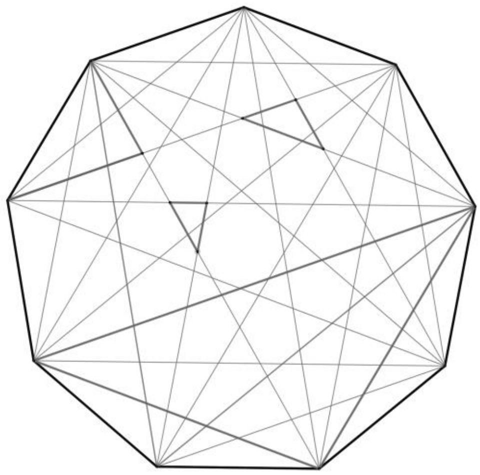
(Regular polygons are both cyclic and tangential.)

5.2.1 Triangles among the diagonals



Triangles like those picked out by thick lines in the figure are made by joining three diagonals of a regular *n*-gon, or segments of them defined by diagonal intersections. *Show that in such a triangle, unless all three sides are equal, no more than two of the sides can stand in integer ratio.*

We situate the polygon in its circumcircle.

Lemma A: As a consequence of the 'same segment' theorem, equal segments of a circle subtend equal angles at the same apex. Thus consecutive diagonals from a given vertex of a regular *n*-gon are separated by an angle $\frac{\pi}{n}$.

Lemma B: From **Lemma A** the angle between any two diagonals from the same vertex is a multiple of $\frac{\pi}{n}$. The whole set of diagonals can be generated by rotation about the centre, in a chosen sense, of the set of diagonals from a single vertex by consecutive multiples of $\frac{2\pi}{n}$. As a result, the p^{th} diagonal from vertex *l* cuts the q^{th} diagonal from vertex *m* at an angle of $\frac{2(m-l)\pi}{n} + \frac{(q-p)\pi}{n}$, an integer multiple of $\frac{\pi}{n}$. In a triangle whose vertices are those of the regular *n*-gon, all angles between diagonals are therefore integer multiples of the same unit, and thus stand in integer ratio.

Lemma C = Niven's theorem: For angles in integer ratio, there are just two values of sine giving an integer ratio.

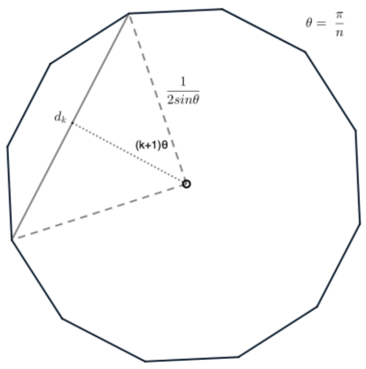
Proof of the general case:

By **Lemma B** the angles of our triangle stand in integer ratio. Let our triangle have side lengths a, b, c, opposite angles in integer ratio α, β, γ. By the sine rule we have: $\frac{a}{b} = \frac{\sin\alpha}{\sin\beta}$, $\frac{b}{c} = \frac{\sin\beta}{\sin\gamma}$, $\frac{c}{a} = \frac{\sin\gamma}{\sin\alpha}$. By **Lemma C** only one of the three ratios can be integral. The required result follows.

Proof in the restricted case of *complete* diagonals:

By **Lemma B** the angles of our triangle stand in integer ratio.

From the figure we have: $d_k = \frac{sin\left[\frac{(k+1)\pi}{n}\right]}{sin\left(\frac{\pi}{n}\right)}$.



Thus each side of our triangle corresponds to a different ratio of sines. By **Lemma C** only one can be integral. The required result follows.