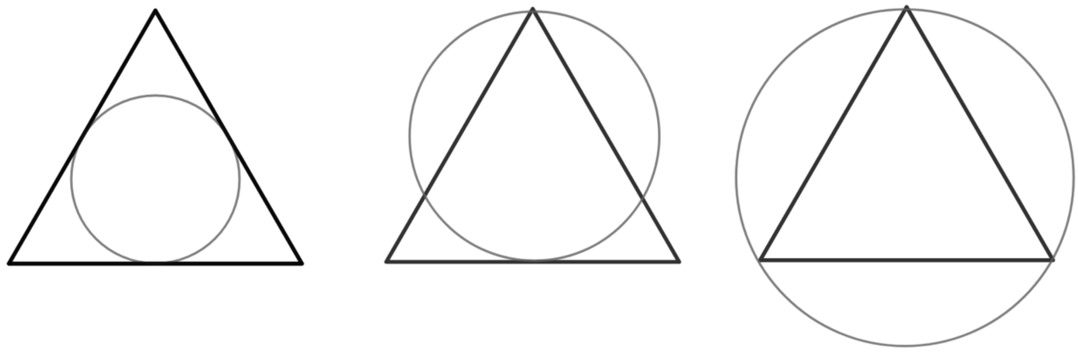
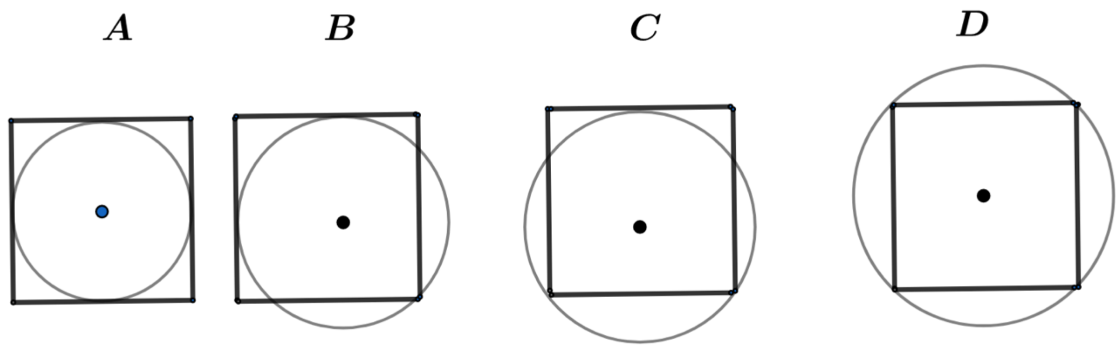
5.2.8 Cyclic/tangential hybrids

We have three congruent equilateral triangles with circles differently arranged in each. Show that their radii stand in the ratio 2 : 3 : 4.



If the square has side 2, find the radius r of each of these circles:



Hints:

C Use the intersecting chord property.

B Box the circle in a square of side $2r$, let the diagonal of the old square have length $2r + d$, so that the diagonal of the new square has length $2r + 2d$. This will give you two equations which you can solve for r.

You should obtain these results:

A: $r = 1$, ***B***: $r = 2(2 - \sqrt{2})$, ***C***: $r = \frac{5}{4}$, ***D***: $r = \sqrt{2}$. As you would expect, these increase from left to right.

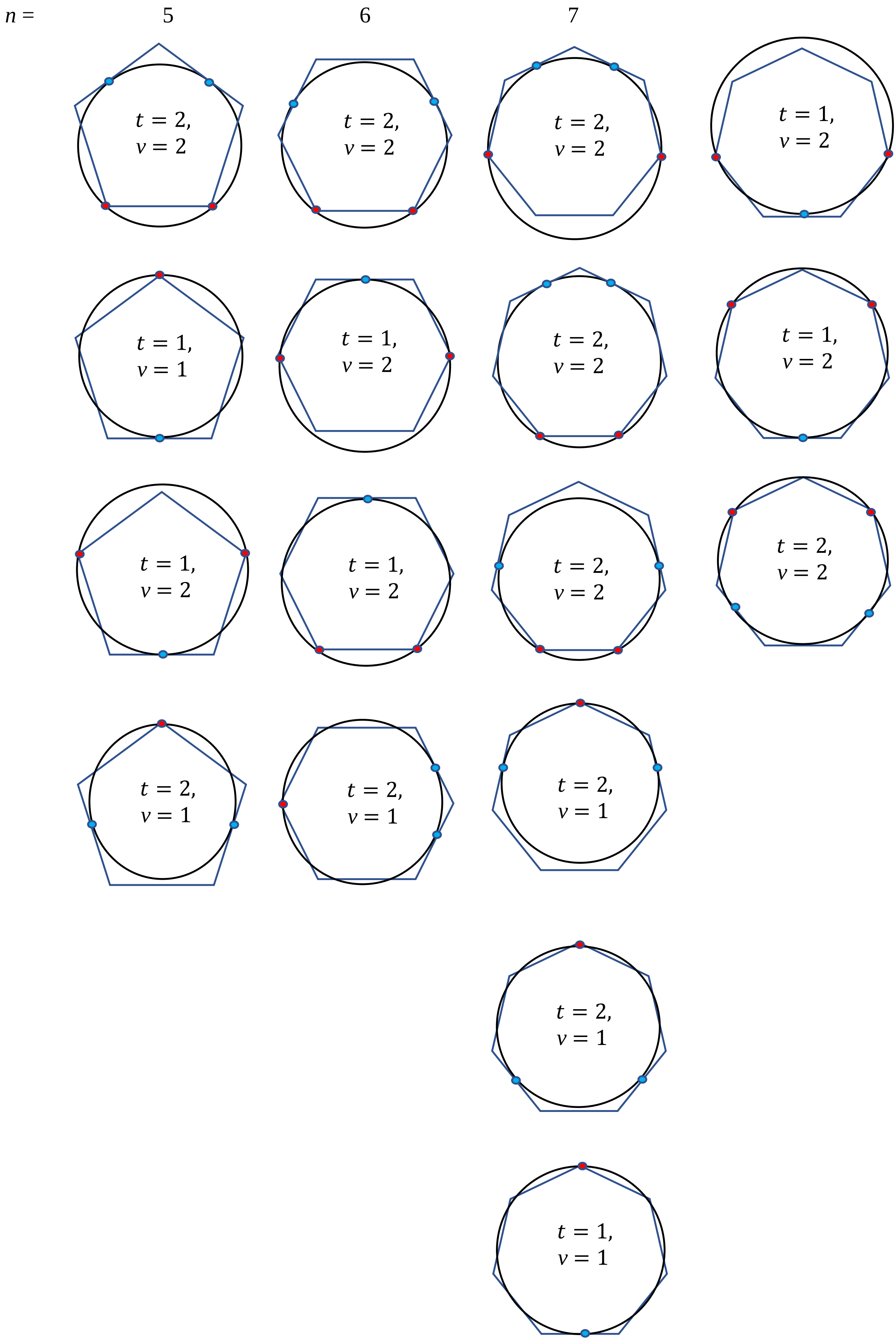
Notice how the circles are defined:

A: 4 tangents - *How many would be sufficient?*
B: 2 tangents, 1 vertex
C: 1 tangent, 2 vertices

D: 4 vertices - *How many would be sufficient?*

Notice also how, in terms of tangents and vertices, we have two dual pairs: (***B***, ***C***) and (***A***, ***D***). (Go to the section **Dual polygons** for duals defined in a different way.) The symmetry of this arrangement suggests there are no other possible relations between a circle and a square in terms of tangency and vertex coincidence.

The following table shows the possible arrangements of vertex and tangent for the next few regular n-gons. Let the number of points of tangency be t and the number of coincident vertices be v. We omit the incircles ($t = n$, $v = 0$) and the circumcircles ($t = 0$, $v = n$).


n =
5
6
7
t = 2,
v = 2
t = 2,
v = 2
t = 2,
v = 2
t = 1,
v = 2
t = 1,
v = 1
t = 1,
v = 2
t = 2,
v = 2
t = 1,
v = 2
t = 1,
v = 2
t = 1,
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t = 2,
v = 1
t = 2,
v = 1
t = 2,
v = 1
t = 2,
v = 1
t = 1,
v = 1

If t = 0 we need v to be 3, but those three points define the circumcircle. If v = 0 we need t to be 3, but those three tangents define the incircle. Therefore t must be 1 or 2, v must be 1 or 2. Thus the possible (t, v) pairs are just 4:

(1, 1): tangent point and vertex on symmetry axis
(2, 1): vertex on symmetry axis
(1, 2): tangent point on symmetry axis
(2, 2): both tangent point and vertex off symmetry axis

Notice these restrictive conditions:

1. Vertices cannot be end points of sides containing tangent points.
2. A (1, 1) pairing is only possible for $n = 2k + 1$, $k > 0$, where there is only one case.
3. Tangent points on parallel sides define the incircle, and therefore lead to an inadmissible (2, 1) pairing.

We take configurations related by a symmetry as equivalent. The numbers of cases are shown in this table. The entries are justified below.

n =	$2k,\ k > 1$	$2k + 1$, $k \geq 1$
Number of (t,v) pairs of each of the four possible types		
(1,1)	0	1
(2,1)	k - 2	k - 1
(1,2)	k - 1	k - 1
(2,2)	$\frac{(k-2)(k-1)}{2}$	$(k-1)^2$

In the six cases of interest, we arrive at the numbers as follows. We pair sides and vertices about a symmetry axis.

$n = 2k$

(2,1): There is one for each pair of sides except the parallel pair and that containing the vertex, so $k - 2$.
(1,2): There is one for each vertex pair except that defining the tangent, so $k - 1$.
(2,2): The vertex pair must not belong to an admissible side pair. This excludes the two pairs of adjacent vertices, so we have $k - 2$ admissible vertex pairs. Each is combined with an admissible side pair. These exclude only the pair of parallel sides. So we have $k - 1$ admissible side pairs. Each combination is duplicated in a horizontal symmetry axis. So, up to symmetry, we have only $\frac{(k-2)(k-1)}{2}$ cases.

$n = 2k + 1$

(2,1): There is one for each pair of sides except that containing the vertex, so $k - 1$.
(1,2): There is one for each pair of vertices except that containing the tangent, so $k - 1$.
(2,2): We have two cases to consider:
(a) The tangents are the pairs of adjacent sides. This pair is combined with $k - 1$ possible vertex pairs.

(b) The tangents are not adjacent sides. There are $k-1$ such pairs. The admissible vertex pairs exclude the two adjacent to each tangent pair. Each tangent pair can therefore be combined with $k-2$ possible vertex pairs.
The grand total is therefore $(k-1)+(k-1)(k-2)=(k-1)^2$.

These studies would be of little interest except for one intriguing historical possibility. To determine π Archimedes famously used a method he ascribes to Eudoxus. He takes regular polygons and draws incircles and circumcircles. In what we would now call the limiting case he argues that the radius cannot be smaller than the greatest possible inradius and cannot be greater than the smallest possible circumradius (the method of 'double contradiction'). Might it be in writings now lost that he considered one of our hybrid forms? This possibility is considered in the following footnote, which concludes that it is unlikely.

Pi by three plausibly ancient means

Archimedes famously approximated a circle by alternately inscribing and circumscribing regular *n*-gons. Below we calculate π in three ways:
Let $\frac{\pi}{n}=\theta$.
1) Following Archimedes, we calculate the area of the unit circle as the mean of the areas of the inscribed and circumscribed regular *n*-gons, $A_n, = \frac{n(\sin 2\theta + 2\tan\theta)}{4}$.
2) Inspired by a problem of Jenny Golding, relayed by the late Peter Neumann in his 2016 presidential address, Gareth Williams considers hybrid cyclic/tangential regular *n*-gons [1]. The circle is tangent to one edge but passes through the opposite vertex (where *n* is odd), or pair of vertices (where *n* is even). Figure 1 illustrates the respective cases *n* = 3, *n* = 4. He remarks, "one cannot help but reminisce about Archimedes' *Measurement of a Circle*." Again we take the unit circle. His hint yields for the odd and even cases these areas for the polygon:
$B_{no} = \frac{4n\tan\theta}{(1+\sec\theta)^2}$, $B_{ne} = \frac{16n\tan\theta}{(4+tan^2\theta)^2}$.

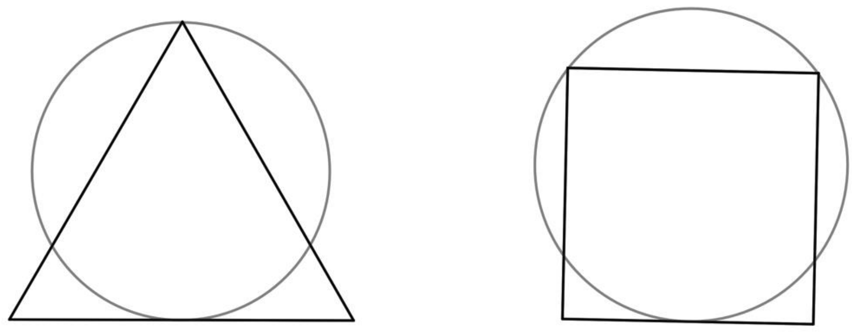

FIGURE 1

3) We take the sides and diagonals of the regular *n*-gon, inscribed in the unit circle, as altitudes of isosceles triangles which pack round a chosen vertex as shown in Figure 2. The circle's area approximated in this way, C_n, = 4 $\tan\frac{\theta}{2}\sum_{j=1}^{j=n-1}\sin^2 j\theta$.

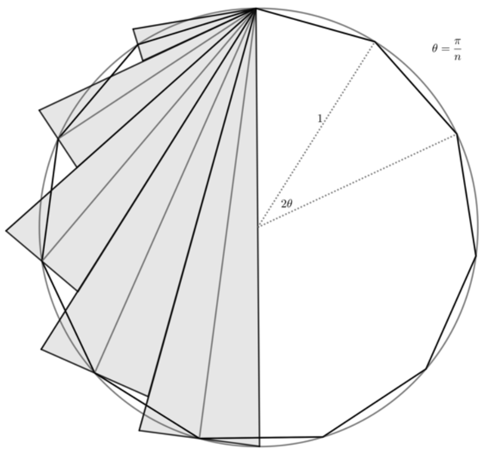

FIGURE 2

In Table 1 we divide these estimates by the value of π (accurate to 10 significant figures), then give the difference from unity to allow comparison.

n	$\frac{A_n}{\pi} \sim 1$	$\frac{B_n}{\pi} \sim 1$	$\frac{C_n}{\pi} \sim 1$
3	+0.033741680	-0.264894806	+0.102657790
6	-0.035174433	-0.060457267	+0.023490520
12	-0.010789909	-0.012285063	+0.005751000
24	-0.002816537	-0.002908707	+0.001430350
48	-0.000711499	-0.000717240	+0.000357130
96	-0.000178334	-0.000178692	+0.000089250

We see that B_n is not superior to A_n for the 96-sided polygons Archimedes took. C_n is twice as accurate, a gain which hardly justifies the amount of calculation involved, but it is interesting that $\left|\frac{A_n - \pi}{C_n - \pi}\right|$ converges on 2.