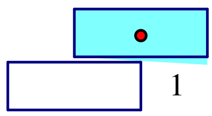
The Overarch v. The Wobbly Wall

The Overarch

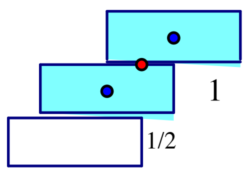
In the days when this magazine was just 'Plus', not 'Symmetry Plus', Nick Lord posed the following problem:

You are given so many identical planks and must stack them to produce the greatest overhang.

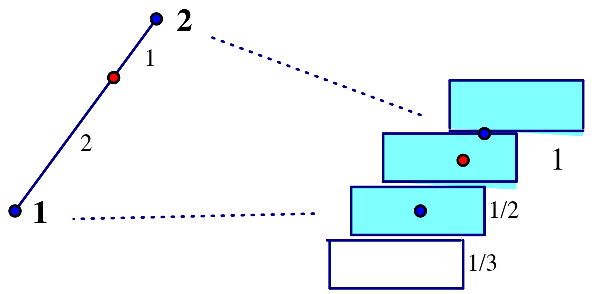
There's no need to dig out the original issue because you can also find the problem here: nrich.maths.org/299 ('Overarch 2'). The solution hinges (literally) on building the arch from the top down! Here are the top two planks/bricks:



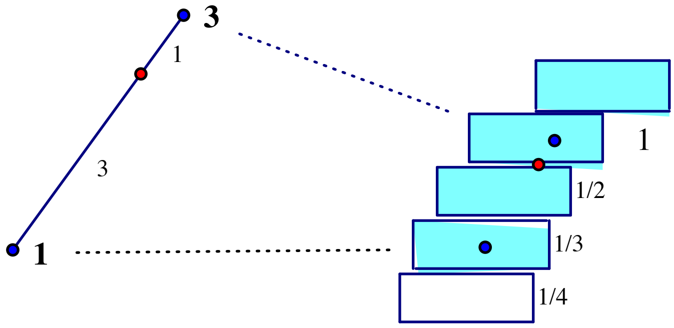
The centroid (centre of mass) of the upper brick is as near the overhang as we can get it. We now take that pair and sit it over a third brick:



... and take that set of 3 and sit it over a fourth:



On the left we show that the position of the new centroid divides the line joining the old ones in inverse proportion to the masses. The overhang in the above case = $\frac{1}{1+2} = \frac{1}{3}$.



This time the overhang is $\frac{1}{1+3} = \frac{1}{4}$. Notice that the new centroid drops half a brick height with every new brick.

By drawing the new straight line for every new brick, we can trace the form of the arch (the curve the centroids would follow if the bricks were infinitessimally thick) as an *envelope* curve. Each straight line would be a tangent to the true curve.

We can work out the total overhang by adding the fractions:

2 bricks: 1

3 bricks: $1+\frac{1}{2}$

4 bricks: $1+\frac{1}{2}+\frac{1}{3}$

5 bricks: $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}$

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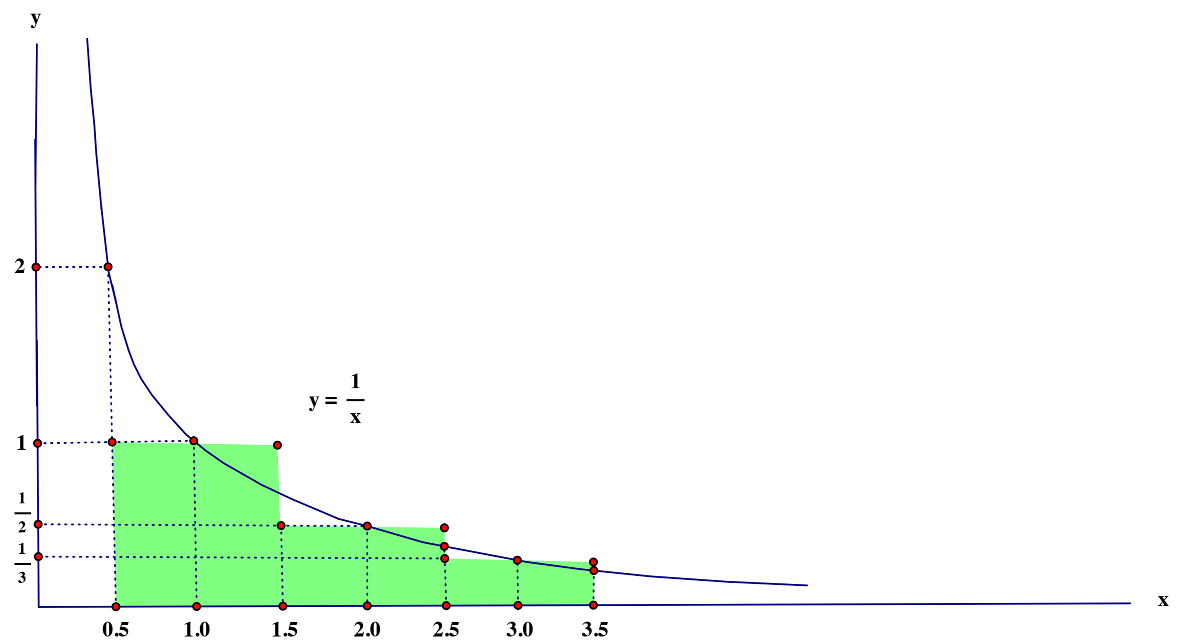
$(n + 1)$ bricks: $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+...+\frac{1}{n}$

The expression on the right is called the *harmonic series*. The name derives from the sequence of pitches obtained when you divide a stretched string in the ratios 1:1, 1:2, 1:3, ... The fractions follow the curve $y = \frac{1}{x}$:



If you've begun to study integral calculus, you'll know a general rule for turning $f(x)$ into $\int f(x)dx$, but also that $\int \frac{1}{x}dx$ doesn't obey it! So you have to learn that $\int \frac{1}{x}dx = \ln(x)$, the natural logarithm of x. So we could work out the area under our curve from $x = 0.5$ to $x = 3.5$: $\int_{0.5}^{3.5} \frac{1}{x}dx = [\ln(x)]_{0.5}^{3.5}$. But what we want is the green area: $1+\frac{1}{2}+\frac{1}{3}$. As we go along the curve, we must subtract the little white regions and add the little green ones. The further we go, the more equal in size these are and the closer the sum gets to our integral. Near the start, however, we have little idea how to compare the white shapes and the green shapes. Euler found that, overall, the green area exceeds the white area and that, the further you go, the closer the difference gets to a particular constant *K,* with a value around 0.577:

$$1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n} \to \ln(n)+K \quad \text{as} \quad n \to \infty \ .$$

(In his piece 'Sum that sequence – I don't believe it!' Nigel Bufton showed a simple way to prove that the sum of the series just goes on getting bigger: the series *diverges* - as opposed to *converging* on a *limit*.)

You can avoid using K and still get good estimates by starting your integration at $x = 1.5$ and just adding 1 to the total, or, better, starting at $x = 2.5$ and adding 1 + $\frac{1}{2}$ or, better still, starting at $x = 3.5$ and adding 1 + $\frac{1}{2}+\frac{1}{3}$, and so on:

$$1+[\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}] \approx 1+[\ln(n+0.5)-\ln(1.5)] \ .$$

$$1+\frac{1}{2}+[\frac{1}{3}+...+\frac{1}{n}] \approx 1+\frac{1}{2}+[\ln(n+0.5)-\ln(2.5)] \, .$$

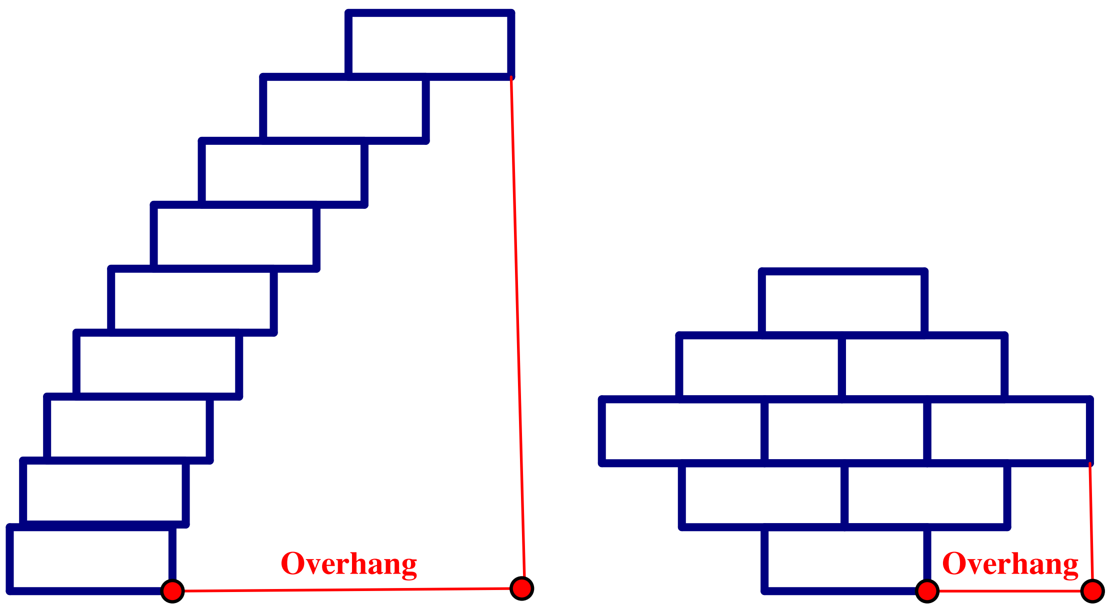
$$1+\frac{1}{2}+\frac{1}{3}+[...+\frac{1}{n}]\approx 1+\frac{1}{2}+\frac{1}{3}+[\ln(n+0.5)-\ln(3.5)]\ .$$

Using your scientific calculator, make a table comparing these estimates with the figure obtained by adding the fractions one at a time.

The Wobbly Wall

People build 'overarches' in our touring maths lab. Sometimes, however, they break the rules and use more than 1 brick in some layers. They do this to achieve an anticlockwise *turning moment* (or *torque*) to balance the clockwise moment due to the overarch. (The moment of a force about a pivot point is the size of the force multiplied by the perpendicular distance to the pivot.) In other words, they build a *cantilever*. But the challenge is still to achieve the greatest overhang for a given number of bricks. Using more than 1 brick in a layer is expensive in terms of bricks. Is it ever justified?

Let's build a cantilever of a very symmetrical kind: a piece of 'wall'. When bricks are laid lengthwise, they're called 'stretchers'; when edgewise, 'headers'. In the arrangement known as 'stretcher bond' all the bricks are laid lengthwise and we have the usual brick wall. But we're going to build our (dry) brick wall on a single brick. Compare this overarch (**A**) and 'wobbly' wall (**B**):



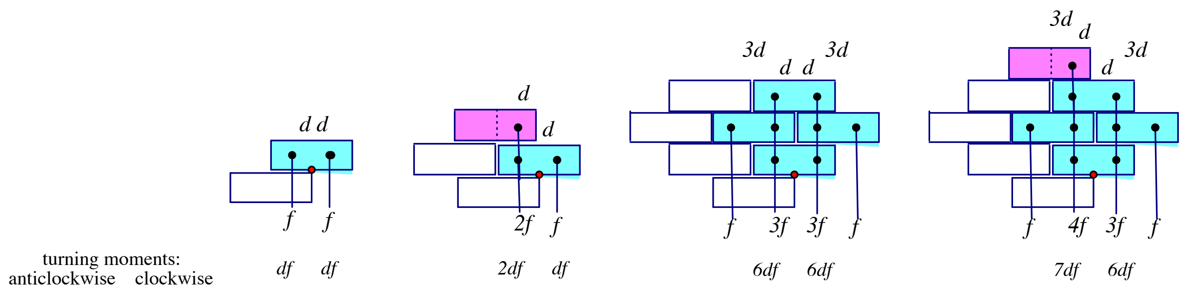
A **B**

Both use 9 bricks but the overhang is far greater in (**A**). However, each brick added to (**A**) results in a smaller and smaller increase in the overhang; whereas, with (**B**), every time we move from one square number of bricks to the next, we increase the overhang by 1 complete unit. This means that - unless our wobbly wall has collapsed long before - there will come a point where (**B**) takes the lead. *From your work in the first part of this piece you are now in a position to calculate where this will be.*

Will our wall collapse?

Here's my attempt to build the wobbly wall. With $4^2 = 16$ bricks, things were fine. But with 5^2 bricks ...

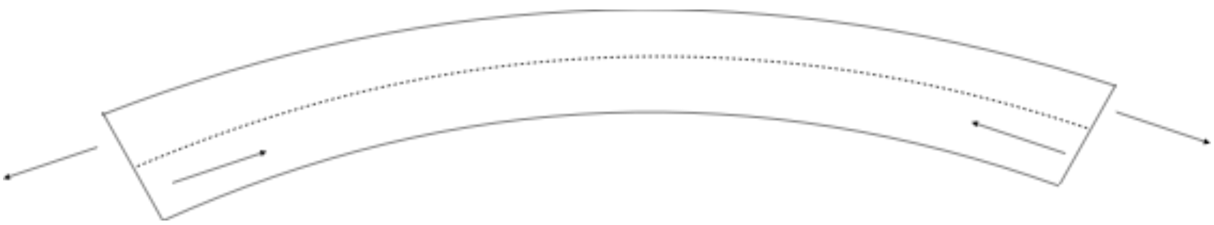
The wobbly wall is symmetrical so we'll just take one side. We'll think of a brick as 2 half-bricks joined together, each of weight f and length $2d$:



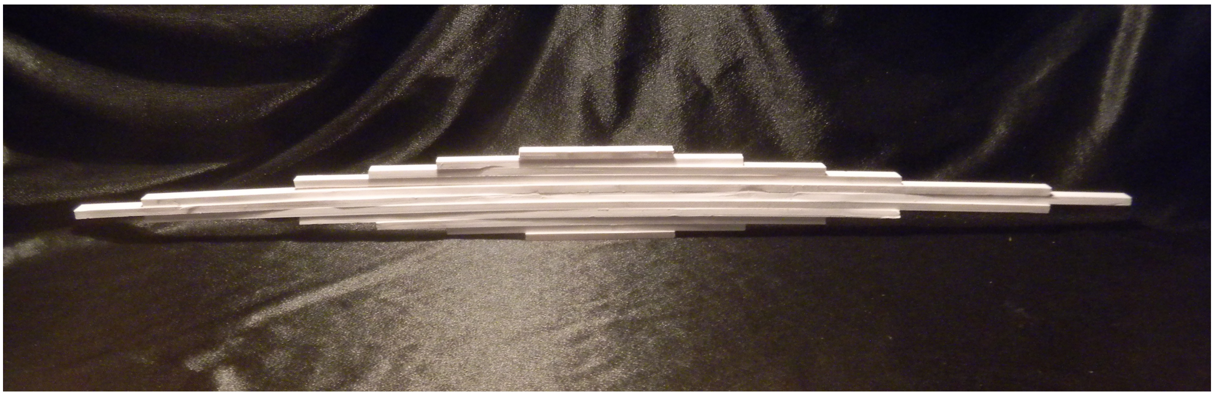
Starting on the left, the blue brick is delicately poised but the addition of the pink brick restores stability. In the third picture, the blue set is again delicately poised and again the addition of the pink brick restores stability. As long as the bricks are absolutely uniform and perfectly positioned, could we continue indefinitely?

Only if, in addition, each brick was absolutely rigid.

Imagine each course of bricks icemented together so that we have a series of horizontal beams. An unsupported beam bends under its own weight:



There is no change of length along the dotted line, the *neutral axis*, but, as shown by the arrows, above it the beam is stretched; below, compressed. Here is a wobbly wall made of 'foamex'. The central beam is 63 cm long. And there's a noticeable bend: the outer edge is 5 mm lower than the centre.



It looks, then, as if we'll have to make a compromise in the design of our wall. We'll do what the medieval stonemasons who built our cathedrals did. Above the outer edge of a flying buttress, you'll often see a decorative pinnacle. (Google 'images of flying buttress' to see a wide variety.) This serves the purpose of loading the outside of the arch (equivalent to the centre of our wobbly wall). The result is to move the neutral axis of each layer upwards, reducing the *tensile* (stretch) forces. (Unlike steel, stone has little tensile strength.)

If you've got Jenga bricks (or anything else which serves the purpose), experiment with your own design of cantilever. Can you beat the overarch?

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