24**	**36**	**48**	**60**	**72**	**84**	**96**	**108**	**120**	**132**	**144**	**156**	**168**	**180**
13	**13**	**26**	**39**	**52**	**65**	**78**	**91**	**104**	**117**	**130**	**143**	**156**	**169**	**182**	**195**
14	**14**	**28**	**42**	**56**	**70**	**84**	**98**	**112**	**126**	**140**	**154**	**168**	**182**	**196**	**210**
15	**15**	**30**	**45**	**60**	**75**	**90**	**105**	**120**	**135**	**150**	**165**	**180**	**195**	**210**	**225**

In the each of the diagonal crosses we subtract the total of the red numbers from the total of the blue crosses. What do we find?

We shall distinguish crosses enclosed by a square with an even number, $2n$, of cells along an edge, and those enclosed by a square with an odd number of squares, $2n + 1$, along an edge.

2*n* cells

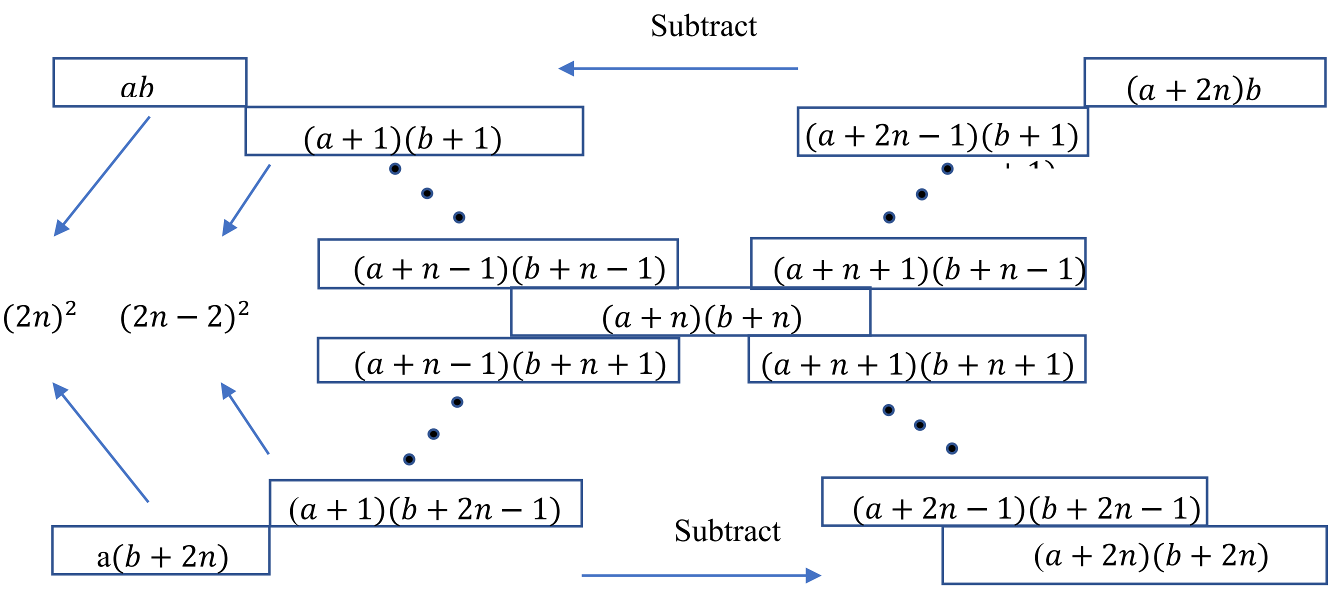


When we've made the subtractions shown, we find we can pair the expressions in such a way that we produce the sequence of odd squares down to 1 (produced by the block in the middle of the diagram). The square number S_k is the sum of the adjacent triangular numbers T_k and T_{k-1}. This is the result:

$$\begin{array}{cccccccc} \multicolumn{2}{c}{S_{2n-1}} & \multicolumn{2}{c}{S_{2n-3}} & \multicolumn{2}{c}{S_{2n-5}} & \cdots & S_1 \\ T_{2n-1} & T_{2n-2} & T_{2n-3} & T_{2n-4} & T_{2n-5} & T_{2n-6} & \cdots & T_1 \end{array}$$

We obtain all the triangular numbers from T_{2n-1} down to T_1. The sum of the first *k* triangular numbers is the *k*th tetrahedral number, Tet_k, so in our case Tet_{2n-1}

2*n* + 1 cells



This time we obtain the even squares, leading to Tet_{2n}.

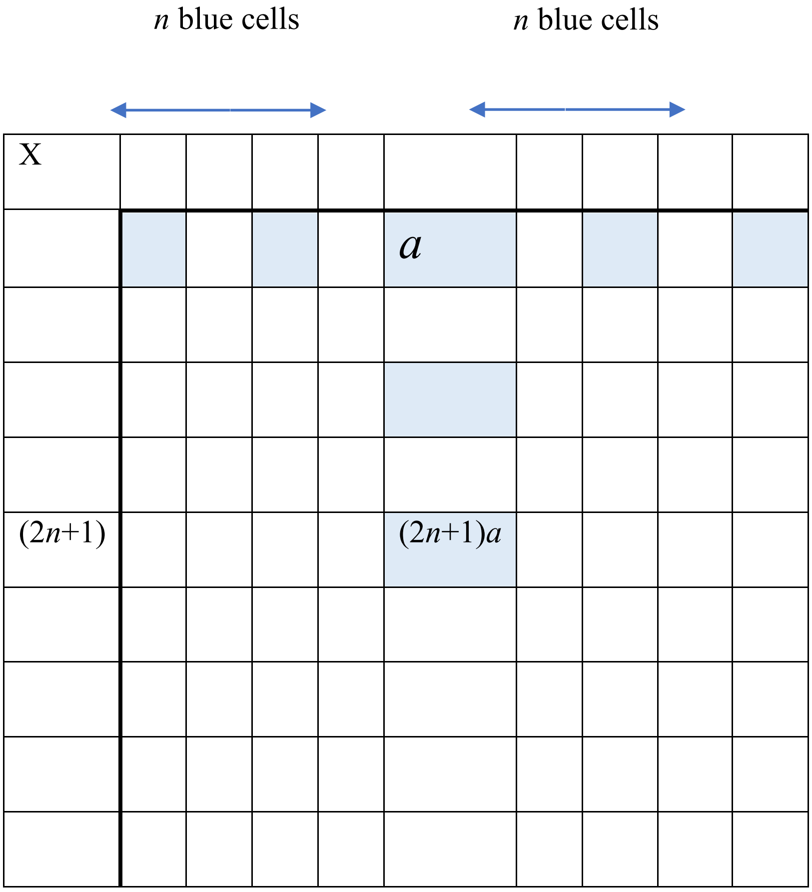
We can summarise the two results: if the enclosing square has side s, the difference of the sums is Tet_{s-1}.

Look at the cross in the top left hand corner of the multiplication square. The blue numbers are squares so they sum to the 6th pyramidal number, Pyr_6.
The red numbers must therefore sum to $Pyr_6 - Tet_5 = Tet_6$. This is what we would expect from **2.5.1**.

43. *We move a set square around on the multiplication square and find the following. We note the cells at the edge which the set square cuts and fill in alternate ones between, e.g. the pale blue squares below. We find that the right angle of the set square lies in the cell which is their sum.*

X	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
2	**2**	**4**	**6**	**8**	**10**	**12**	**14**	**16**	**18**	**20**	**22**	**24**	**26**	**28**	**30**
3	**3**	**6**	**9**	**12**	**15**	**18**	**21**	**24**	**27**	**30**	**33**	**36**	**39**	**42**	**45**
4	**4**	**8**	**12**	**16**	**20**	**24**	**28**	**32**	**36**	**40**	**44**	**48**	**52**	**56**	**60**
5	**5**	**10**	**15**	**20**	**25**	**30**	**35**	**40**	**45**	**50**	**55**	**60**	**65**	**70**	**75**
6	**6**	**12**	**18**	**24**	**30**	**36**	**42**	**48**	**54**	**60**	**66**	**72**	**78**	**84**	**90**
7	**7**	**14**	**21**	**28**	**35**	**42**	**49**	**56**	**63**	**70**	**77**	**84**	**91**	**98**	**105**
8	**8**	**16**	**24**	**32**	**40**	**48**	**56**	**64**	**72**	**80**	**88**	**96**	**104**	**112**	**120**
9	**9**	**18**	**27**	**36**	**45**	**54**	**63**	**72**	**81**	**90**	**99**	**108**	**117**	**126**	**135**
10	**10**	**20**	**30**	**40**	**50**	**60**	**70**	**80**	**90**	**100**	**110**	**120**	**130**	**140**	**150**
11	**11**	**22**	**33**	**44**	**55**	**66**	**77**	**88**	**99**	**110**	**121**	**132**	**143**	**154**	**165**
12	**12**	**24**	**36**	**48**	**60**	**72**	**84**	**96**	**108**	**120**	**132**	**144**	**156**	**168**	**180**
13	**13**	**26**	**39**	**52**	**65**	**78**	**91**	**104**	**117**	**130**	**143**	**156**	**169**	**182**	**195**
14	**14**	**28**	**42**	**56**	**70**	**84**	**98**	**112**	**126**	**140**	**154**	**168**	**182**	**196**	**210**
15	**15**	**30**	**45**	**60**	**75**	**90**	**105**	**120**	**135**	**150**	**165**	**180**	**195**	**210**	**225**

Along the top row we see below that we have $(2n + 1)$ cells. Their mean is a and therefore their total, $(2n + 1)a$. But we also see that the number beneath lies in row $(2n+1)$. So a is the multiplicand, $(2n + 1)$ the multiplier, the product therefore $(2n + 1)a$:



44. *We investigate the pairings of 'snakes' and 'ladders' on the multiplication square. Take two rows. For every 'ladder' joining a number in the first row to one k bigger in the second, there's a 'snake' joining a number in the first row to one k smaller in the second.*

On the square below we have chosen pairs of rows, for example 5 and 6. For each pair we have chosen a small number, in that case 2, and joined cells with a blue 'ladder' showing that number as an increase, and cells with a red 'snake' showing that number as a decrease. Symmetry dictates that the mean of the numbers at the heads of the arrows is equal to the mean of the numbers at the tails of the arrows, in our example, 30. We've shown these numbers in green. Correspondingly, the green cells are equidistant from the blue and red cells in their row. In the case of rows 13 and 14 we see that the mean does not label a cell in row 14 since it falls midway between 84 and 98. In rows 2 and 4 the snake and ladder arrive at the same square.

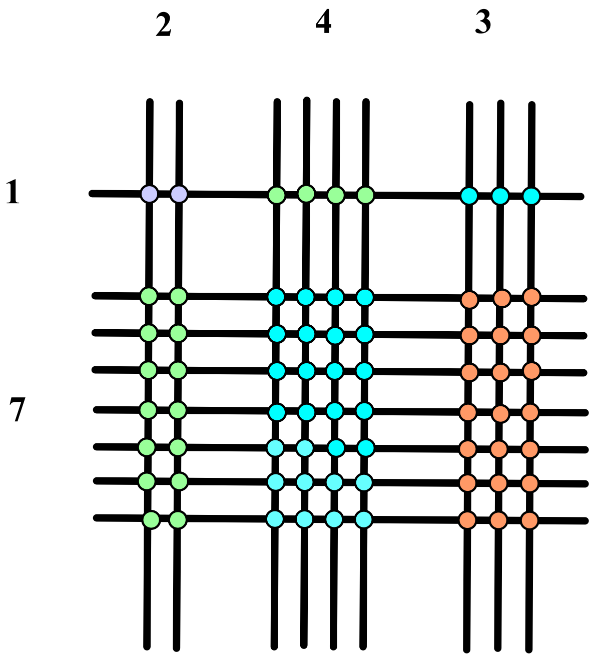
X	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
2	**2**	**4**	**6**	**8**	**10**	**12**	**14**	**16**	**18**	**20**	**22**	**24**	**26**	**28**	**30**
3	**3**	**6**	**9**	**12**	**15**	**18**	**21**	**24**	**27**	**30**	**33**	**36**	**39**	**42**	**45**
4	**4**	**8**	**12**	**16**	**20**	**24**	**28**	**32**	**36**	**40**	**44**	**48**	**52**	**56**	**60**
5	**5**	**10**	**15**	**20**	**25**	**30**	**35**	**40**	**45**	**50**	**55**	**60**	**65**	**70**	**75**
6	**6**	**12**	**18**	**24**	**30**	**36**	**42**	**48**	**54**	**60**	**66**	**72**	**78**	**84**	**90**
7	**7**	**14**	**21**	**28**	**35**	**42**	**49**	**56**	**63**	**70**	**77**	**84**	**91**	**98**	**105**
8	**8**	**16**	**24**	**32**	**40**	**48**	**56**	**64**	**72**	**80**	**88**	**96**	**104**	**112**	**120**
9	**9**	**18**	**27**	**36**	**45**	**54**	**63**	**72**	**81**	**90**	**99**	**108**	**117**	**126**	**135**
10	**10**	**20**	**30**	**40**	**50**	**60**	**70**	**80**	**90**	**100**	**110**	**120**	**130**	**140**	**150**
11	**11**	**22**	**33**	**44**	**55**	**66**	**77**	**88**	**99**	**110**	**121**	**132**	**143**	**154**	**165**
12	**12**	**24**	**36**	**48**	**60**	**72**	**84**	**96**	**108**	**120**	**132**	**144**	**156**	**168**	**180**
13	**13**	**26**	**39**	**52**	**65**	**78**	**91**	**104**	**117**	**130**	**143**	**156**	**169**	**182**	**195**
14	**14**	**28**	**42**	**56**	**70**	**84**	**98**	**112**	**126**	**140**	**154**	**168**	**182**	**196**	**210**
15	**15**	**30**	**45**	**60**	**75**	**90**	**105**	**120**	**135**	**150**	**165**	**180**	**195**	**210**	**225**

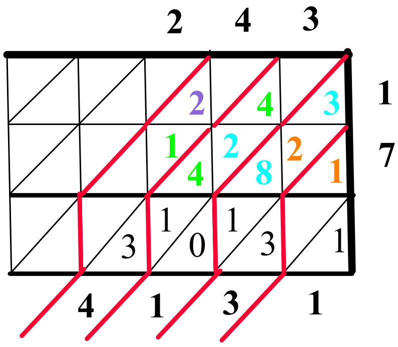
45 exploits the idea of *place value*, the principle on which our number system is based.

45. *We find a way to use a small square to multiply big numbers.*

We can extend the range of our square.

In our place value number system each digit multiplies a different power of 10.
For example, 243 = $2\times10^2 + 4\times10^1 + 3$, 17 = $1\times10^1 + 7$.
As a result, when we multiply the two numbers, each product of digits multiplies a particular power of 10.
For example, the '2' in the first number times the '1' in the second number multiply $10^2 \times 10^1 = 10^{2+1} = 10^3$.
We can represent the product by bunching sticks in this way:





On the right we have collected up the 6 products in the cells upper right. Where they exceed 10, there is a 'carry' but, by taking advantage of the diagonal stripes, we ensure that these line up correctly. The boxes in the third row total the stripes. But again there may be a carry so we need another set of diagonal stripes in these before we can make the final addition in bold print.

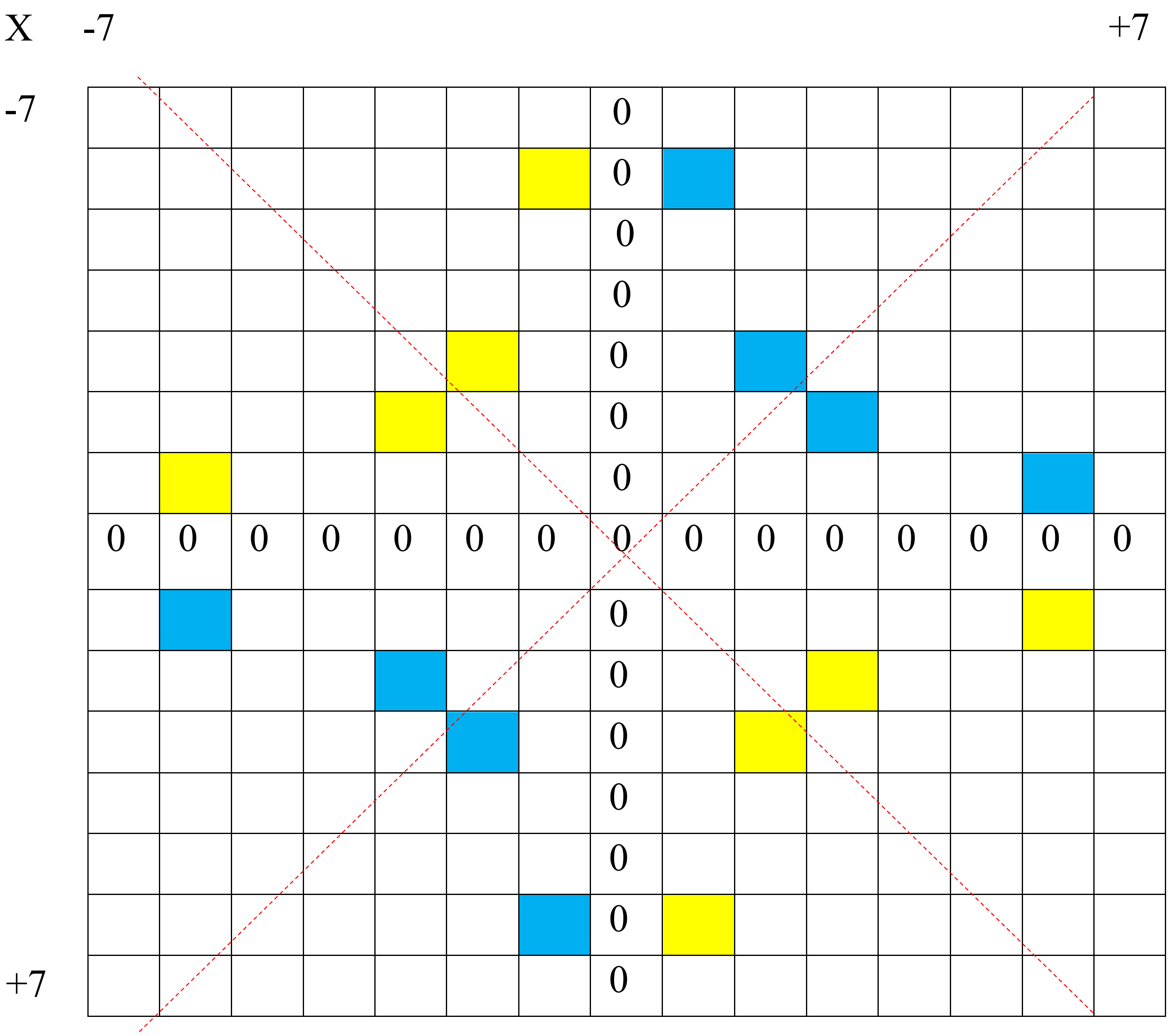
In the 'Cuisenaire product finder' you have a a strip of acetate, the width of a row or column on the multiplication square, of a distinct colour for each decimal place. In the above example, where the '4' strip crosses the '7' strip, the physical process of colour subtraction will result in a new colour, representing the number of 10s, 28. This is the same as the colour observed where the '3' strip crosses the '1' strip, 3. Where strips of the same colour cross, in our case for example the '4' and the '1', both of which stand for 10s, there is a deepening of the shade. Where the '2' strip crosses the '7' strip would ideally give the same colour as where the '4' strip crosses the '1' strip, as in our stick diagram, but that's not physically possible. The result is 6 different colours or shades. It remains to enter the 6 totals in a lattice like ours to compute the result. Here is the square with those settings:

X	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
2	**2**	**4**	**6**	**8**	**10**	**12**	**14**	**16**	**18**	**20**	**22**	**24**	**26**	**28**	**30**
3	**3**	**6**	**9**	**12**	**15**	**18**	**21**	**24**	**27**	**30**	**33**	**36**	**39**	**42**	**45**
4	**4**	**8**	**12**	**16**	**20**	**24**	**28**	**32**	**36**	**40**	**44**	**48**	**52**	**56**	**60**
5	**5**	**10**	**15**	**20**	**25**	**30**	**35**	**40**	**45**	**50**	**55**	**60**	**65**	**70**	**75**
6	**6**	**12**	**18**	**24**	**30**	**36**	**42**	**48**	**54**	**60**	**66**	**72**	**78**	**84**	**90**
7	**7**	**14**	**21**	**28**	**35**	**42**	**49**	**56**	**63**	**70**	**77**	**84**	**91**	**98**	**105**
8	**8**	**16**	**24**	**32**	**40**	**48**	**56**	**64**	**72**	**80**	**88**	**96**	**104**	**112**	**120**
9	**9**	**18**	**27**	**36**	**45**	**54**	**63**	**72**	**81**	**90**	**99**	**108**	**117**	**126**	**135**
10	**10**	**20**	**30**	**40**	**50**	**60**	**70**	**80**	**90**	**100**	**110**	**120**	**130**	**140**	**150**
11	**11**	**22**	**33**	**44**	**55**	**66**	**77**	**88**	**99**	**110**	**121**	**132**	**143**	**154**	**165**
12	**12**	**24**	**36**	**48**	**60**	**72**	**84**	**96**	**108**	**120**	**132**	**144**	**156**	**168**	**180**
13	**13**	**26**	**39**	**52**	**65**	**78**	**91**	**104**	**117**	**130**	**143**	**156**	**169**	**182**	**195**
14	**14**	**28**	**42**	**56**	**70**	**84**	**98**	**112**	**126**	**140**	**154**	**168**	**182**	**196**	**210**
15	**15**	**30**	**45**	**60**	**75**	**90**	**105**	**120**	**135**	**150**	**165**	**180**	**195**	**210**	**225**

In **46** and **47** we extend the multiplication square to the signed integers.

46. *What patterns do we find?*

The blue cells contain the product + 6, the orange cells, - 6.



Picture this as a landscape, where ‘0’ is sea level. We see two limbs of the yellow rectangular hyperbola, two limbs of the blue one. The yellow contour shows land lying as far above sea level as the blue contour shows land lying below it. The model in the photgraph below uses Centicubes for the product towers.

47. *Which of the relations we have found in **40, 41** hold on the enlarged square?*

The symmetry of the table ensures that the relations we found in **40** and **41** still hold.

X	-7														+7
-7								0							
			30					0	-6						
								0							
			20					0	-4						
								0			-9				
								0		-4					
								0	-1						
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
					-3			0	1		3		5		
						-4		0		4				12	
							-3	0	3		9				21
								0				16			
								0			15				
								0		12					
+7								0	7						

Upper KS4

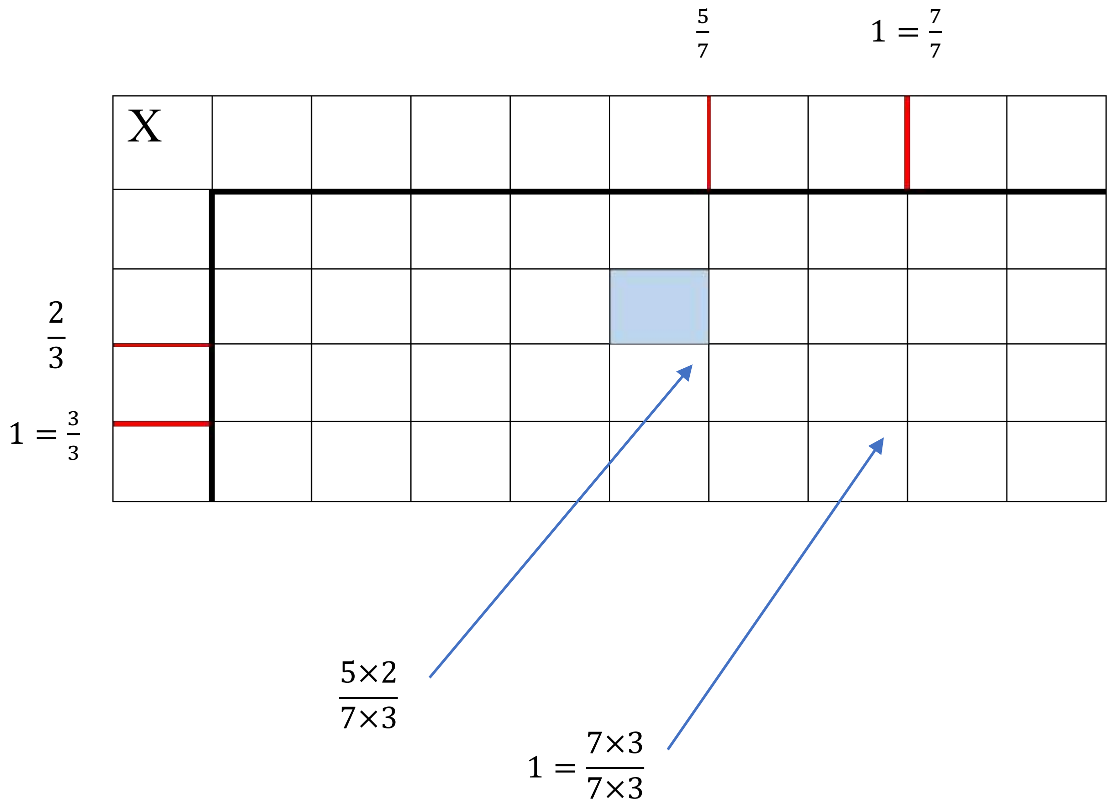
In **48**, our final activity, we fill in the gaps in our ‘integer’ surface by showing the products of all the real numbers.

48. *Where are the hyperbolas? Where are the parabolas?*

The multiplication square we used for the negative integers was pixelated: we used cells for the integers. If we are to fill in all the real numbers, for each axis we must swap the counting strip for the number line, where each number is represented by a point. To put it another way, the resolution of our pixelated image must be infinite.

We begin with the rationals.

On the first square we show the product of 5/7 and 2/3. We could retain the counting strips, so that the blue cell represents a product. For teaching purposes this is handy since we can determine numerator and denominator simply by counting cells, as we can for the integers. The snag is, we need a different strip for each denominator. Better then to use number lines, so that our product is the vertex arrowed in the bottom right hand corner:

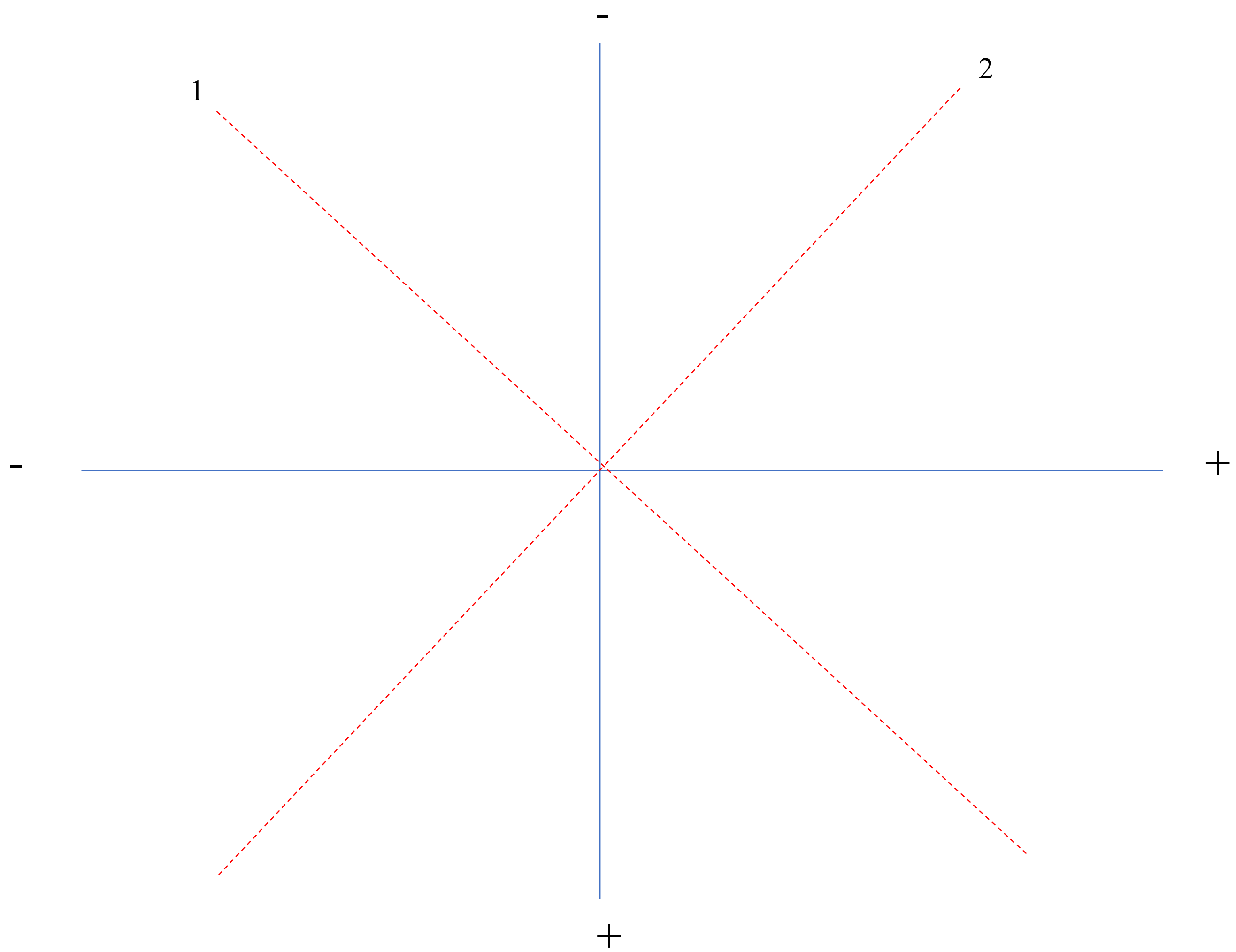


The same remarks apply to terminating decimals.

When we consider irrational numbers, by definition no cell is small enough, so we must use number lines.

The landscape we identified in **3.1** is now continuous. In the two planes of symmetry containing the square numbers we can now trace parabolas. In fact we can do so in all the planes parallel to these. The result is a surface whose contour lines are rectangular hyperbolas and whose vertical transects are parabolas: the hyperbolic paraboloid.

The red dashed lines indicate the vertical planes of symmetry. Parabolas with their vertices directed downwards lie in planes parallel to '1'; those the other way up, in planes parallel to '2'. The blue lines indicate planes which contain the asymptotes of the rectangular hyperbolas, which lie in horizontal planes.



This picture shows the surface for the signed integers on the left, the reals on the right. In the second model we have taken advantage of the fact that the hyperbolic paraboloid is a ruled surface. We can think of a ruling as a particular 'times table' (singular), though a right triangle rather than a staircase since we have filled in all the numbers on the number line.