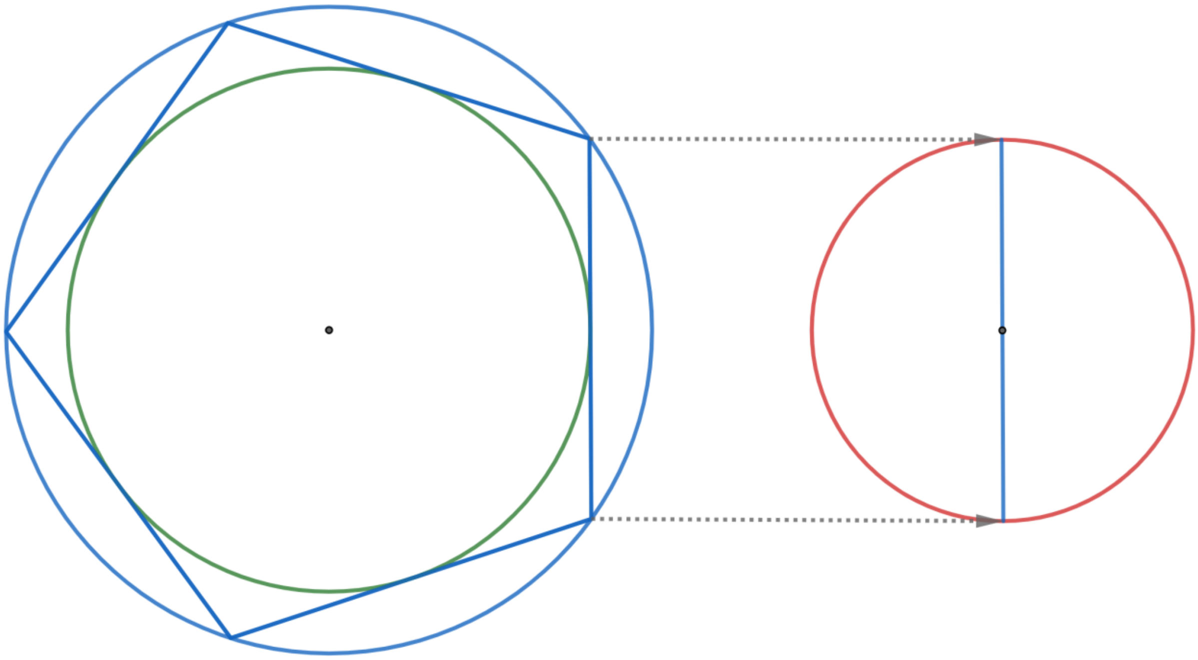
5.2.6 Du Fay’s area theorems

For more than two thousand years the world has shared the textbook which brought together the mathematical knowledge of the Ancients: Euclid’s Elements. Imagine all the ideas it has prompted in those exposed to it. There must be countless theorems lost to us, most not even recorded. 99.9% of these results would not be deep, but a small fraction - *a small fraction of a very large number, therefore significant* - would be original.

If you bring up the Wikipedia entry for Charles François de Cisternay du Fay (1698-1713), you will see him described as a ‘chemist’ but, running down the list of publications, you will see that most concern what we would now call ‘physics’. No scientist at the time would be pigeon-holed in that way. Indeed the term ‘scientist’ is itself a nineteenth-century term. And we are right not to pigeon-hole du Fay because the fifth item down is a work of pure mathematics. English-speaking readers would have remained ignorant of this work had it not been for the researches of the late John Sharp of the London Knowledge Lab.

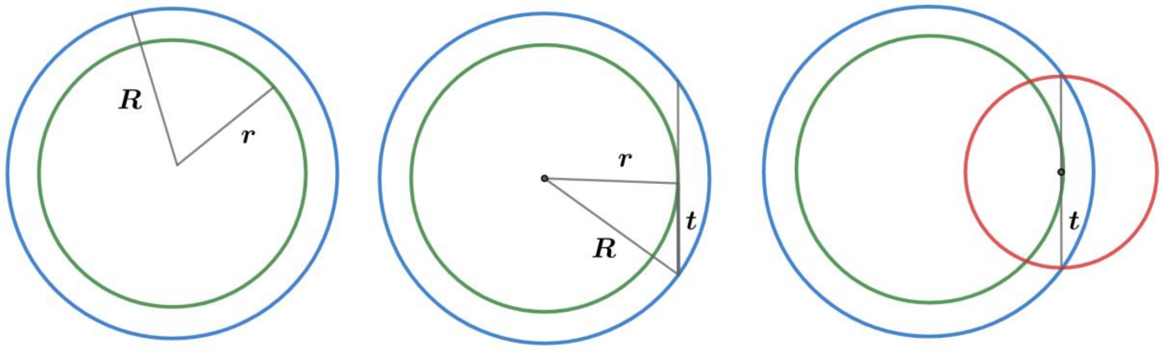
Though du Fay will be our focus, we shall make one or two digressions along the way.

The figure below shows a regular polygon trapped between two circles. (Such figures are called *bicentric* and include all regular polygons. Du Fay asks, “What is the area between the inner and outer circle?” and answers, “It is the area of a circle on a polygon side as diameter”.

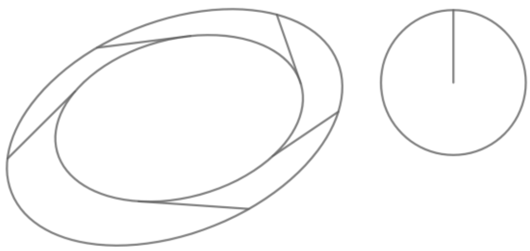


We shall not give du Fay’s proof but take one step back and two steps forward in time.

Generations of schoolchildren asked for the area of the annulus, the ring between two concentric circles, have argued from the figure below left that it is $\pi(R^2 - r^2)$. The young Mamikon Mnatsakanian (born 1942) was more interested in the middle figure, because it shows that this area, πt^2 by Pythagoras, is that of a circle with a line segment as diameter which is a chord of the outer circle tangent to the inner.

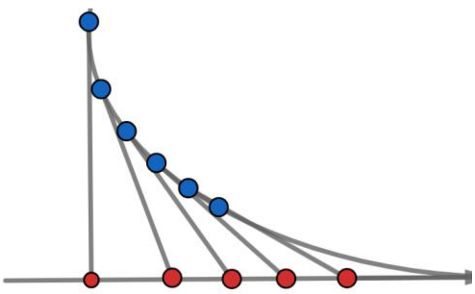


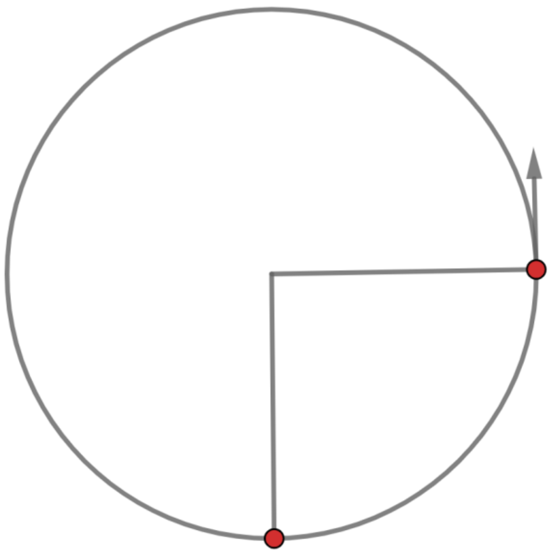
This explains du Fay's result. The sides of du Fay's polygon are snapshots of the one chord/tangent as it sweeps round. We shall look at further ones but first return to Mnatsakanian. He realised that the inner curve need not be a circle. The line is simply a tangent of fixed length sweeping out an area. If it arrives back at its original position, it will have swept out a circle with the half-tangent as radius:



Christening his insight 'the method of sweeping tangents', he applied it to all sorts of things, for example the area between two bicycle tyre tracks as the rider turns a corner. A particularly neat example is this. A child (red dot) pulls a toy truck (blue dot) out of the garden gate and along a kerb. (From that description, the curve it follows is called a *tractrix*.)

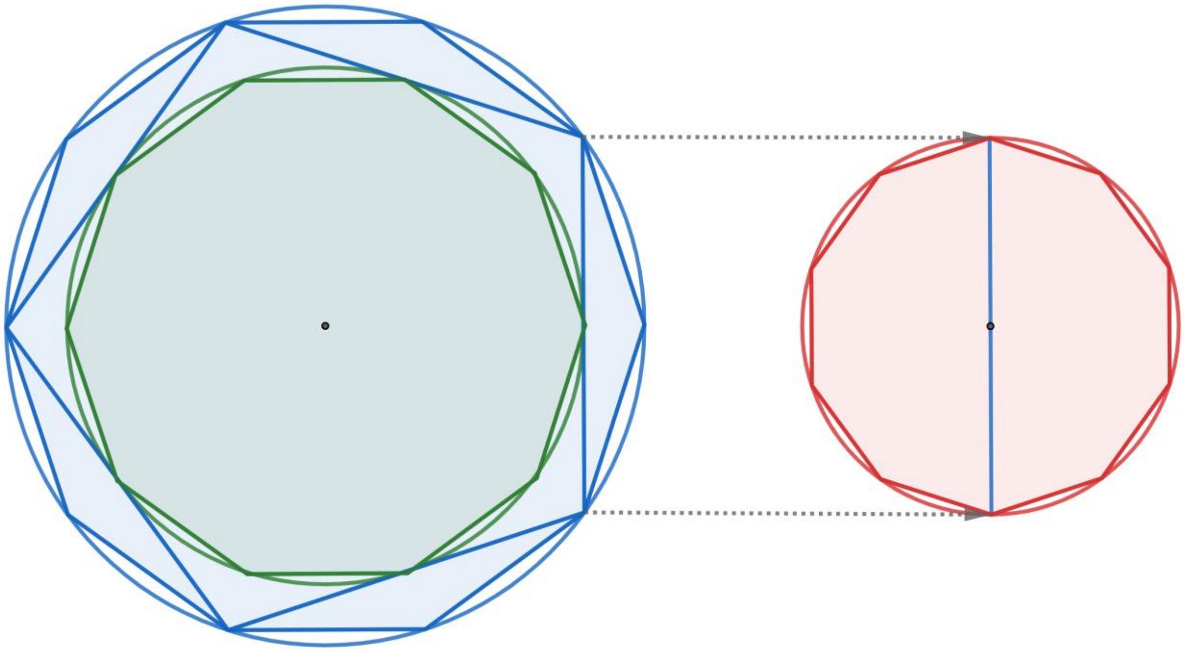
What is the area between the tractrix and the kerb? Answer: that of the quarter circle shown, whose radius is the length of the string.





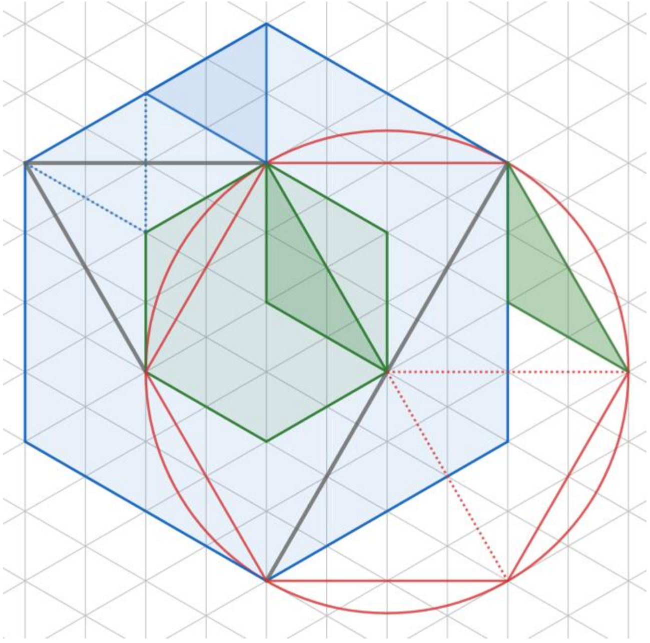
Back to du Fay. Here is another result.

He takes an even-sided polygon (here a 10-gon) and joins alternate vertices to form one with half the number of sides (here 5 therefore). Within that, in the same orientation as the original, he draws a copy. He now asks, "What is the area between the inner 10-gon and the outer 10-gon?" (the blue border) and answers, "It is the area of the 10-gon inscribed in a circle whose diameter is equal to a side of the 5-gon."



We've given each shape its circumcircle. A side of the pentagon is our sweeping tangent. The circumcircles play the role of the circles in the first result. We can now invoke the principle of similarity. The ratio of areas of the similar polygons is the same as the ratio of areas of their circumcircles. And we have proved the claim.

We can test it in a simple case:

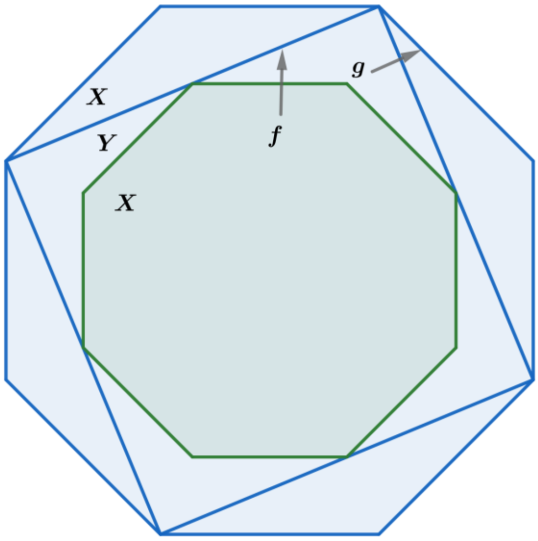


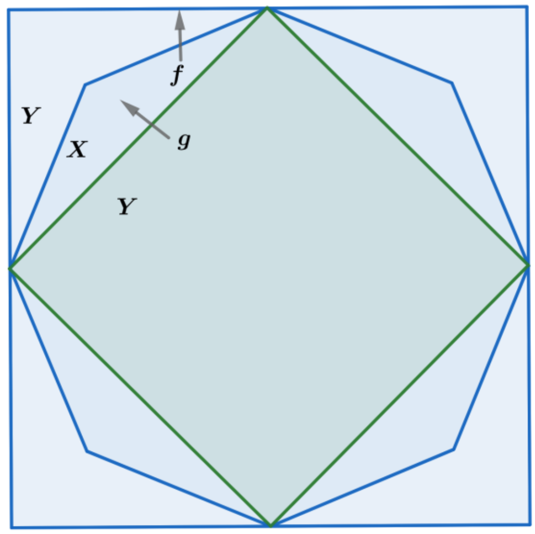
The outer hexagon is the inner scaled by 2, so the outer has 4 times its area, and the blue border therefore 3 times its area.

We can observe that the red hexagon is the inner scaled by $\sqrt{3}$, so again has 3 times its area.

.. or we can simply count triangles. Note that the dark blue equilateral triangle and the dark green isosceles triangle have the same area.

A principle which we didn't need apply there but is useful in trickier cases is what we may call the 'cycled nest' principle. We apply it elsewhere in these notes but describe it more fully here. All it does is instance the commutative rule of multiplication, as we shall see.



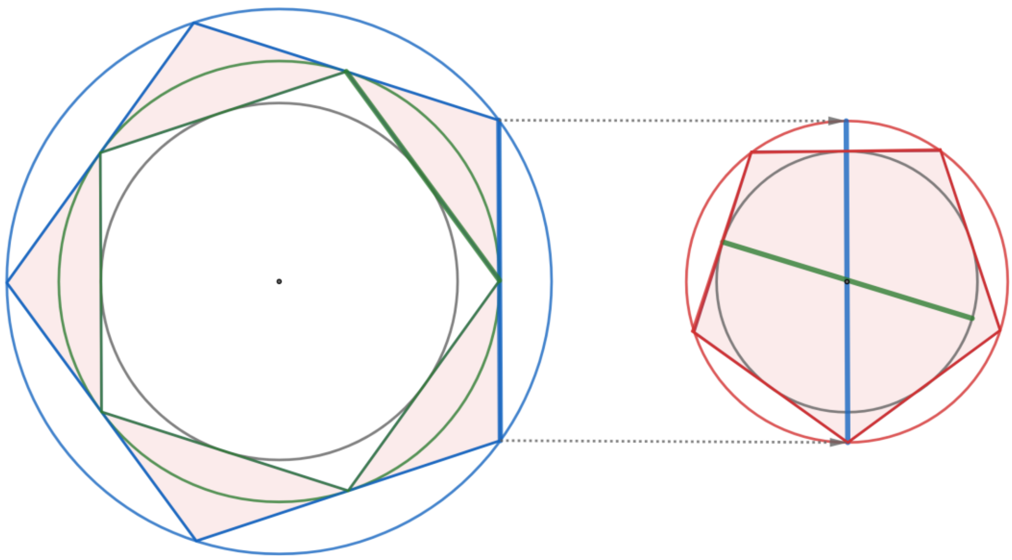


On the left, the inner octagon is a fraction f of the square, which is a fraction g of the outer octagon.

On the right, the inner square is a fraction g of the octagon, which is a fraction f of the outer square.

In both cases the inner figure is a fraction $fg = gf$ of the outer. Though it might take us some work to enumerate the separate fractions f and g, a simple dissection on the right gets us straight to the product fg. The figures are *nested* and we have permuted the nest *cyclically*: (octagon, square, octagon) ⟶ (square, octagon, square).

Back again to du Fay. In the next figure the inner polygon and the outer polygon are the same (here a pentagon). The vertices of the inner are the side midpoints of the outer. This means that the circumcircle of the inner is the incircle of the outer.



A side of the outer pentagon is our sweeping tangent. Arguing as before, we find that the area difference between the green pentagon and the blue pentagon (that of the five red isosceles triangles) is that of the red pentagon on the right. But we can say more. For the same regular polygon, the ratio circum-diameter to in-diameter is a constant. And this is equal to the ratio of a side length of the outer polygon to a side length of the inner. This means that the thick green line on the left has the same length as the thick green line on the right.

We finish with a principle which extends du Fay's results into the third dimension. It is named for Bonaventura Cavalieri (1598 - 1647) and states that, if two solids have the same cross-sectional areas at the same heights, they have the same volume.

These three cones are the same height. We ask, "What is the volume of the space between the blue cone and the green cone?" The answer is, "That of the red cone".