# Changing one uniform tiling into another

Among the most commonly studied tilings are those where every tile is a regular polygon and every vertex is the same: *uniform* tilings. Because each vertex is the same, we can name the tilings in a simple way: we list in order the polygons we meet as we make a circuit of a vertex. This way, for example, we can identify and distinguish the two tilings of squares and triangles: 3.3.3.4.4 and 3.3.4.3.4. If we cycle the numbers or reverse their order, we still get the same tilings. Only five regular polygons contribute to uniform tilings. Their interior angles are all multiples of 15°. In this diagram each tiling is an annulus and the polygons which pack round a vertex are shown with their correct interior angles:
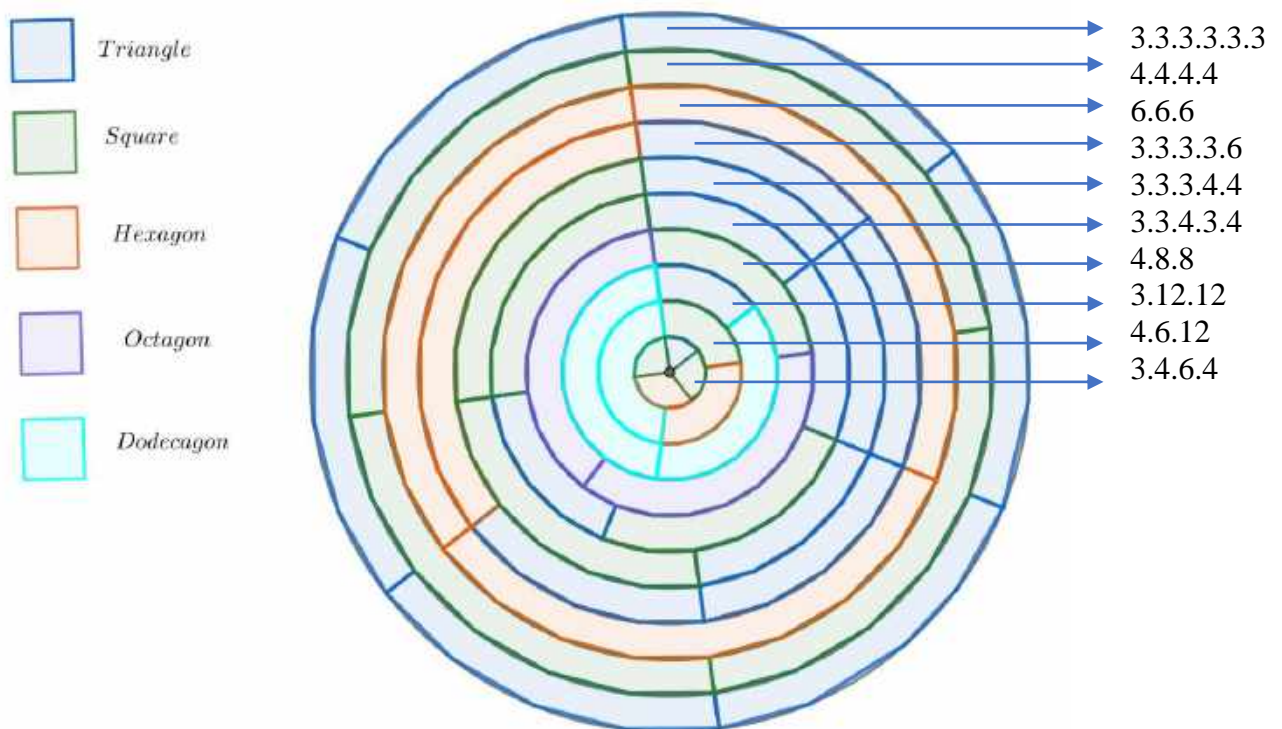
The centres of rotation symmetry are of only four orders: 2, 3, 4 & 6. These facts suggest that it should be possible to derive one tiling from another by a simple change. This will be our task in this investigation. The operations we shall use are our own. Other authors use different ones. We shall distinguish *static* operations, where we derive the new tiling by picking out features on the old one, from *dynamic* operations, where we create the new tiling by moving something.

Unless specified otherwise, 'triangles' will mean 'equilateral triangles', 'hexagons', 'regular hexagons', etc.

(A) Static operations

1. Adding and removing tiles

(a) In the first example all we do is remove tiles from 3.3.3.3.3.3, the isometric grid. (Equivalently we could remove edges.) The result is 3.3.3.3.6. The hexagons are bordered

with triangles. The tiling has a left- and right-handed form, shown here in blue and green. First we mark the centres of rotation symmetry and their orders. The rhombuses are lattice cells which produce the whole tiling by translation.

3.3.3.3.6
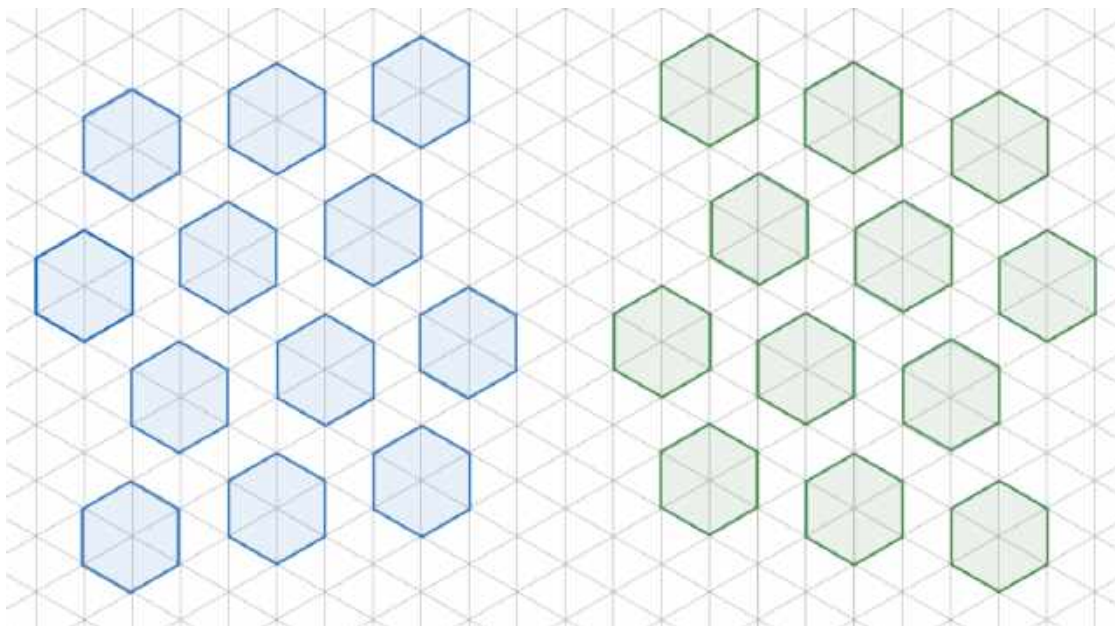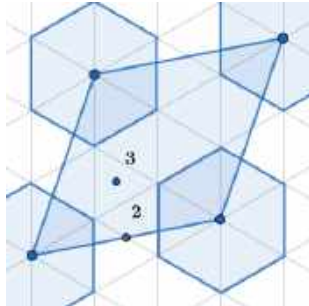
(b) In the next three examples we begin with 3.12.12. We then inscribe a square in a 12-gon. If we inscribe squares in neighbouring 12-gons so that each shares a vertex with the last, we find we have produced 3.4.6.4. Finally we remove squares to leave 12-gons again, but larger ones this time, bordered by the remaining squares and hexagons, to form 4.6.12.
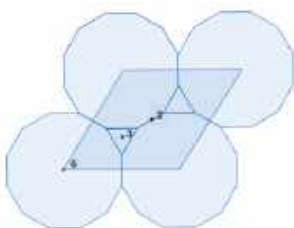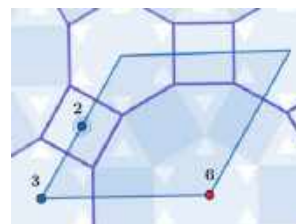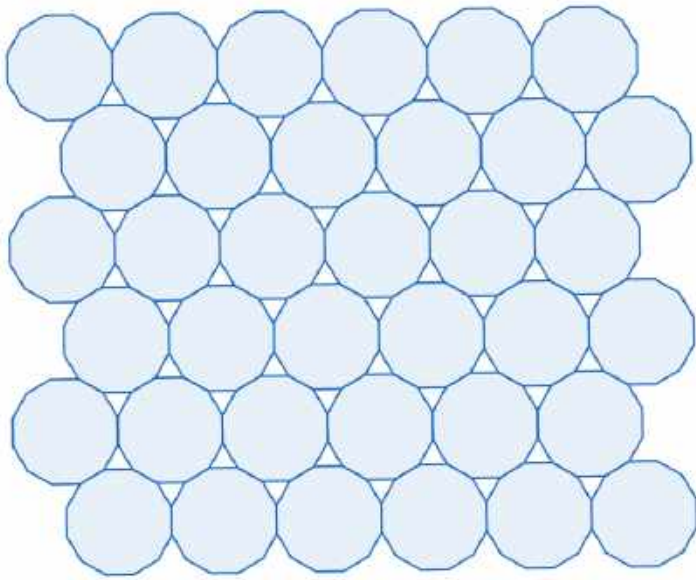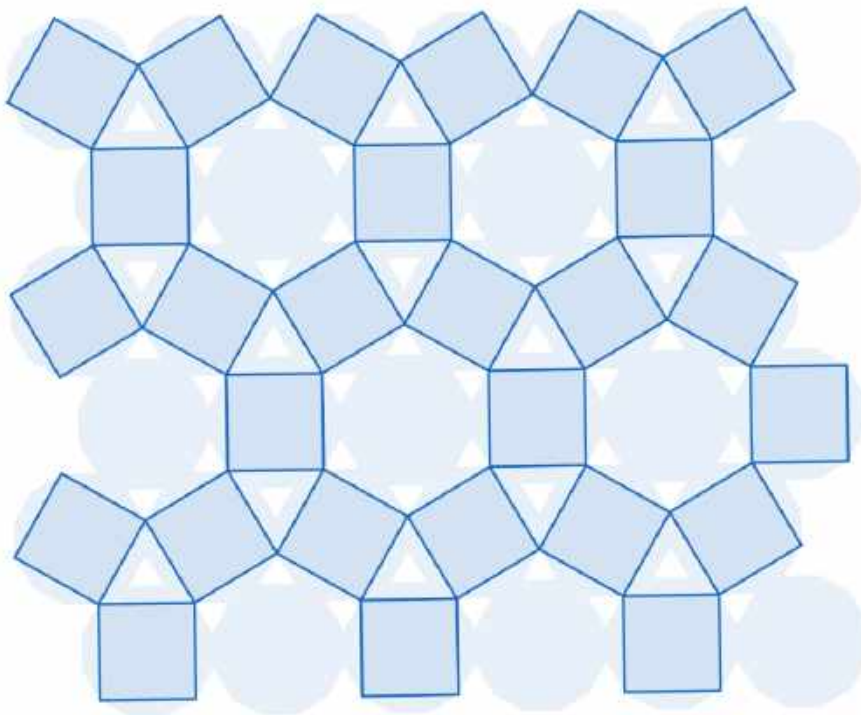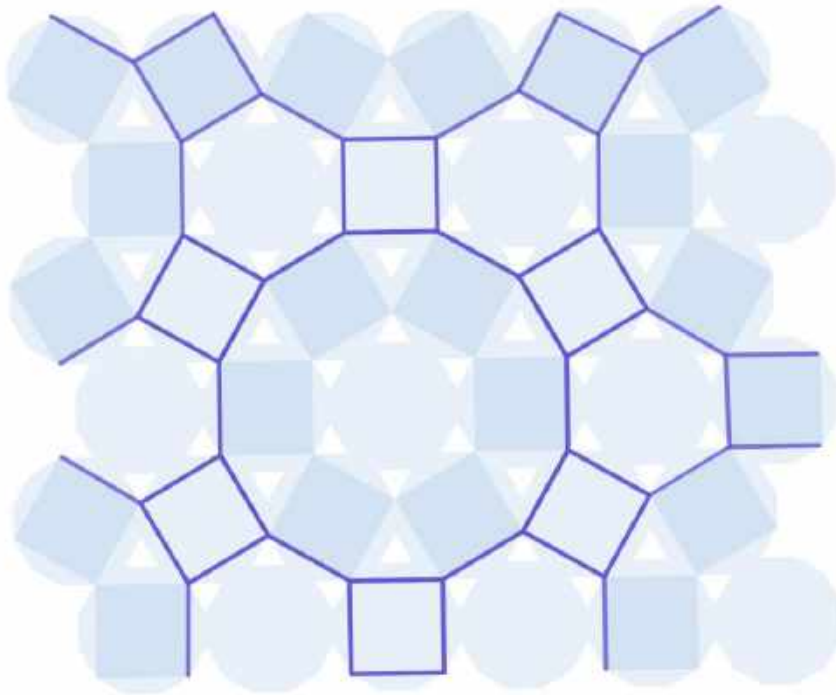
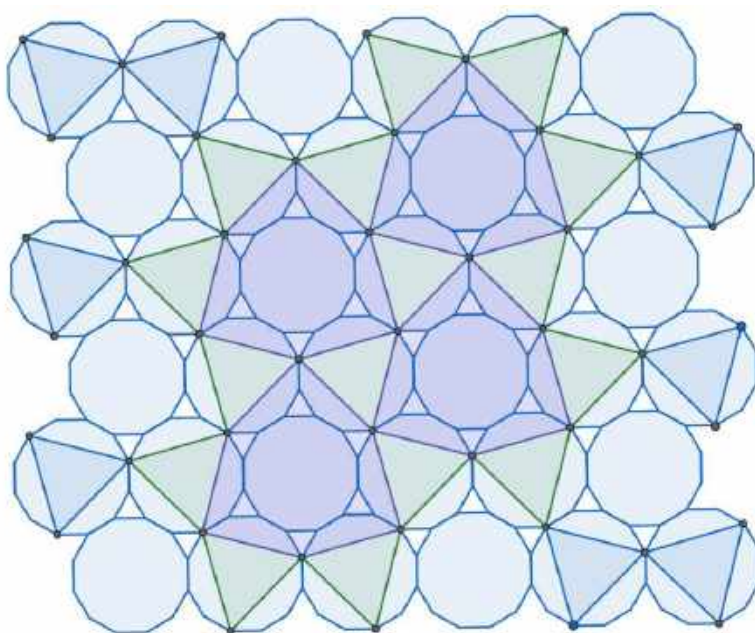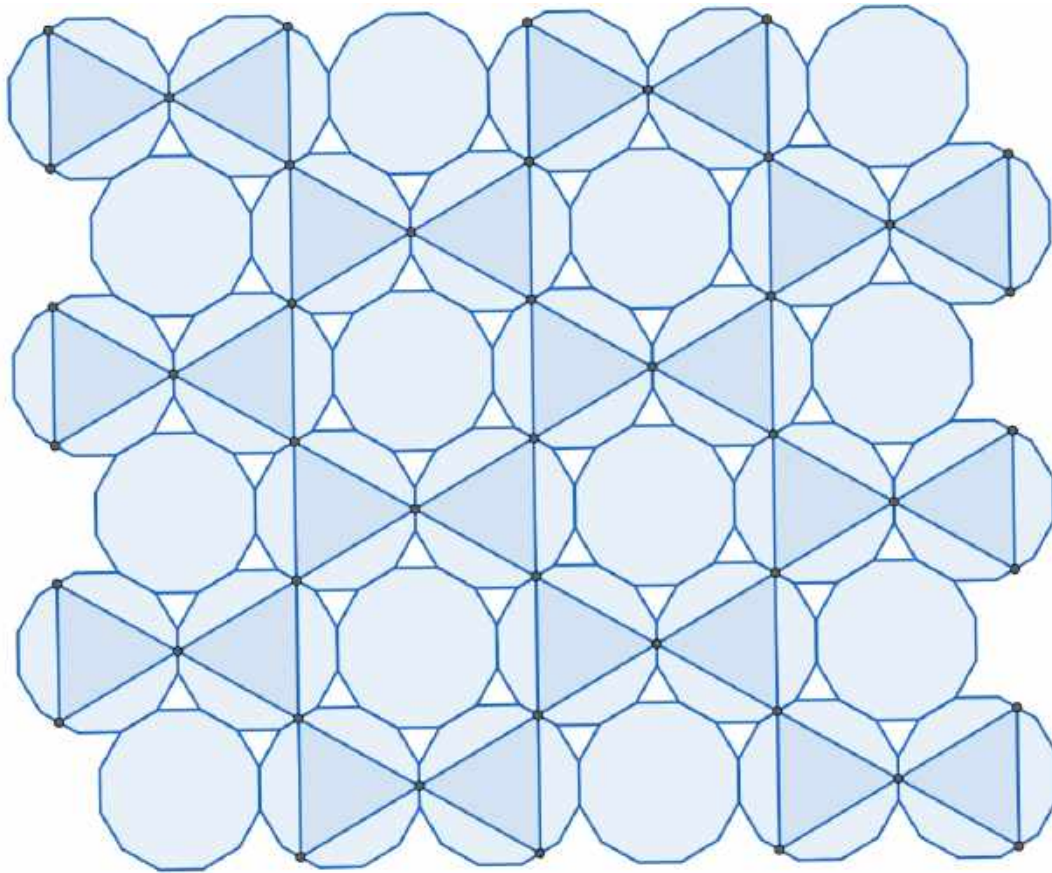| 3.12.12 | 3.4.6.4 | 4.6.12 |

3.12.12:

3.4.6.4:

4.6.12:

If we inscribe triangles in the 12-gons of 3.12.12 instead of squares, we get this distorted 3.6.3.6, a tiling of triangles and hexagons with equal sides and alternating angles of 90° and 150°:

For 3.6.3.6 itself we need to locate the triangle vertices at midpoints of 12-gon sides:


It is always possible to remove tiles (or equivalently edges) from a uniform tiling to create a motif which tiles the plane by translation. Here our original is 3.3.4.3.4:
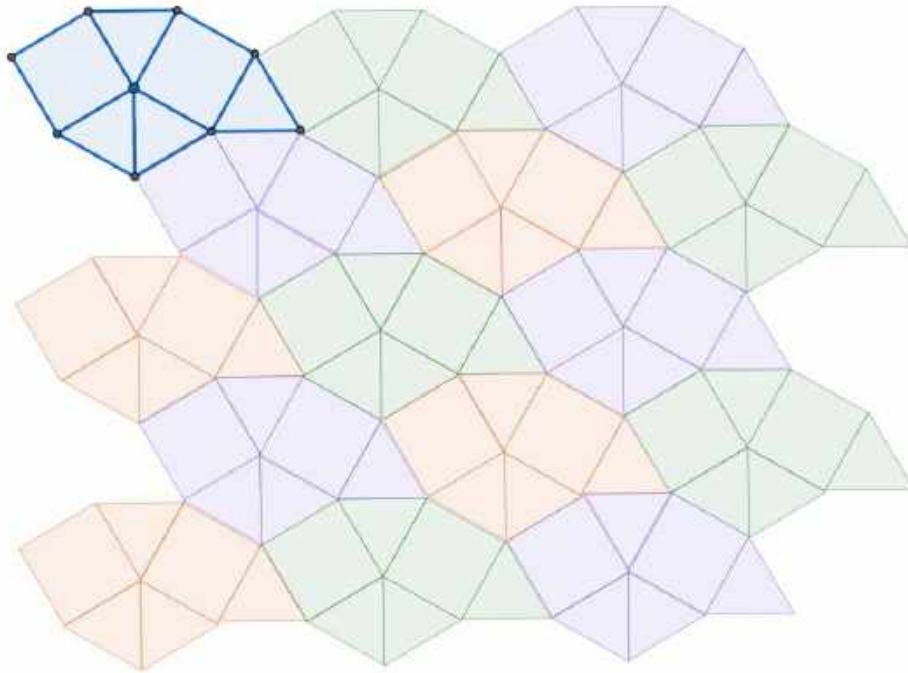
2. Joining tile centres

In section **1** we preserved edges in moving from one tiling to the next. In this section we join centres.

(a) Again we start with 3.12.12. In the upper part of the figure we join 12-gon centres to form 3.3.3.3.3.3. In the lower part we join triangle centres to form 6.6.6. In the middle we show that one set of edges bisects the other set at right angles. We say that 3.3.3.3.3.3 and 6.6.6 are *dual* tilings. This being the case, the symmetry elements are the same and identically arranged so we can show them on the same figure:
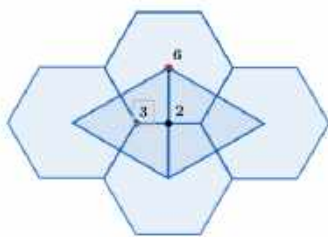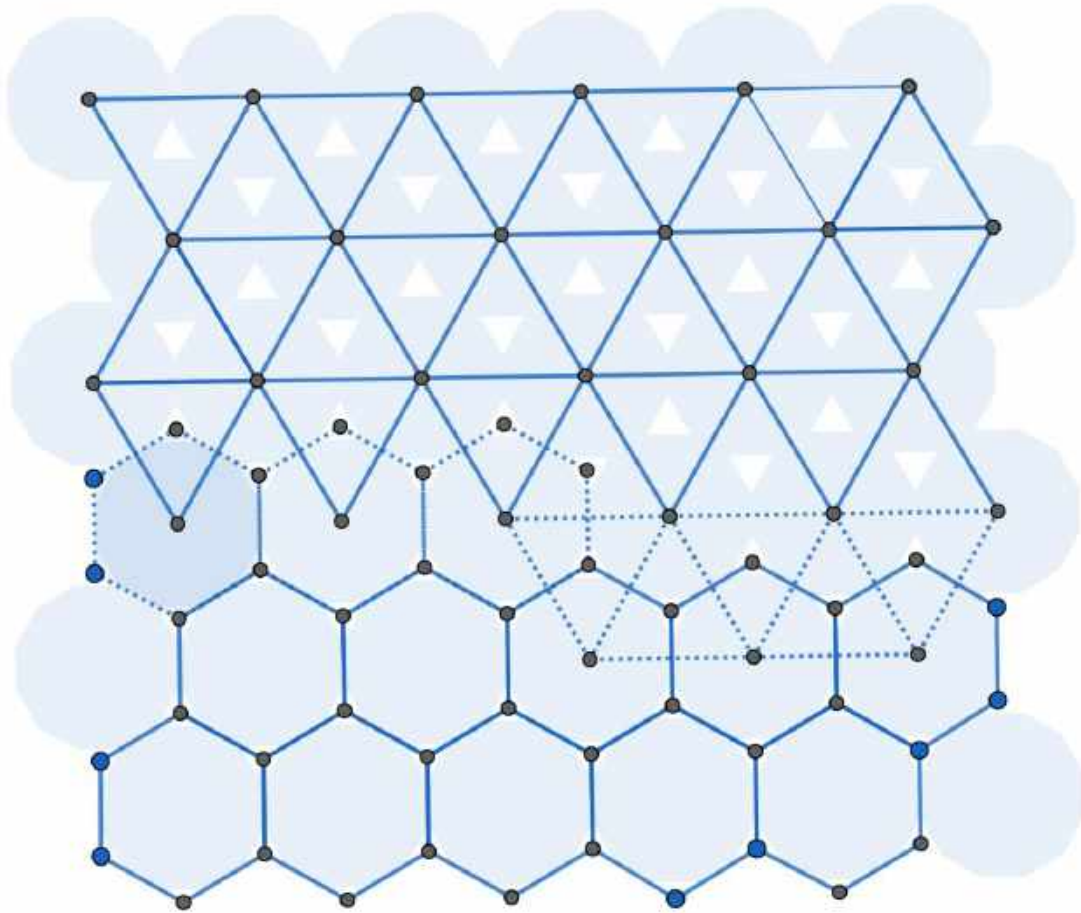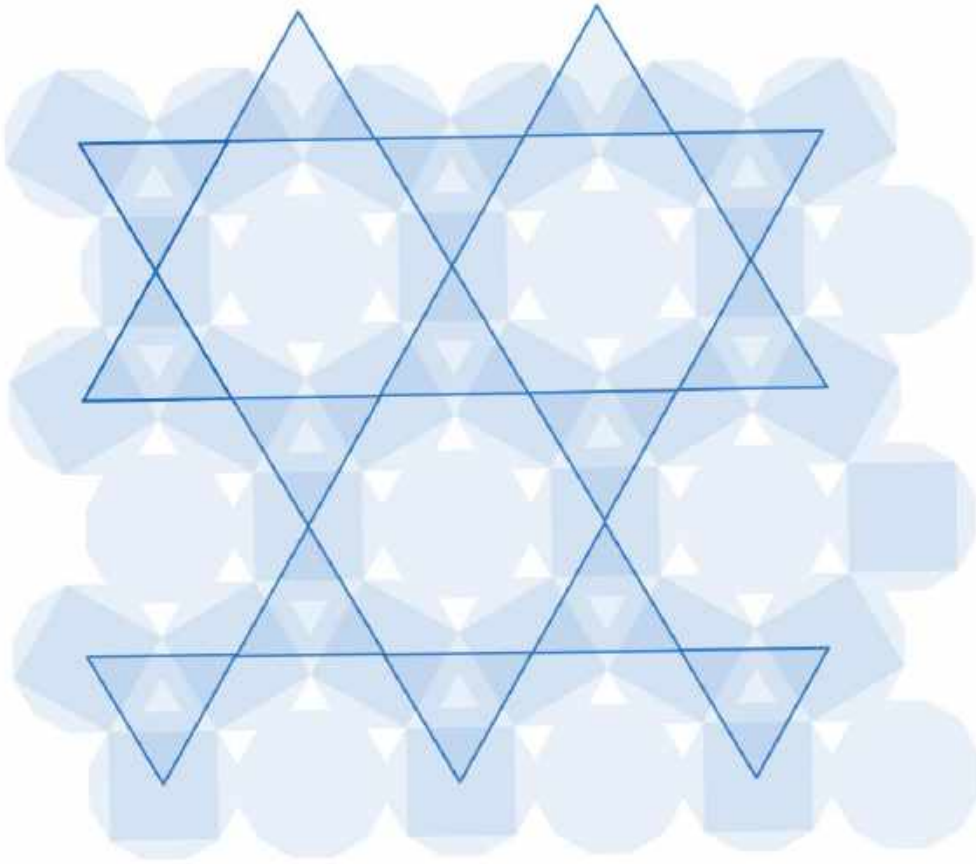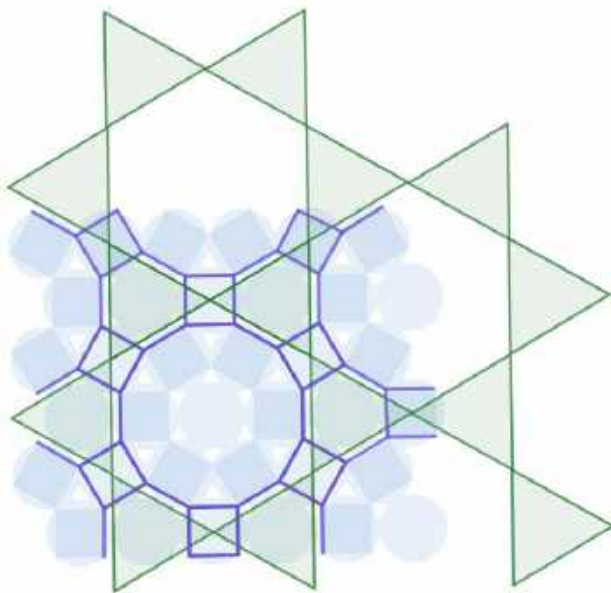
What happens if we join polygon centres in the tilings we derived from 3.12.12?

(b) If we join the square centres in 3.4.6.4, we get 3.6.3.6.
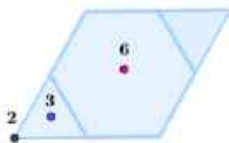
If we join the square centres in 4.6.12, we also get 3.6.3.6, but with an increase in scale:

3. Using algebra

The code for a tiling guarantees that the interior angles round a vertex sum to 360°. Unfortunately, this *necessary* condition for a tiling is not *sufficient.* But let us see how many valid tilings we can derive just using the associative and commutative laws of algebra and cycling the numbers.

We have these angular equivalences: $(3.3) \equiv 6$, $(3.4) \equiv 12$, $(4.4.4) \equiv (8.8)$, $(4.4) \equiv (3.6)$.
Blindly following the algebra, this gives us:

$3.3.3.3.3.3 \equiv 3.3.3.3.(3.3) \equiv 3.3.3.3.6.$
$\quad\quad\quad \equiv 3.3.(3.3).(3.3) \equiv 3.3.6.6$: not a valid tiling.
$\quad\quad\quad \equiv 3.(3.3).3.(3.3) \equiv 3.6.3.6.$
$3.3.3.4.4 \equiv (3.3).(3.4).4 \equiv 6.12.4 \equiv 4.6.12.$
$\quad\quad\quad \equiv 3.3.(3.4).4 \equiv 3.3.12.4$: not a valid tiling.
$\quad\quad\quad \equiv 3.3.3.(4.4) \equiv 3.3.3.(3.6) \equiv 3.3.3.3.6.$
$\quad\quad\quad \equiv 3.3.(3.4).4 \equiv 3.3.(4.3).4 \equiv 3.3.4.3.4.$
$\quad\quad\quad\quad\quad\quad\quad \equiv 3.4.3.3.4 \equiv 3.4.(3.3).4 \equiv 3.4.6.4.$
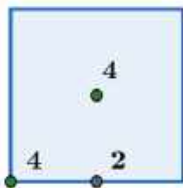$\quad\quad\quad\quad\quad\quad\quad \equiv 3.(3.4).(3.4) \equiv 3.12.12.$
$4.4.4.4 \equiv (4.4).(4.4) \equiv (3.6).(3.6) \equiv 3.6.3.6.$
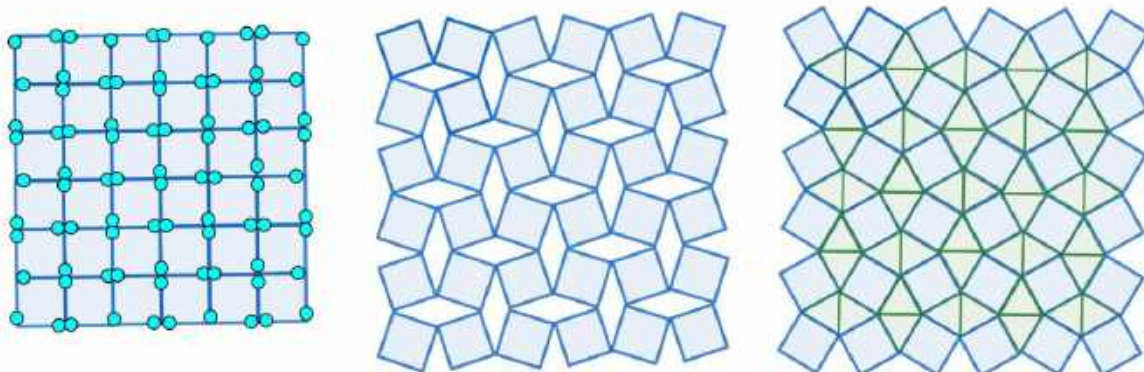$4.4.4.4 \equiv 4.(4.4.4) \equiv 4.(8.8) \equiv 4.8.8.$

(B) Dynamic operations

1. Hinged tiles

**(a)** We start with 4.4.4.4, a grid of squares.

We hinge the square tiles at corners. The cyan dots show which pairs are joined. We stretch the grid vertically. Rhombic holes develop until these have an angle of $60°$, at which point we have pairs of equilateral triangles and consequently the tiling 3.3.4.3.4:

If we continue to stretch the tiling, we pass through a stage where again we have 4.4.4.4 but each tile is bordered by 4 square holes. Past this point we go through a stage where we return to 3.3.4.3.4, the rhombuses lying at right angles to their former orientation, until finally the grid closes up again.

Here is how to set up a GeoGebra animation.
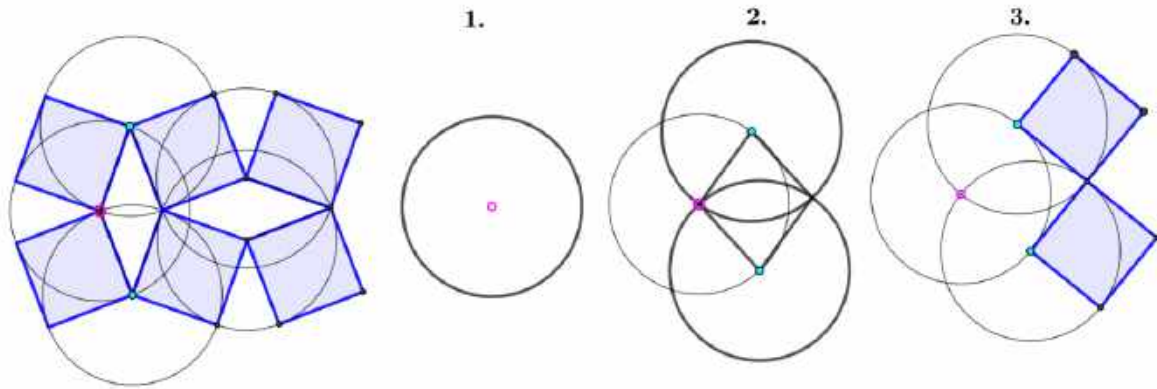
1. Decide on a unit of length and use 'Circle: centre & radius'. The pink centre will be your fixed point.

2. On the circumference centre two equal circles. 'Intersect' these new circles. You have now got your basic rhombus.

3. Using 'Regular polygon' you can construct squares outwards on all four sides, of which the two on the right are shown.

If you now centre two more equal circles on their right hand vertices and intersect them, you will be able to complete a rhombus which moves 90° out of phase with the first.

Move either of the cyan points to articulate the figure through the whole cycle described above.
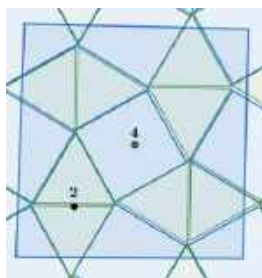

Returning to **(A) 1**, it's interesting to start with 4.8.8 instead of 3.12.12. If again we inscribe squares in the octagons joined according to our previous rule, we get this:
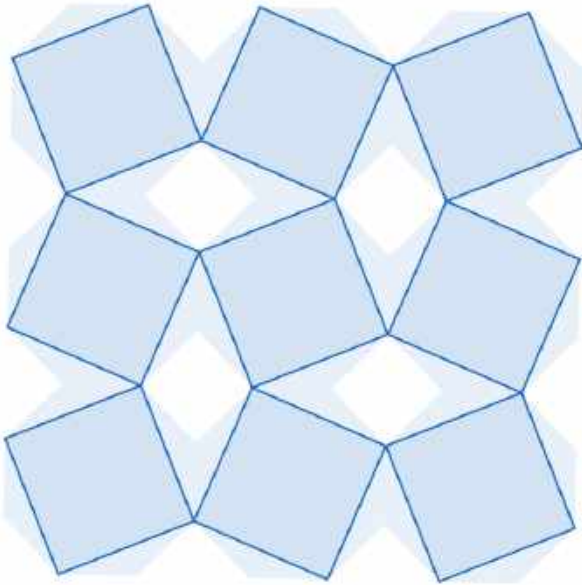
It is a stage in the unfolding of our grid of square tiles where the angle of the rhombic hole is 45°.

(b) For our second example under this heading we begin again with 4.4.4.4 but omit certain squares. A cyan dot shows a join. A red circle shows that there is no join. Note in particular the case where a tile is joined to another diagonally but not to the tile between them. The result is 3.4.6.4. Notice that the grid to be flexed has translational symmetry only.
But, bearing in mind the symmetry of the result, we realise that we could give the figure quarter-turns and draw it in any of those four orientations. We deform the grid uniformly until we find the detached vertices joining up. The colouring shows how each tromino becomes a triangle with a square on each side.
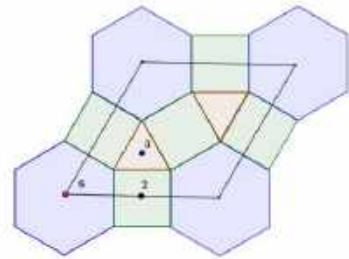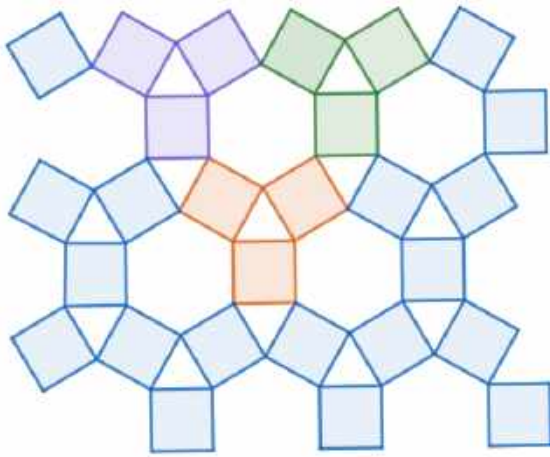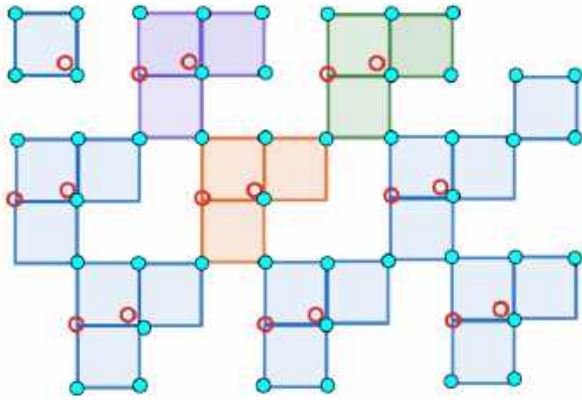
This is how to animate the mechanism using GeoGebra:
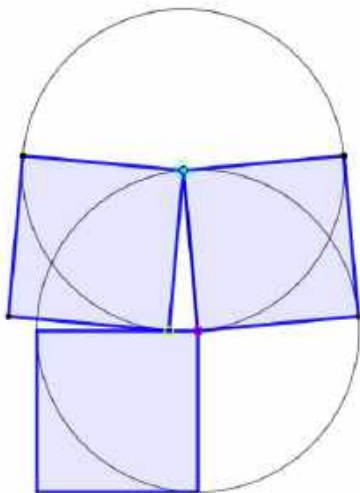
Using 'Circle: centre & radius' and 'Regular polygon', construct the pink-centred circle and square lower left.

On another radius construct the square upper right.

Construct an equal circle centred on the cyan vertex.

'Intersect' the new circle and the square bottom left.

Construct the third square.

Move the cyan point and the green vertex will slide on the upper edge of the first square.

2. Hinged rods

**(a)** This first example is taken from Hugo Steinhaus.

Again we begin with a grid of squares. But this time they have no interiors. We omit edges so as to leave staggered dominoes, like the bricks in a wall. We stretch the grid horizontally. The result is 6.6.6:
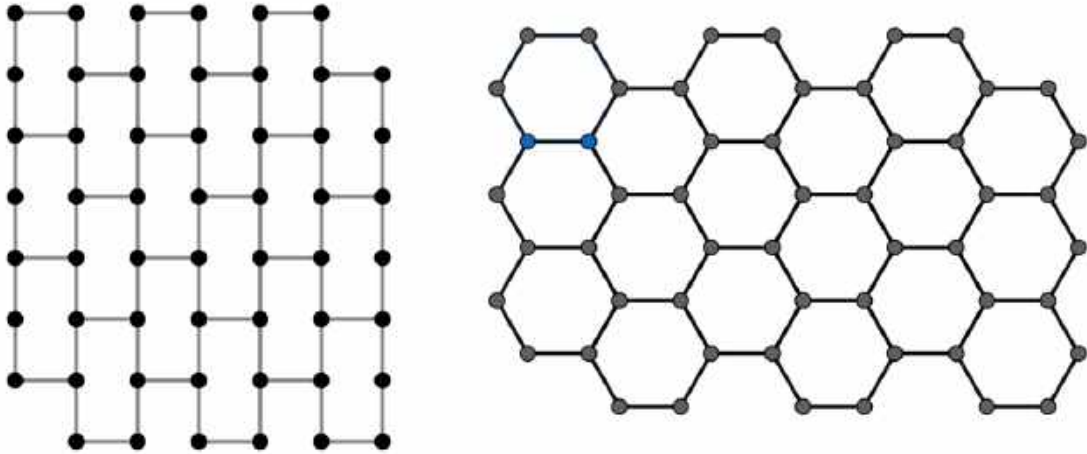

(b) Now for stage 2. We take Steinhaus' 6.6.6. We preserve hexagons separated by one edge and distort the others into triangles two units on a side. Their subdivision into 4 smaller triangles gives us 3.3.3.3.6 again:
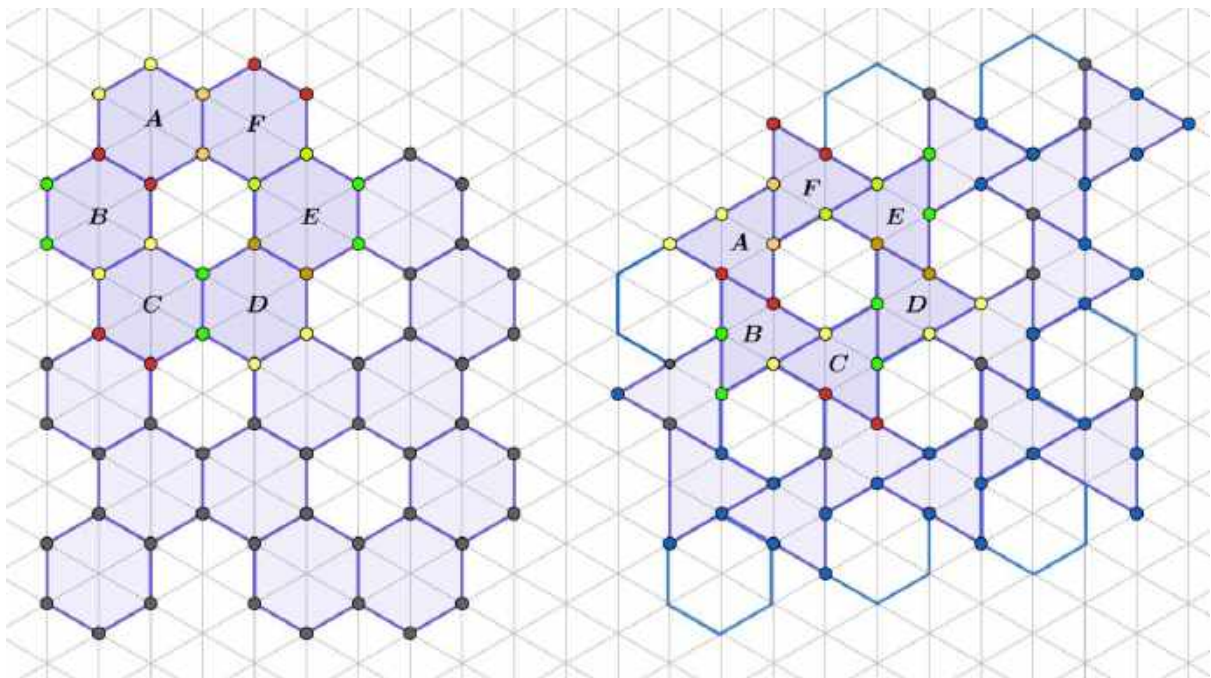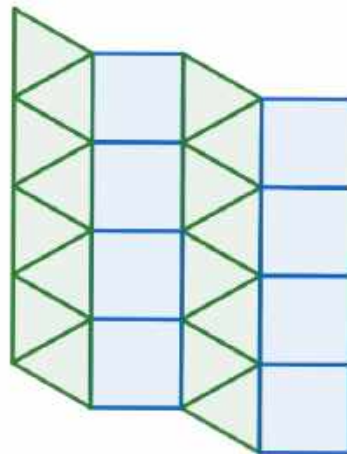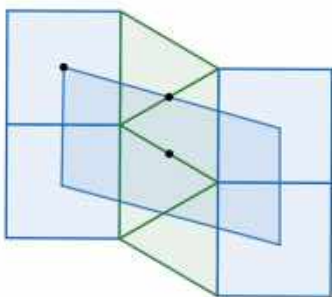

We demonstrate this transformation with Geomag:
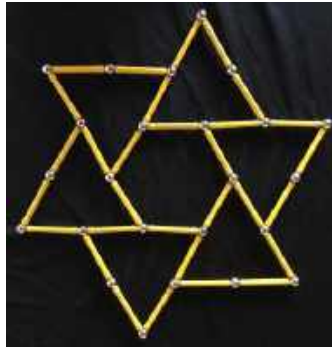
(c) As an alternative stage 2, we continue to stretch 6.6.6 horizontally until the inclined edges meet at 60°. At this point each hexagon consists of a central square bordered by two triangles and we have 3.3.3.4.4:
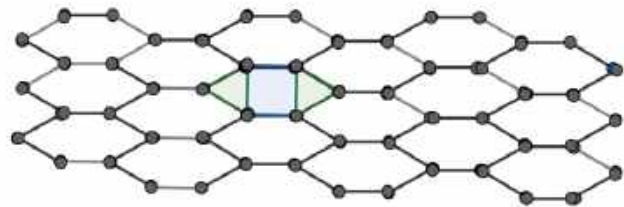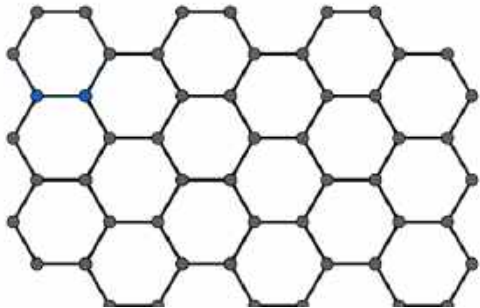

The orders of rotation symmetry are 2 in each case.

With GeoGebra one way to animate the grid is as shown below. We set up a horizontal and a vertical reference line. Thereon we create our unit of distance with the 'Circle: centre & radius' tool. We anchor a circle to a horizontal line with the 'Intersect' tool. (This will

produce two points. We hide the one we do not want with the 'Show/Hide' tool.) We need our vertical reference line to line up vertices vertically. We click on 'Parallel line' and locate the point where the vertical reference line cuts the upper left hand circle. We deform the whole grid by clicking on the pink point, which pivots round the cyan point.
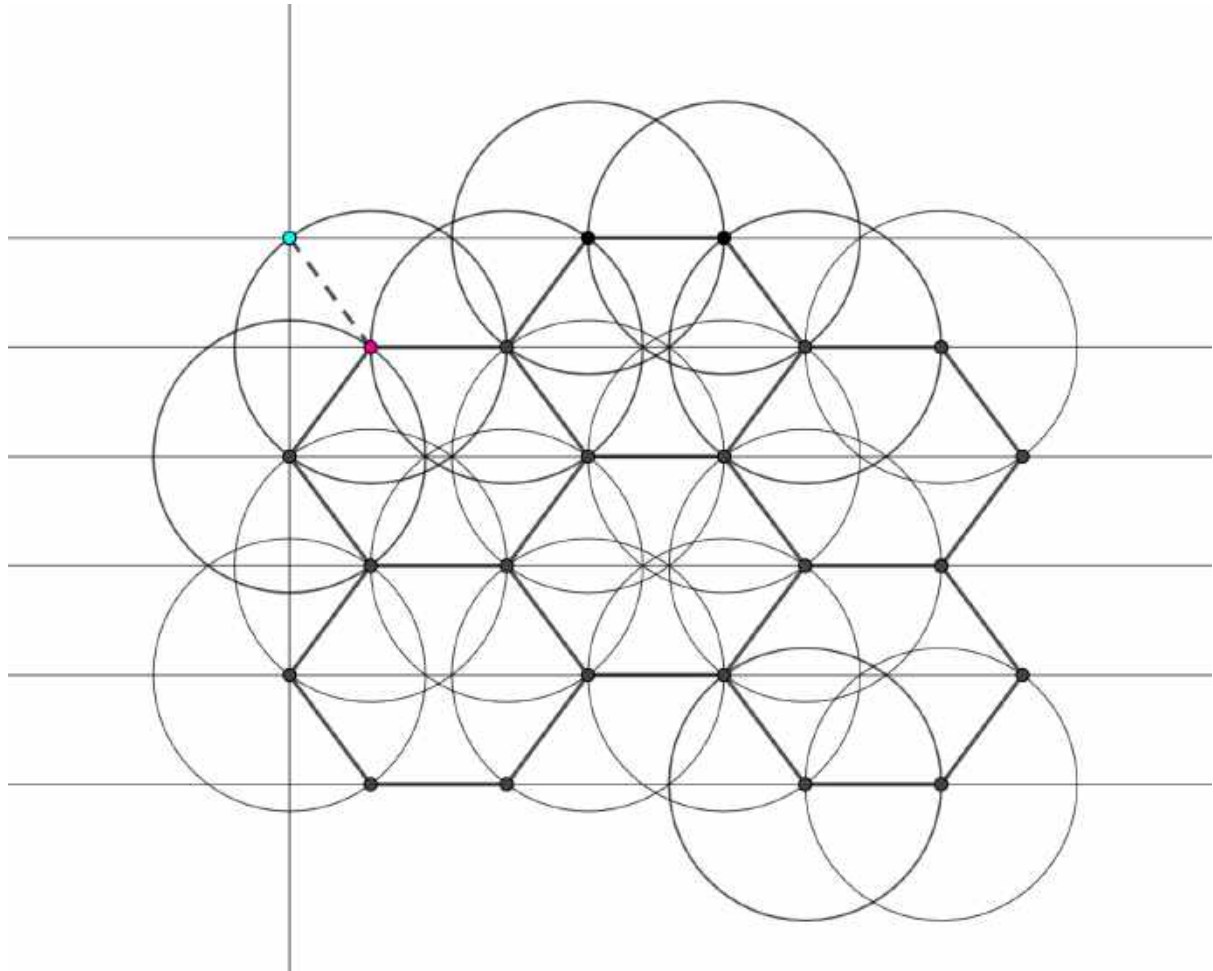
(d) If we replace Steinhaus' wall, where the bricks are dominoes, with this wall, where monominoes alternate with trominoes, we get 4.8.8:
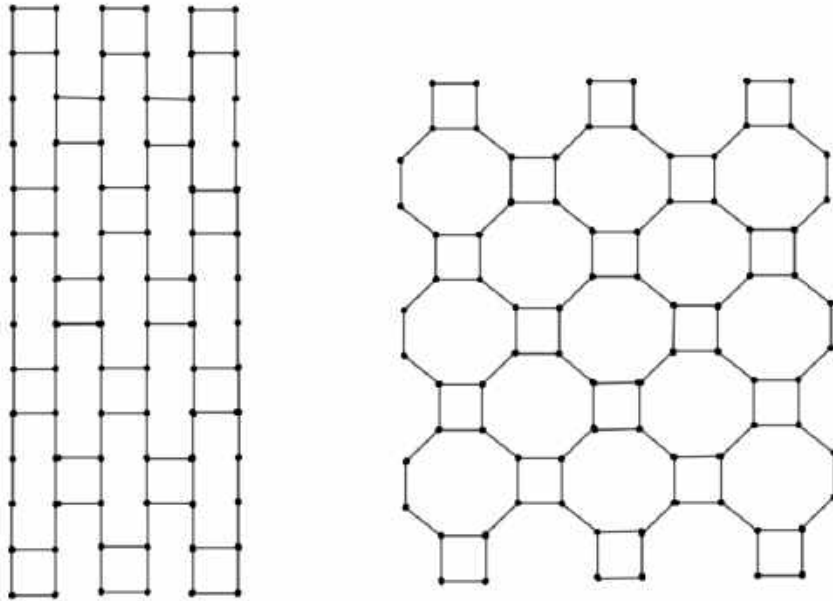
(e) We can model the distorted 3.6.3.6 in **(A) 1. (b)** with hinged rods. It is an example of a tiling where a polygon with an even number of equal sides has angles which alternate in size. The result is a tiling which preserves the rotation symmetry of the original but loses all mirror lines. Here is a distorted 4.8.8 where the octagon has angles of 120° and 150°. The octagon centres and square centres shown still define equal squares:
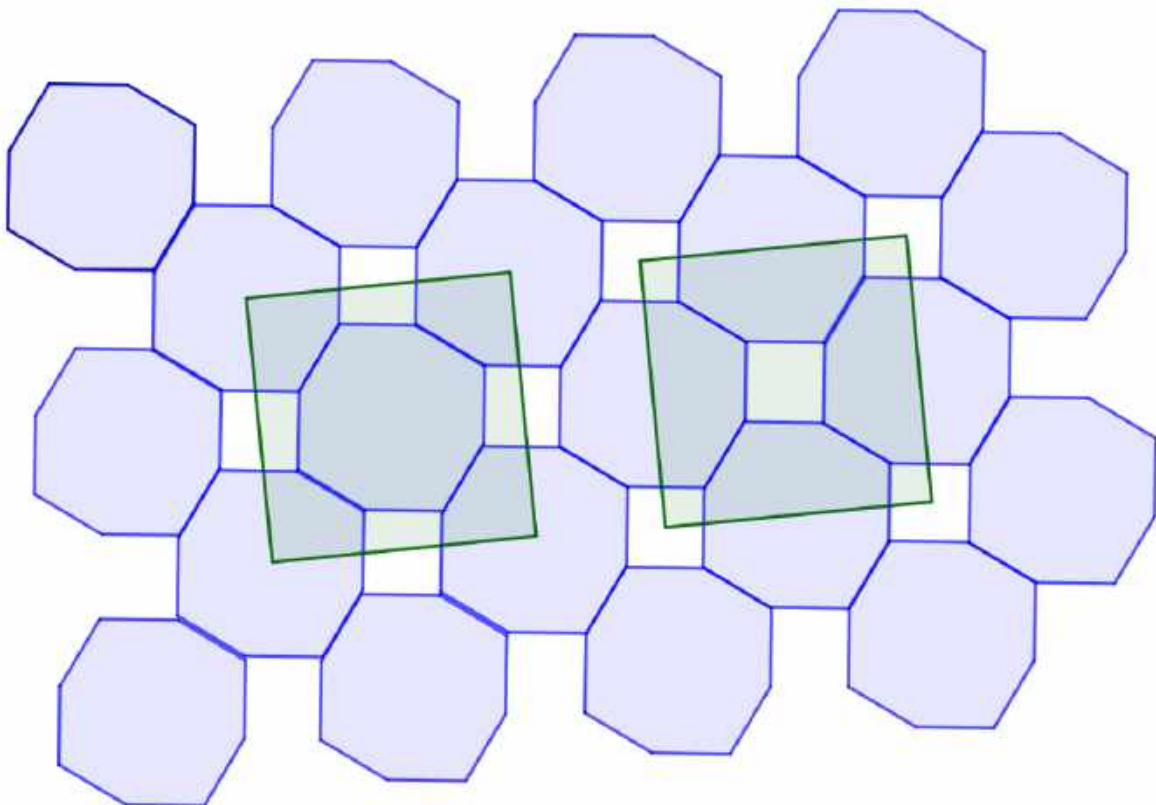
In this distorted 4.6.12 the squares have been preserved but the hexagons have angles of 110° and 130°, and the 12-gons, 140° and 160°:
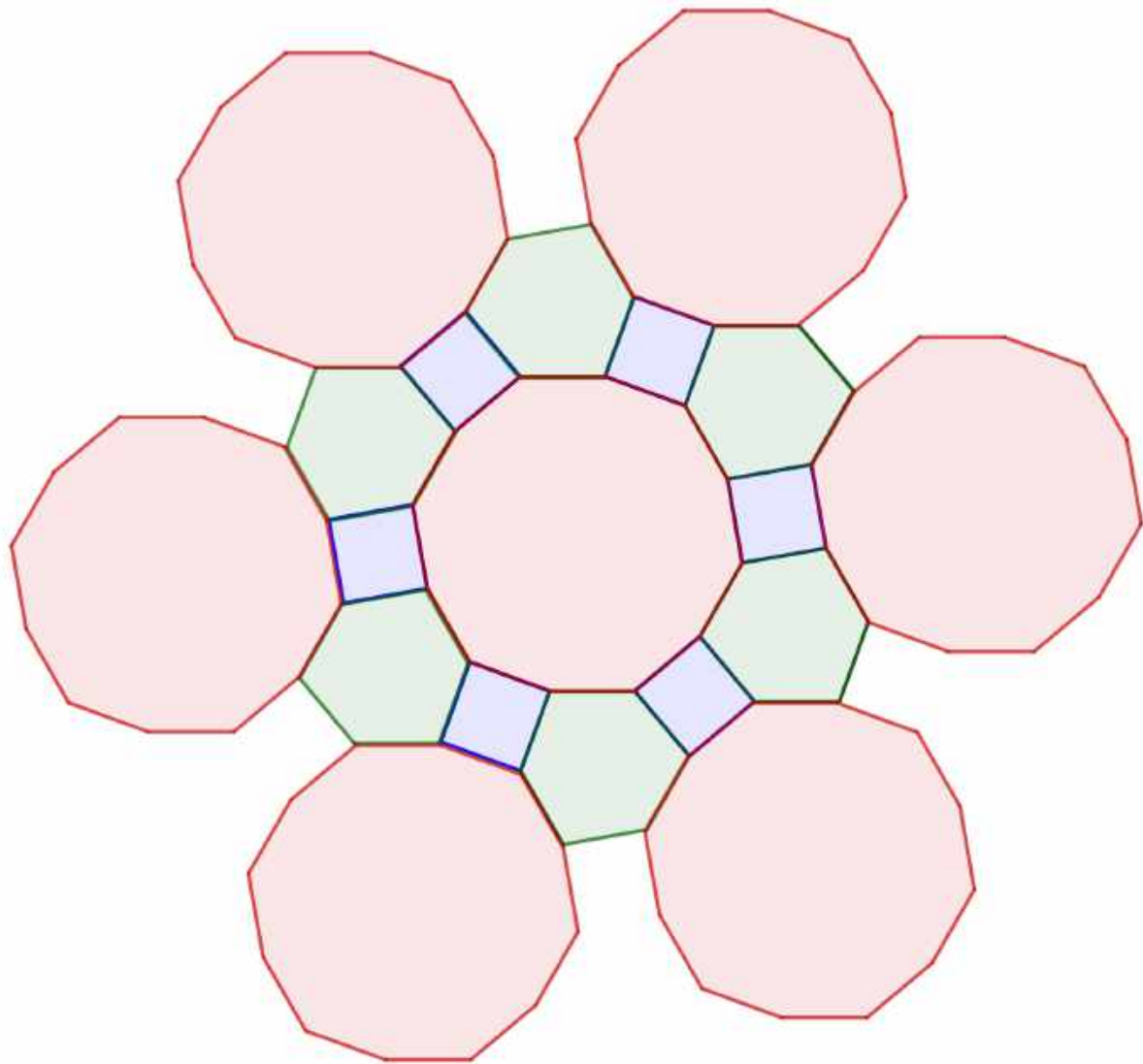
3. Stretching a sheet with embedded tiles

The tile centres are stuck to a rubber sheet, which we stetch uniformly, that is, by the same factor in two perpendicular directions. This is the two-dimensional equivalent of an expanding universe with tiles instead of galaxies. (Alternatively, instead of a rubber sheet being stretched, we can fix the tile centres and shrink the tiles.)

(a) We begin with 4.4.4.4 but remove alternate squares to leave a checkboard pattern. We stretch our sheet until the distance between tile corners is equal to a tile side. At that point we have 4.8.8.
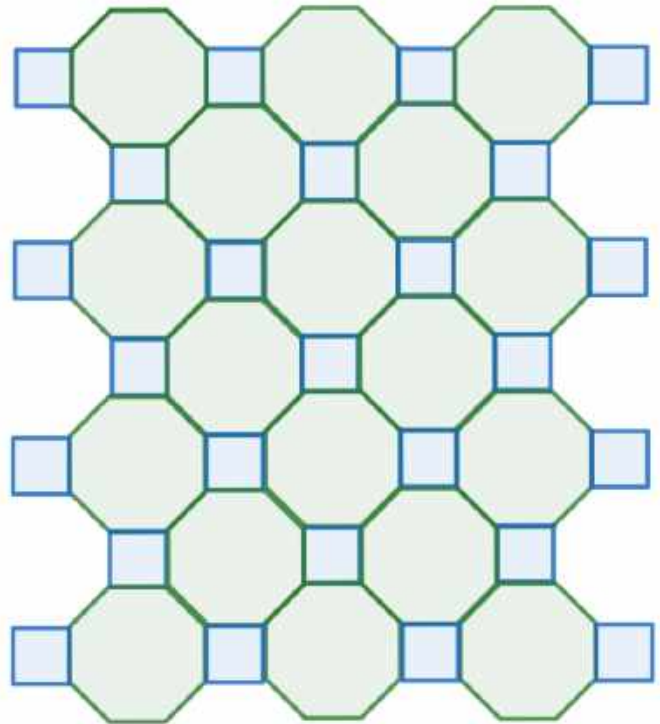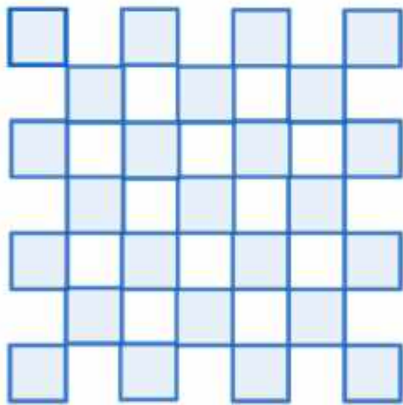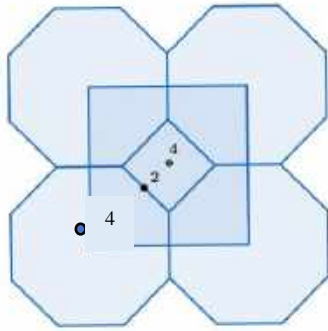
(b) We can produce 3.4.6.4, starting from 6.6.6. We stretch the rubber till the hexagons are separated by one side length:
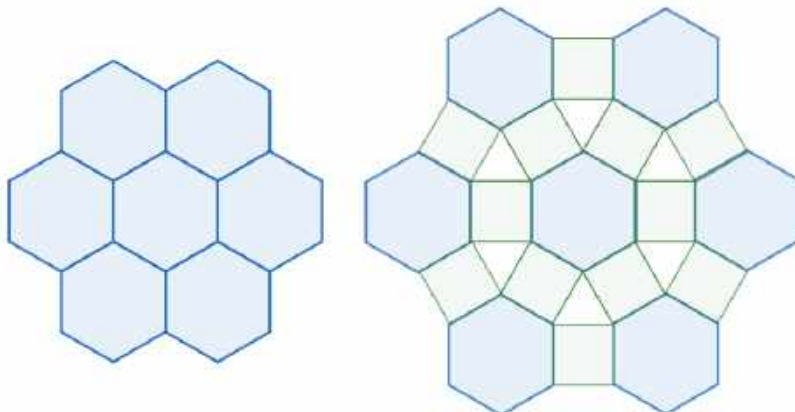
(c) We can derive 3.12.12 from 3.6.3.6 by stretching the rubber till the triangles are separated from the nearest hexagons by one side length: