5.4 Dual polygons and tilings

An *analogy* takes the form

A is to *B* in *X* as *A'* is to *B'* in *X'*.

A *duality* takes the more specific form

A is to *B* in *X* as *B* is to *A* in *X'*.

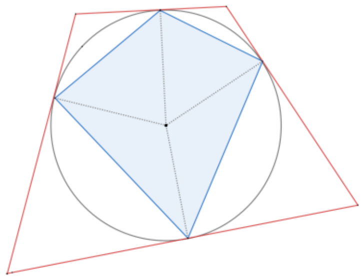
In projective geometry duality is fundamental and can be given a formal definition, that is to say, one for which we can use a sentence as a template:

If *points A, B, C,* ... lie on *line k*, *lines a, b, c,* ... pass through *point K*.

Collinearity of points corresponds to coincidence of lines. In Euclidean geometry we must specify the correspondence.

If one polygon is the dual of another, a vertex in one corresponds to a side in the other so that the size of an angle in one corresponds to the length of a side in the other.

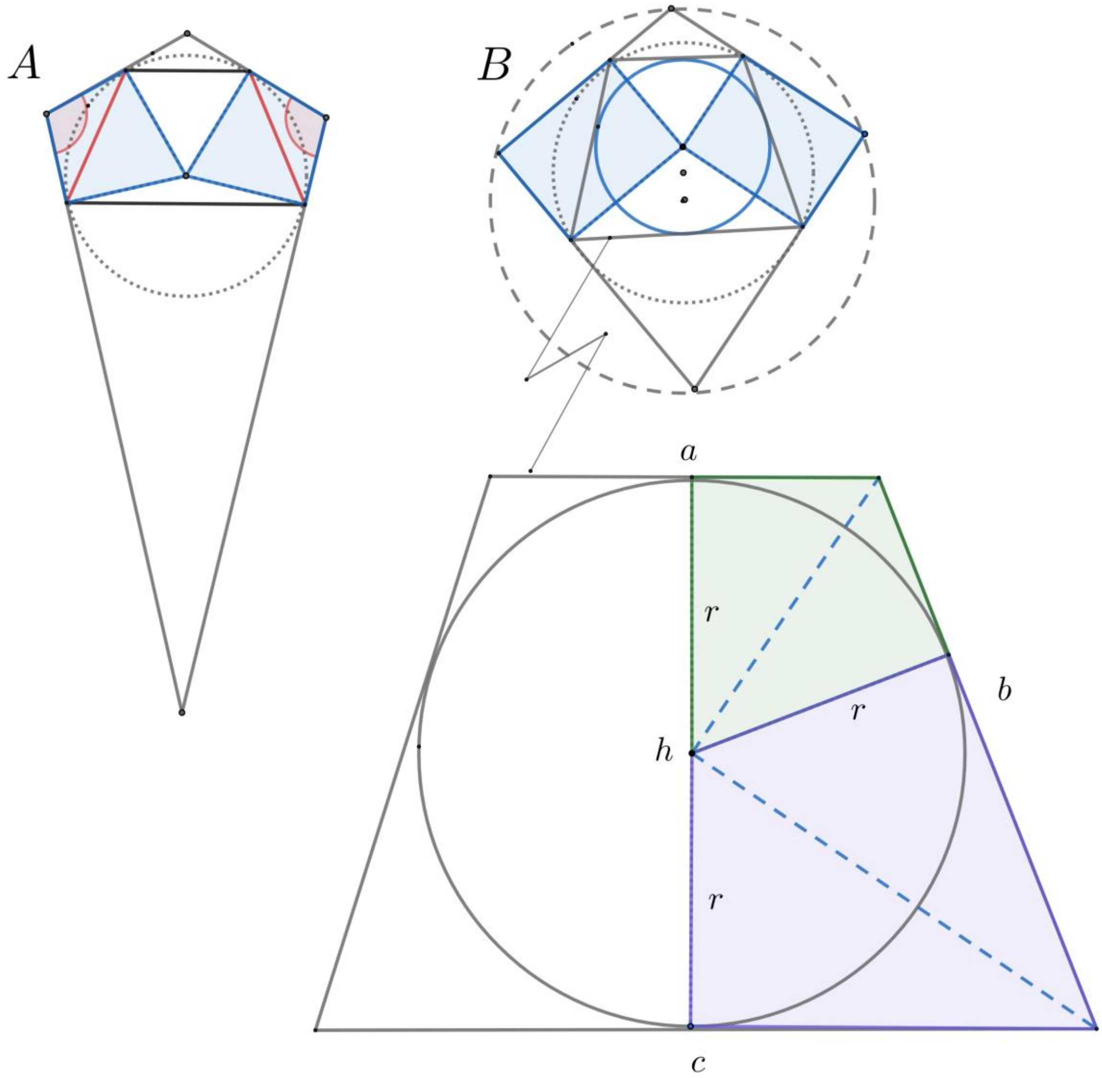
If we have a *cyclic* polygon, we can create a *tangential* polygon like this.



Each is the dual of the other.

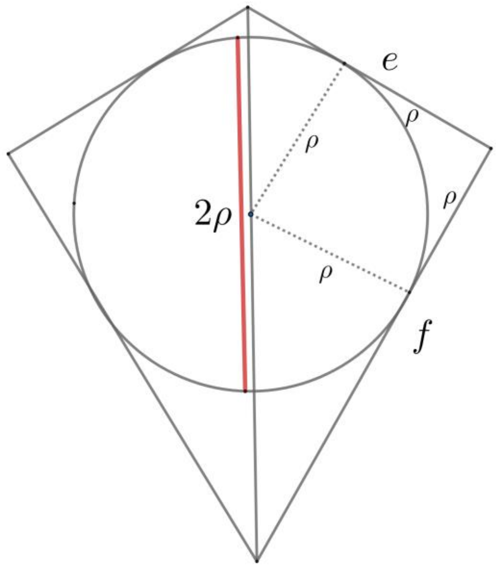
In this section we shall look at specific pairs of dual quadrilaterals obtained in this way.

All isosceles trapezia are *cyclic*; all kites are *tangential*. We can obtain the second as a dual of the first as shown in ***A***. As shown in red, the isosceles trapezium has two equal *sides*; the kite has two equal *angles*.



Since radii are equal and tangents from a point are equal, the blue figures in ***A*** are kites. In ***B*** we have taken an isosceles trapezium which is also *tangential* . (It is therefore *bicentric*.) The red angles have become right angles with the result that the kite is also bicentric; the blue figures have become rectangles. We can see this by considering the similar green and lilac kites in the enlarged figure. *Show that the blue lines are perpendicular*.

In the bicentric trapezium $b = \frac{a+c}{2}$. *Why?* That is, *b* is the arithmetic mean of *a* and *c*. Comparing corresponding sides in the similar kites, $\frac{a/2}{r} = \frac{r}{b/2}$, so that $\frac{ab}{4} = r^2 = \frac{h^2}{4}$, $h = \sqrt{ab}$. That is, *h* is the *geometric* mean of *a* and *b*. Does the 90° kite have a special property? It does.



Three similar right triangles are shown. We choose two and compare corresponding sides.

$\frac{e-\rho}{\rho} = \frac{e}{f}.$

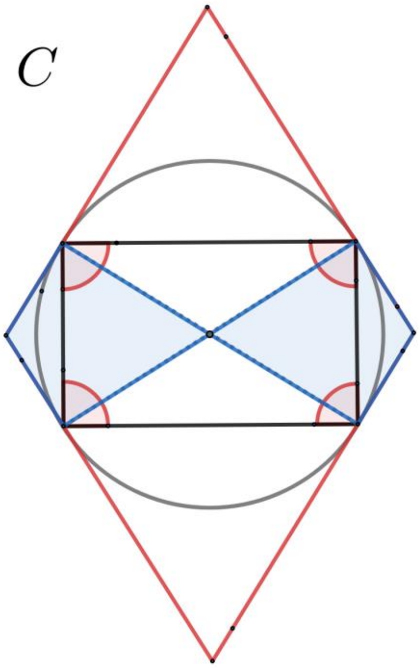
Do the algebra to show that

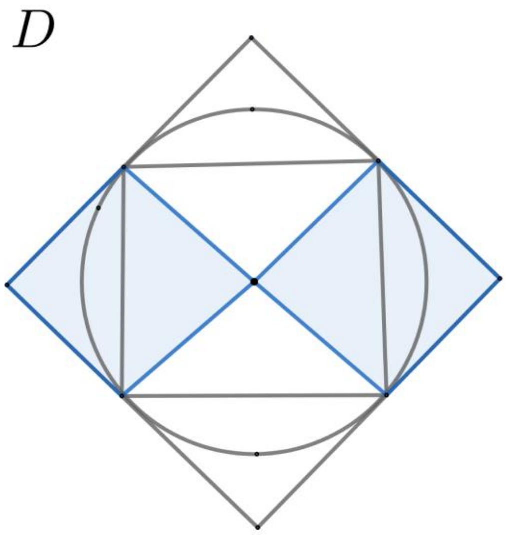
$2\rho = \frac{2ef}{e+f}$.

That is to say, the diameter of the incircle is the *harmonic* mean of *e* and *f*.

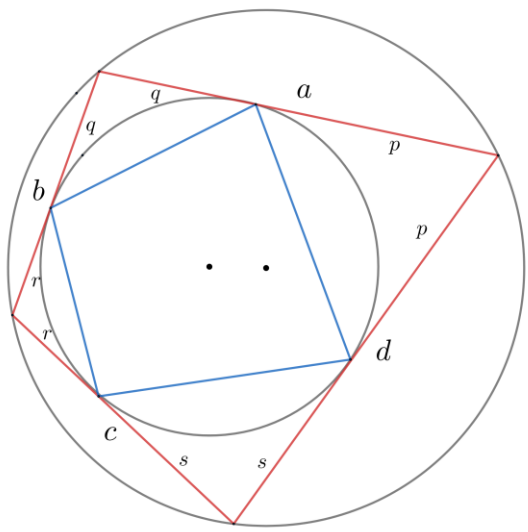
C Returning to ***A***, we now make the top and bottom sides of the trapezium equal. It has become a rectangle; the kite has become a rhombus. To the four equal sides of the first, there correspond the four equal angles in the second. The blue figure is a kite.

D A tangential rectangle is a square; a cyclic rhombus is a square. The blue figure is a square.





Here is a little more on *bicentric* quadrilaterals (type *CT* in the notation of section **4.1**).

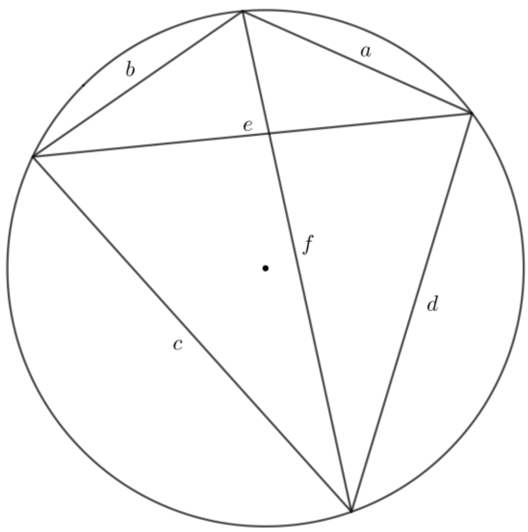


In the section **4.2: From Heron to von Staudt** we learn that the area A of a cyclic quadrilateral is given by $A = \sqrt{(s-a)(s-b)(s-c)(s-d)}$, where $s = \frac{a+b+c+d}{2}$.

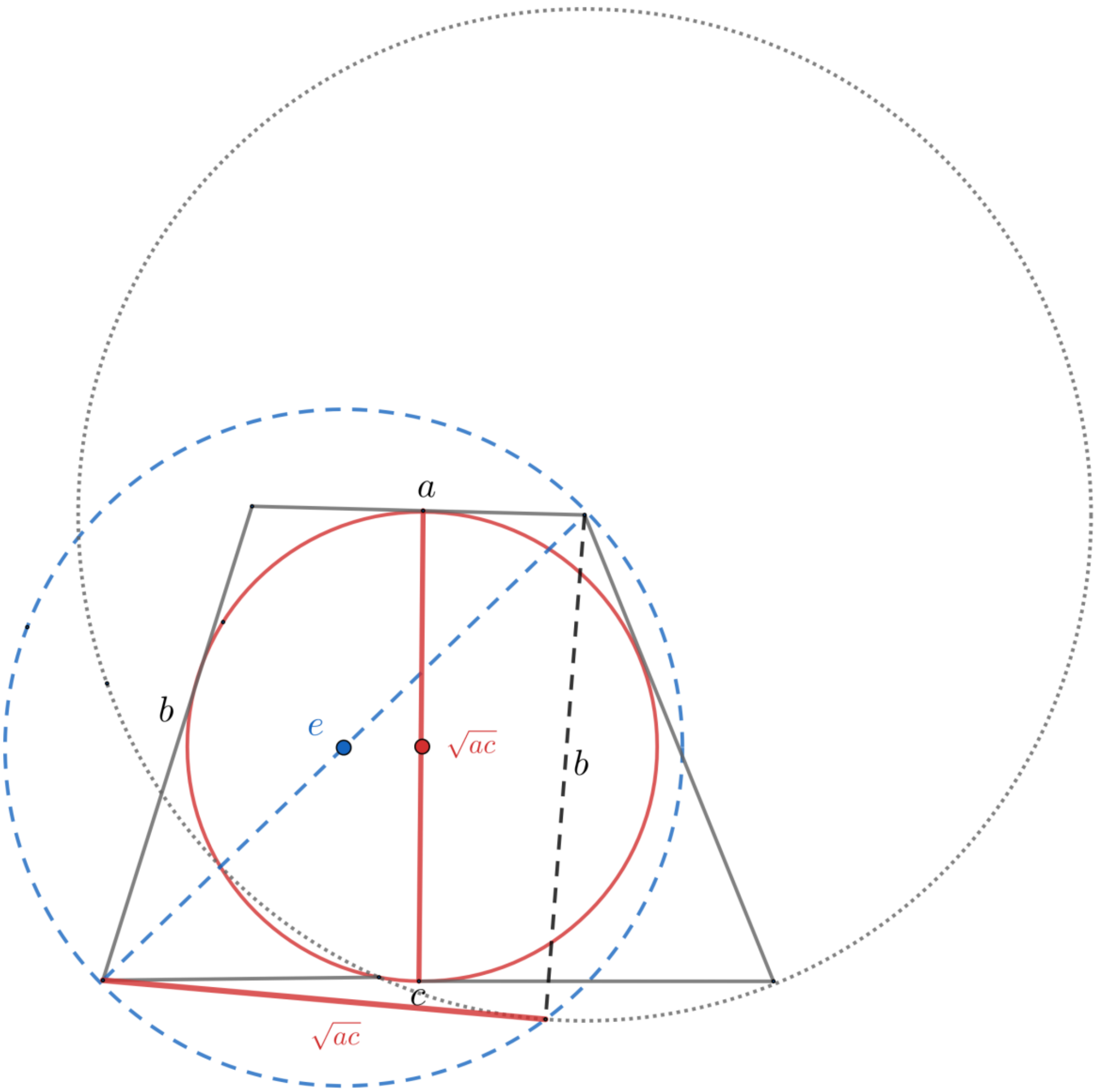
As we find in the section there too, adding up the p, q, r, s segments, we see that, for a tangential quadrilateral, $a + c = b + d$.

Use this fact to show that the area A of a bicentric quadrilateral = $\sqrt{abcd}$.

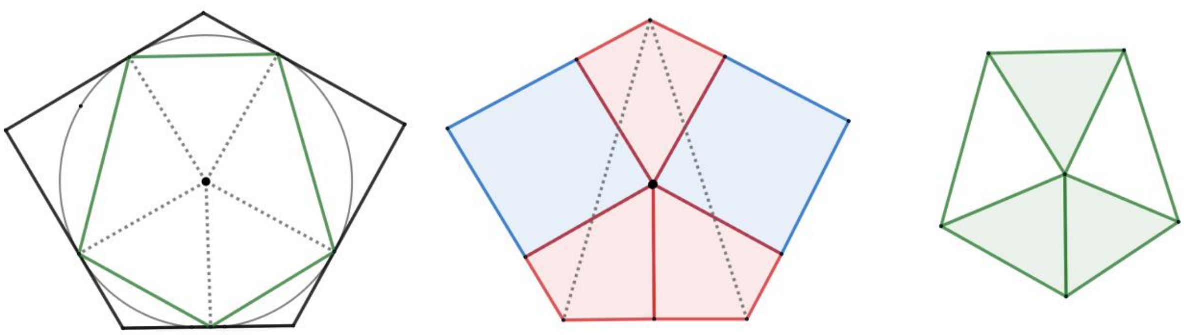
We found a geometric mean in the bicentric trapezium. In fact we can construct a second by use of Ptolemy's theorem. (We can construct this second one in any isosceles trapezium, not just a tangential one.) This relates the lengths of the diagonals and sides of a cyclic quadrilateral: $ac + bd = ef$. We shall not prove it here, though it can be proved by use of similar triangles.



In the isosceles trapezium below, the top right vertex is a centre of the dotted black circle, which also passes through the vertex bottom right.
Ptolemy's theorem gives $ac = e^2 - b^2$. Note the form is that of Pythagoras' theorem. We construct the blue circle on a diagonal as diameter. The black dotted circle allows us to transfer the length b to the blue circle. Since the angle in a semicircle is a right angle, we have a right triangle and the red length is $\sqrt{ac}$. (As you can see, we could draw another such segment to the second crossing point of the black and blue circles.)



The black pentagon below is known as the Cairo tile. The name already suggests that it tessellates, which indeed it does. We shall come to this shortly. It has many nice numerical properties. We have listed some, *which you may like to confirm.* The green figure is the dual.



The Cairo tile comprises:
2 equal squares,
3 congruent right kites.

(The dotted line shows that it may also be dissected
Into 2 equal half-squares and a sector of a regular
12-gon.)

4 sides of one length (1 for comparison),
1 of another ($\sqrt{3}$ - 1)

3 angles of one size (120°),

2 of another (90°)

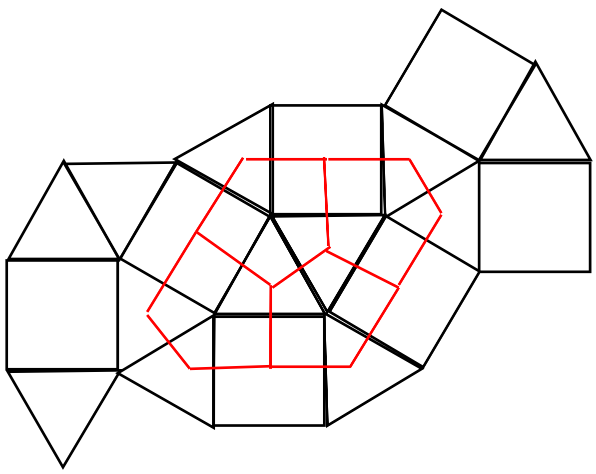
(With the side length 1 as allocated, the area
is rational, namely 3/2.)

Its dual comprises:
2 equal ½-squares,
3 equal equilateral triangles.

4 angles of one size (105°),
1 of another (120°)

3 sides of one length $\left(\frac{\sqrt{3}}{2}(\sqrt{3}-1)\right)$,

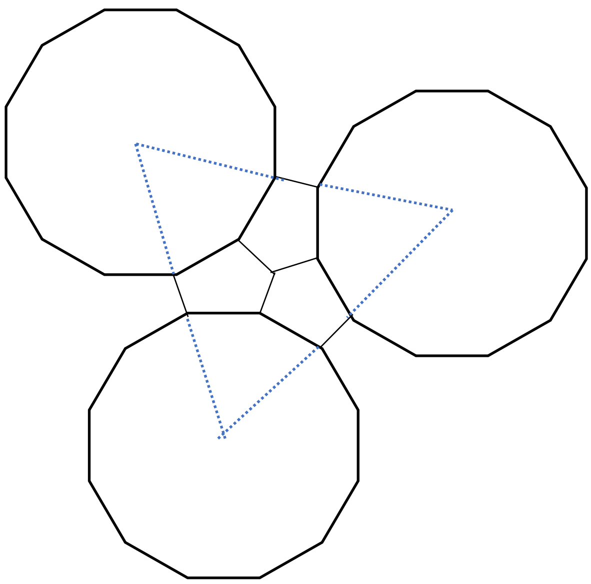
2 of another $\left(\frac{\sqrt{6}}{2}(\sqrt{3}-1)\right)$



The Cairo tiles are shown in red. The figure shows how 4 make up an irregular hexagon which tiles the plane, therefore so do the constituent pentagons.

The figure also shows that the vertices of the Cairo tiles fall on the incentres of the triangles and squares which compose the semiregular tiling 3.3.4.3.4. Correspondingly, the vertices of the triangles and squares fall on the incentres of the Cairo tiles. This reciprocal property defines a *dual tiling.*

Below we have a tiling which comprises regular 12-gons and Cairo duals.



Another dual relation concerns quadrilaterals and the parallelograms formed by joining their side midpoints (Varignon parallelograms). The attributes 'orthodiagonal' and 'equidiagonal' are interchanged.

Quadrilateral:	Varignon parallelogram:
Equidiagonal (a rectangle)	Orthodiagonal (a rhombus)
Orthodiagonal (a rhombus)	Equidiagonal (a rectangle)