5.1 Maximisation problems

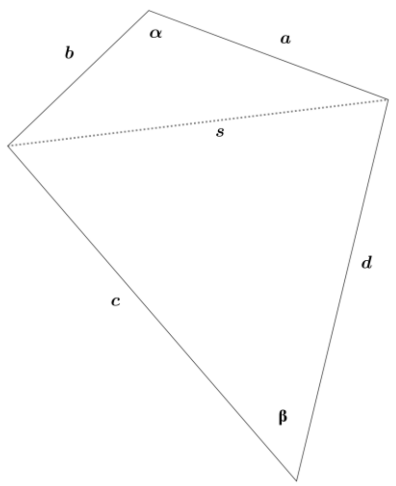
5.1.1 Zenodorus' theorem

We begin by determining

(A) the largest *n*-gon with given side lengths

But we first need the lemma obtained by determining ...

the largest *quadrilateral* with given side lengths



Following Alsina & Nelsen, we write and equate two expressions for s **[Equation 1]** and an expression for the sum of the areas of the two triangles **[Equation 2]**.

$s^2 = a^2 + b^2 - 2ab\cos\alpha = c^2 + d^2 - 2cd\cos\beta$
$a^2 + b^2 - c^2 - d^2 = 2(ab\cos\alpha - cd\cos\beta)$
[Equation 1]

Area $K = \frac{1}{2}(ab\sin\alpha + cd\sin\beta)$
[Equation 2]

Write $cos\ \alpha$ as $c\alpha$, $\sin\alpha$ as $s\alpha$, $\cos\beta$ as $c\beta$, $\sin\beta$ as $s\beta$.

We square **[Equation 1]**:
$4[a^2b^2(c\alpha)^2 - 2abcd(c\alpha)(c\beta) + c^2d^2(c\beta)^2] = [a^2 + b^2 - c^2 - d^2]^2$.

We scale **[Equation 2]** by 4 and square it:

$4[a^2b^2(s\alpha)^2 + 2abcd(s\alpha)(s\beta) + c^2d^2(s\beta)^2] = 16K^2$.

Noting that $(\sin\theta)^2 + (\cos\theta)^2 = 1$, we add those equations:

$4[a^2b^2 + c^2d^2] - 8abcd(\alpha + \beta) = 16K^2 + [a^2 + b^2 - c^2 - d^2]^2$.
$16K^2 = 4[a^2b^2 + c^2d^2] + [a^2 + b^2 - c^2 - d^2]^2 - 8abcd\ c(\alpha + \beta)$. **[Equation 3]**

The right hand side is greatest when $c(\alpha + \beta)$ takes its lowest value. This is -1, when $\alpha + \beta = \pi$.

Thus the quadrilateral with given sides and greatest area is cyclic. **[Lemma]**

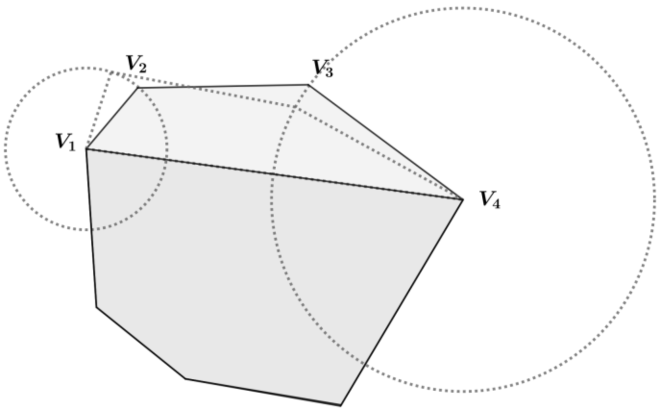
Before we use this result we derive an immediate corollary from **[Equation 3]**.

Corollary

The area of a cyclic quadrilateral with sides *a, b, c, d* is given by
$16K^2 = 4[a^2b^2 + c^2d^2] + [a^2 + b^2 - c^2 - d^2]^2 + 8abcd.$

Incorporate the last term in the first bracket, obtain a difference-of-two-squares, and show the resulting product of four brackets. You should find:
$16K^2 = (a + b + c - d)(a + b - c + d)(a - b + c + d)(-a + b + c + d).$

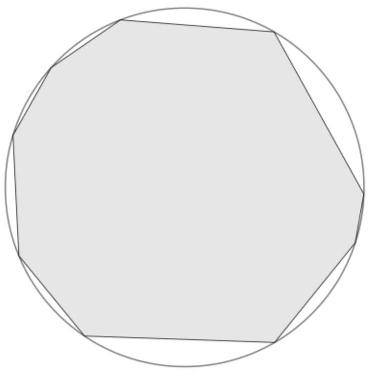
With the semiperimeter s=$\frac{a+b+c+d}{2}$, we have Brahmagupta's formula:
$K = \sqrt{(s-a)(s-b)(s-c)(s-d)}$. [See **From Heron to von Staudt**. (When $\alpha + \beta$ is not necessarily equal to π, that is, the quadrilateral is a general one, we have Bretschneider's formula.)]
We now apply our **Lemma**.



We treat the polygon as a linkage. We keep the dark grey part fixed. We are free to pivot the sides as shown until the quadrilateral $V_1V_2V_3V_4$ is cyclic. By our lemma the light grey region then has maximal area. The same is therefore true of the combined region. The area of the whole polygon will only be maximal when every set of four vertices we choose lie on the same circle, that is the polygon is cyclic.

An elegant way of seeing this is due to David Epstein.

Again we treat our polygon as a linkage. Its area is the area of the circle less the white segments. We can imagine flexing the polygon, which would take the segments to different positions and alter the polygon's area while maintaining its perimeter and the perimeter of the closed curve. Since the total area of the white segments is fixed, by maximising the total area of the whole figure we maximise the polygon's area. The figure of given perimeter enclosing the greatest area is a circle. Therefore the polygon achieves its greatest area when its vertices are concyclic.



So we establish that **the biggest polygon with given side lengths is cyclic.**

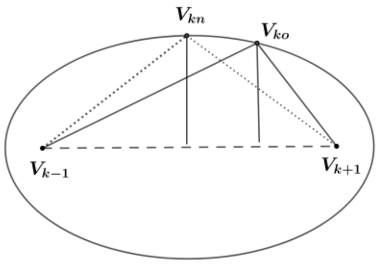
(B) The largest *n*-gon in a given circle

We need the following lemma.

The largest triangle with given base and perimeter is isosceles.

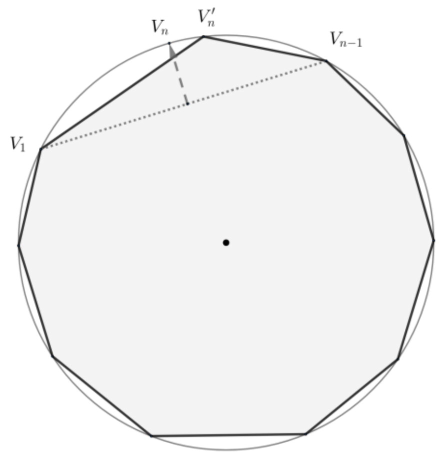
First, **(a)** an algebraic proof, then **(b)** a geometric one.

(a) We have a triangle with given base, a, and given perimeter, $2s = a + b + c$.
Let the triangle have area K, but, more conveniently, we shall maximise K^2.
By Heron's theorem, $K^2 = s(s-a)(s-b)(s-c) = constant \times (s-b)(s-c)$.
We must therefore maximise $(s-b)(s-c) = \frac{[a-(b-c)]}{2} \times \frac{[a+(b-c)]}{2} = \frac{a^2-(b-c)^2}{4}$.
This takes its greatest value when $b = c$, i.e. the triangle is isosceles.



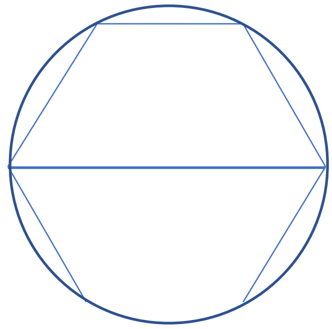
(b) A property of the ellipse is that the perimeters of the two triangles in the figure are equal. The one with the greater area is that with the greater height. The greatest height is achieved when the altitude is a semi-minor axis of the ellipse. The triangle is then isosceles.

Consider a cyclic *n*-gon where the vertices V_1 to V_{n-1} belong to a regular *n*-gon but vertex $V_n{}'$ does not. Consider the area of the triangle defined by this vertex and the diagonal V_1V_{n-1}. It will be increased by moving $V_n{}'$ till it lies on the perpendicular bisector of the diagonal, thus making arcs $V_{n-1}V_n$ and V_nV_1 equal. Therefore, given any cyclic *n*-gon where the vertices are irregularly spaced, the area will always be increased by making the arcs equal. In other words, the **cyclic** *n*-gon of greatest area is the **regular** form.



We now bring **(A)** and **(B)** together. Since the biggest polygon with given side lengths, and therefore perimeter, is cyclic, and the biggest cyclic polygon is regular, **the biggest polygon with given perimeter is regular**. This is Zenodorus' theorem.

5.1.2 The largest polygon in a semicircle



One consequence is that the $(n+1)$-gon of greatest area in a semicircle is given by bisecting a regular $2n$-gon by a diagonal. For, if a bigger $(n+1)$-gon were possible, then, by symmetry, a bigger $2n$-gon would be possible. For example, the biggest quadrilateral in a semicircle is an isosceles trapezium where the base has twice the length of the top, corresponding to half a regular hexagon.