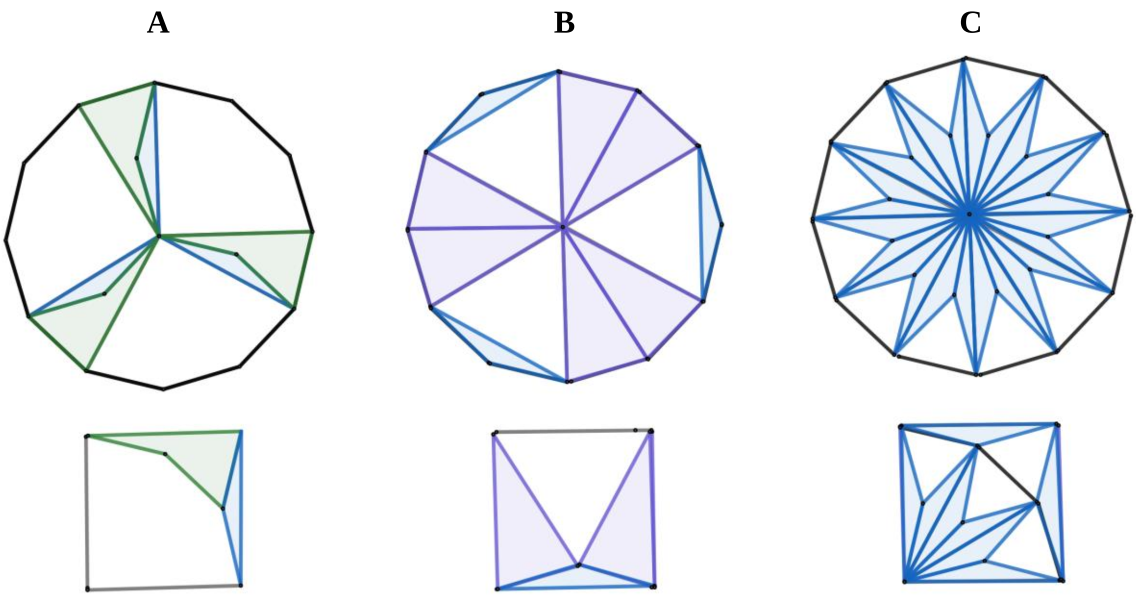
5.2.9 Dissections of the regular dodecagon

Dissecting the regular 12-gon in different ways reveals interesting area relations involving the 12-gon and the square.

(1) Here are three dissections of a regular 12-gon into four squares. They each demonstrate the fact that a regular 12-gon of unit radius has an area of exactly 3.



Liu Hui, C3

9 pieces,
3 shapes

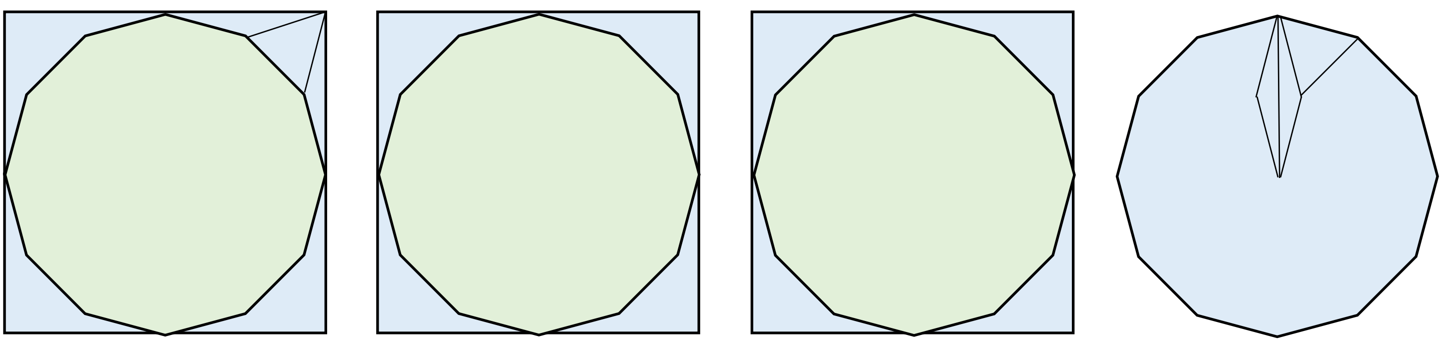
Roger B. Nelsen, C20-21

12 pieces,
3 shapes (all triangles)

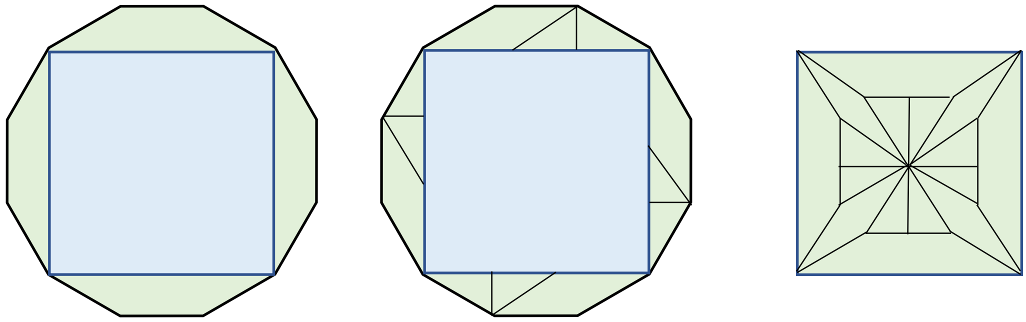
József Kürschák, C19-20

36 pieces,
2 shapes (both triangles)

(2) If we inscribe 3 of the **C** 12-gons in squares, we can use the peripheral pieces to make one complete 12-gon, thus producing a dissection of 3 squares into 4 12-gons:



(3) The biggest possible square in a 12-gon has an area 2/3 that of the 12-gon. We can use that fact to produce a dissection of 2 12-gons into 3 squares:

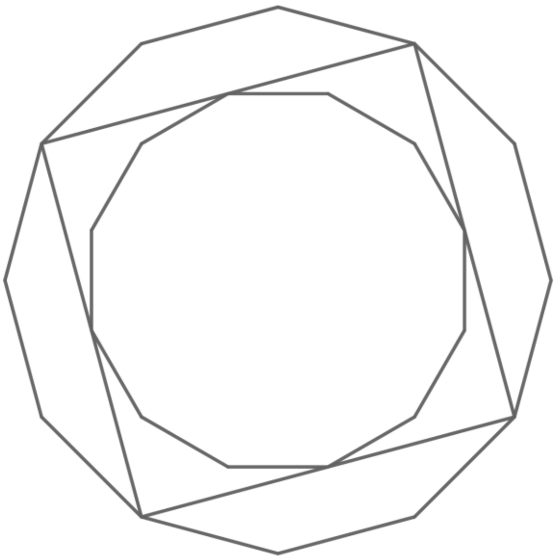


Conversely, the dissection establishes the area fraction:

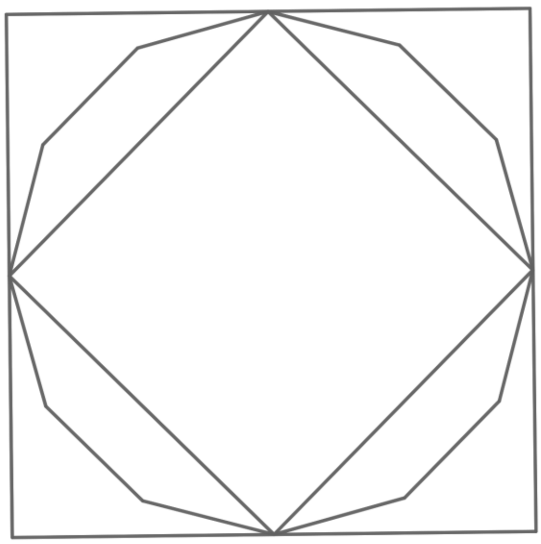
The trapezium *in*side the square is the trapezium *out*side the square flipped about a side. *Count the right triangles and rhombuses inside the square and compare with the totals outside.* You should get a ratio of 2 : 1. (Rotation symmetry only requires you to compare a quarter of the figure inside and a single trapezium outside.)

(4) In **D** we again inscribe a 12-gon in a square so that its vertices coincide with midpoints of square sides, as in **(2)**, but then inscribe the square in a second 12-gon as in **(3)**:

D



E



We know that the inner 12-gon has an area 3/4 that of the square, and the square has an area 2/3 that of the outer 12-gon. Therefore the inner 12-gon has an area $\frac{3}{4} \times \frac{2}{3} = \frac{1}{2}$ that of the outer one. By the commutative law of algebra we obtain the same result if we interchange the shapes as in **E**. There it is clear by dissection into right isosceles triangles that the inner square has half the area of the outer one.

A different dissection also yields this result. We dissect **D** as in **F**. If each right isosceles triangle in the border has area b, a square has area $2b$. This gives a total of $12(a + b)$ for both the border and the centre as required.