5.2.2 Three consecutive diagonals

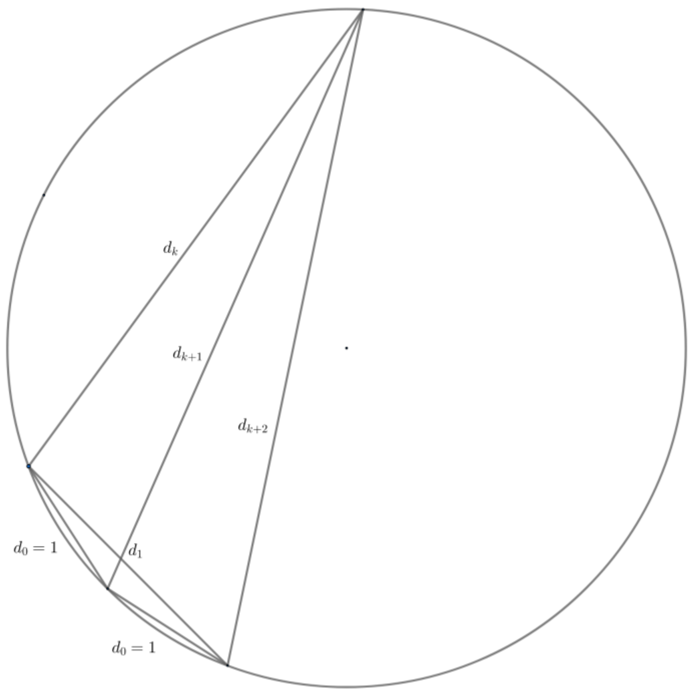
Consider a regular *n*-gon of unit side. Label the lengths of the consecutive diagonals from a given vertex, starting from a side of length $d_0 = 1, d_1, d_2, d_3, \dots, d_k, \dots, 0 \le k \le \left\lfloor \frac{n}{2} \right\rfloor$.
Show that $d_1 d_{k+1} = d_k + d_{k+2}$.

We suggest a number of approaches.

1st proof

A We see that the theorem follows immediately from Ptolemy's theorem for cyclic quadrilaterals.

A



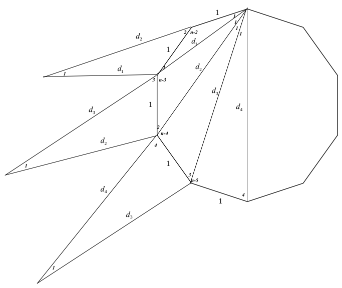
2nd proof

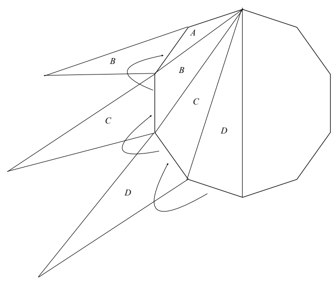
Lemma: As a consequence of the 'same segment' theorem, equal segments of a circle subtend equal angles at the same apex. Thus consecutive diagonals from a given vertex of a regular *n*-gon are separated by an angle $\frac{\pi}{n}$.

B The lengths of the diagonals are labelled and the angles are shown in multiples of π/n. The diagonals divide the polygon into triangular regions. ***C*** shows how rotating a region about a vertex completes an isosceles triangle with an adjacent region similar to that defined by the shortest diagonal. Equating ratios of the longest to a shorter side gives:

$\frac{d_1}{1} = \frac{d_3+d_1}{d_2} = \frac{d_4+d_2}{d_3} = \frac{d_5+d_3}{d_4} = \ ... = \frac{d_{k+2}+d_k}{d_{k+1}}$, whence: $d_1 d_{k+1} = d_k + d_{k+2}$ as required.

B ***C***





3rd proof

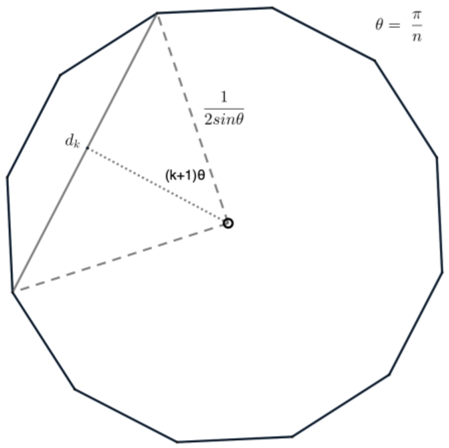
From the figure below we have by direct substitution:

$$\text{LHS} = d_1 d_{k+1} = \frac{sin\left(\frac{2\pi}{n}\right)}{sin\left(\frac{\pi}{n}\right)} \times \frac{sin\left[\frac{(k+2)\pi}{n}\right]}{sin\left(\frac{\pi}{n}\right)} = \frac{2\ sin\left[\frac{(k+2)\pi}{n}\right]cos\left(\frac{\pi}{n}\right)}{sin\left(\frac{\pi}{n}\right)}.$$

$$\text{RHS} = d_k + d_{k+2} = \frac{sin\left[\frac{(k+1)\pi}{n}\right]+sin\left[\frac{(k+3)\pi}{n}\right]}{sin\left(\frac{\pi}{n}\right)} = \frac{2\ sin\left[\frac{(k+2)\pi}{n}\right]cos\left(\frac{\pi}{n}\right)}{sin\left(\frac{\pi}{n}\right)} = \text{LHS as required.}$$

It is interesting to note the implied identity, which is valid for all $k \in$ Z but where $(k+2)\varphi$ is not a multiple of π:

$$cos\varphi = \frac{sin[(k+1)\varphi]+sin[(k+3)\varphi]}{2sin[(k+2)\varphi]}.$$



$\theta = \frac{\pi}{n}$
$\frac{1}{2sin\theta}$
d_k
(k+1)θ