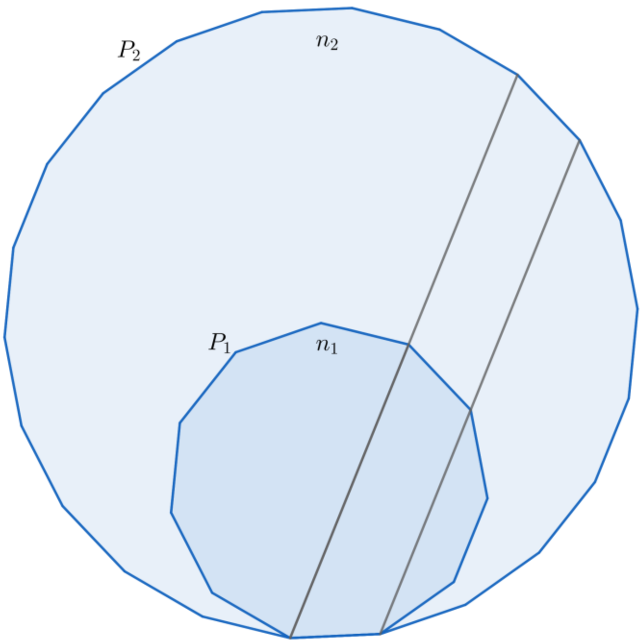
5.2.4 Collinear diagonals of two regular polygons sharing a side

In proving the next result we shall use two properties of diagonals which we have already met.

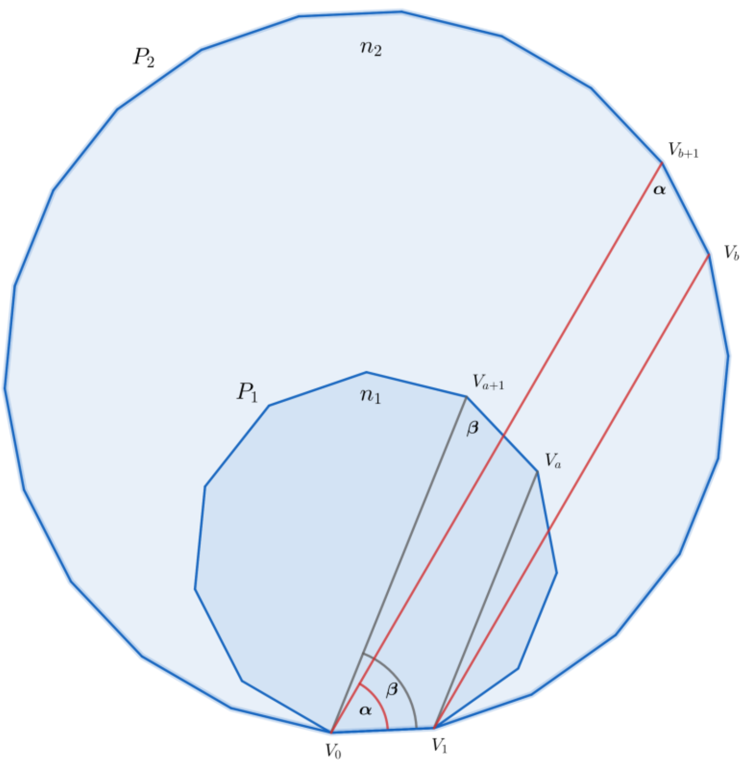


P_1 is a regular polygon of n_1 sides,
P_2 a regular polygon of n_2 sides.
P_1 and P_2 share a side.

Diagonals join two pairs of adjacent vertices in P_1 and two pairs of adjacent vertices in P_2. The pair defining the shared side is common.

We see that the P_1 pair are collinear with the P_2 pair.

What is the condition for this?



Since P_1, P_2 share a side, all the sides in the figure have equal length. P_1 has n_1 sides. We label and join vertices as shown. We require the condition for V_{a+1} to lie on V_0V_{b+1}, V_a to lie on V_1V_b.

By symmetry, two pairs of adjacent vertices in a regular polygon define a figure with equal base angles, be it isosceles triangle, isosceles trapezium or rectangle. Hence the labelling of the two pairs of equal angles in the figure. The condition we must impose is that $\alpha = \beta$.

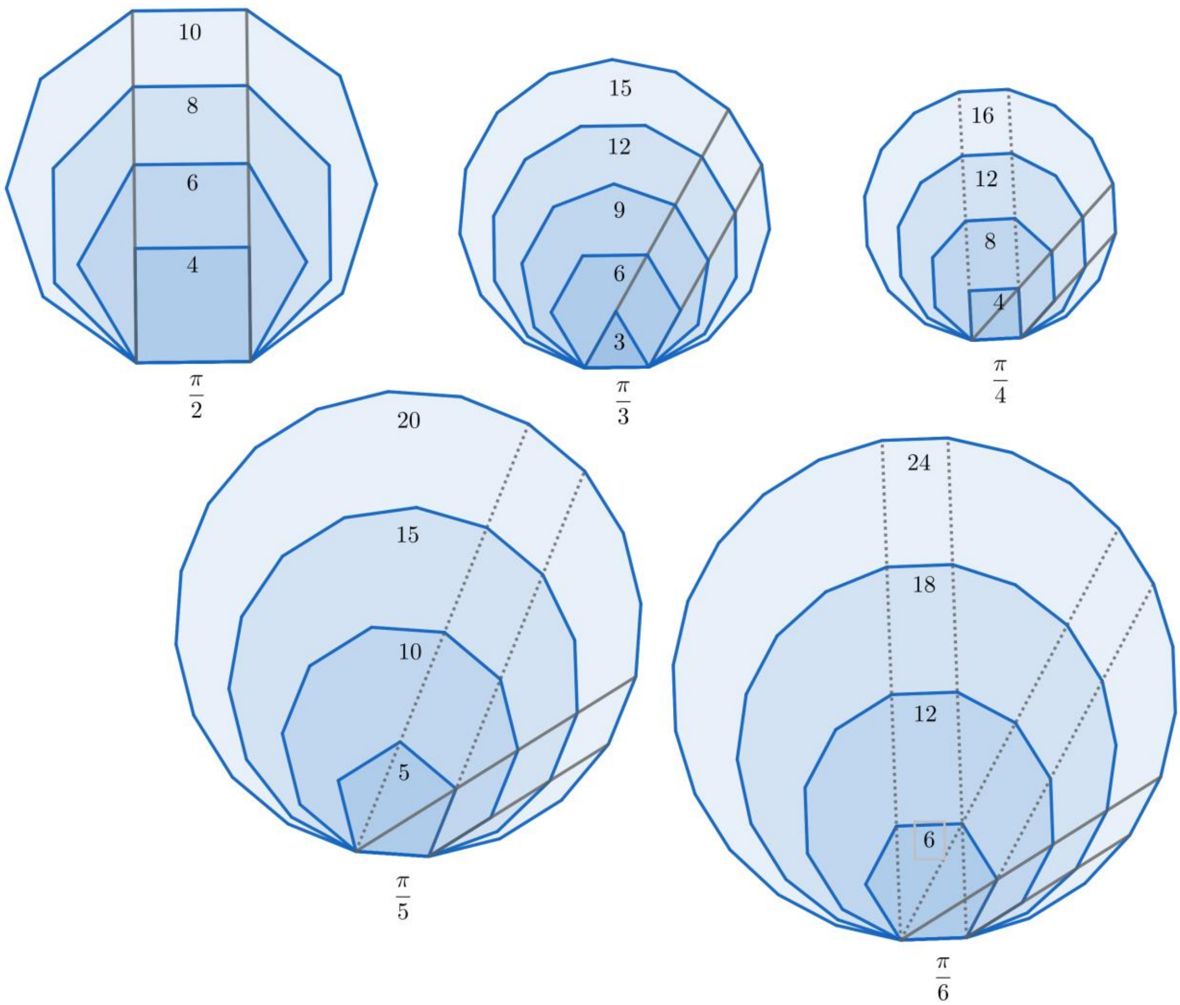
By the symmetry of a regular *n*-gon, the angle at a vertex subtended by *v* sides = $v \times$ the angle subtended by one, $\frac{v\pi}{n}$. Thus $\alpha =$ angle $V_0V_{b+1}V_b = \frac{b\pi}{n_2}$, $\beta =$ angle $V_0V_{a+1}V_a = \frac{a\pi}{n_1}$. Since $\alpha = \beta, \frac{a\pi}{n_1} = \frac{b\pi}{n_2}, \frac{b}{a} = \frac{n_2}{n_1}$. For the alignments to be possible we only require n_1 and n_2 to have a common factor, i.e. $\frac{n_2}{n_1} = k, k > 1$.

The procedure to locate the required vertices on the figure is as follows.

With $\frac{b}{a}$ in lowest terms, from V_o, V_1 we count *a* positions in a given sense round the smaller polygon, and *b* positions in the same sense round the larger one.

The smallest angle which yields a pair of parallels is $\frac{\pi}{n_1}$.
The number of pairs of parallels possible is $t = \frac{\pi}{2} \div \frac{\pi}{n_1} = \frac{n_1}{2}$, where *t* is the largest integer $\leq \frac{n_1}{2}$.

We can nest polygons so that more than two share a side. If they have $n_1, n_2, n_3, \ldots$ sides, we require, as before, that these numbers have a common factor > 1.



10
8
6
4
$\frac{\pi}{2}$
15
12
9
6
3
$\frac{\pi}{3}$
16
12
8
4
$\frac{\pi}{4}$
20
15
10
5
$\frac{\pi}{5}$
24
18
12
6
$\frac{\pi}{6}$