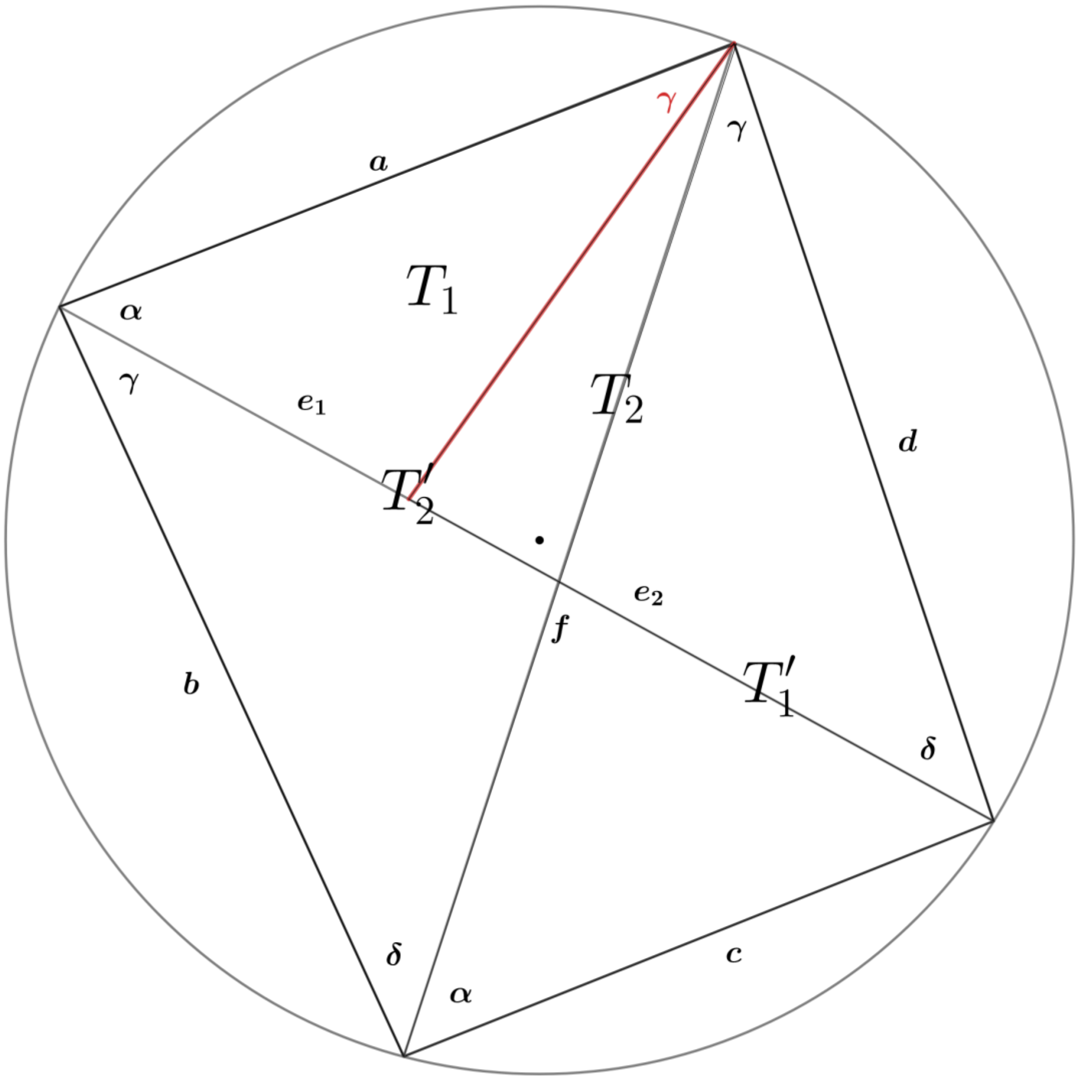
4.5 Cyclic quadrilaterals (*C* in our classification) and their diagonals: Ptolemy's theorem



From the same segment theorem we have the equal angles marked. We have constructed the red line. Because of the angle γ, triangles T_1, $T_1{}'$ (combining two of the smaller triangles) are similar (three angles equal). Similarly, triangles T_2 (combining two of the smaller triangles), $T_2{}'$ (combining three of the smaller triangles) are similar. Writing out ratios of corresponding sides, we have:

$\frac{e_1}{a} = \frac{c}{f}$, $\boldsymbol{e_1 f = ac}$; $\frac{e_2}{d} = \frac{b}{f}$, $\boldsymbol{e_2 f = bd}$.
Adding the two equations, noting that $e_1 + e_2 = e$, the length of the second diagonal, we have:

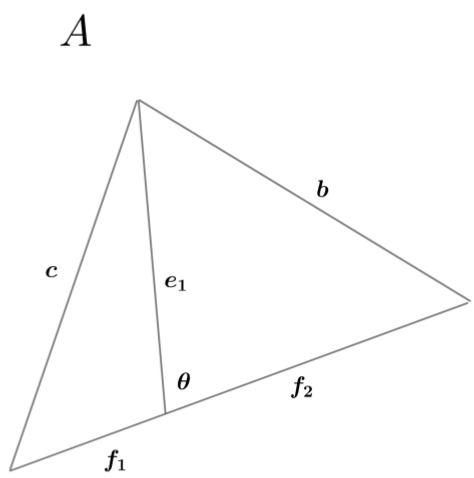
$\boldsymbol{ef = ac + bd}$. This is Ptolemy's theorem: *The product of the diagonals is the sum of the products of the pairs of opposite sides*.

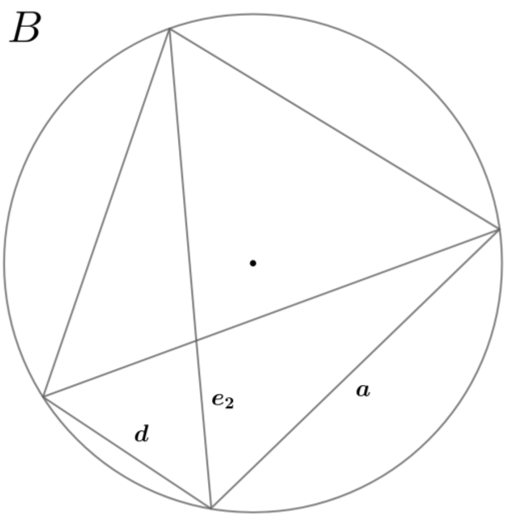
We can derive other relations.

Dividing the diagonal f into two segments, f_1, f_2, the intersecting chord theorem gives us immediately $e_1e_2 = f_1f_2$.

Other relations emerge by establishing a theorem about a triangle median (Stewart's theorem) and extending it to form a diagonal of a quadrilateral.

A So, first Stewart's theorem.





Use the cosine formula in both triangles to obtain equations for b^2 and c^2. Notice that $\cos(\pi - \theta) = -\cos\theta$. *Eliminate $2e_1\cos\theta$ between the equations.*
Simplify to obtain:$b^2f_1 + c^2f_2 = (f_1 + f_2)({e_1}^2 + f_1f_2)$. This is Stewart's theorem.

B By the intersecting chord theorem we can substitute e_1e_2 for f_1f_2. Recalling that $e_1 + e_2 = e$, $f_1 + f_2 = f$, this gives:
$b^2f_1 + c^2f_2 = e_1ef$.
We can form a similar expression involving the other pair of quadrilateral sides:
$a^2f_1 + d^2f_2 = e_2ef$.
Adding the two equations gives:
$(a^2 + b^2)f_1 + (c^2 + d^2)f_2 = e^2f$. **[Equation 1]**
We can form a similar expression splitting the diagonal *e* into its two segments:
$(b^2 + c^2)e_2 + (d^2 + a^2)e_1 = ef^2$. **[Equation 2]**
By Ptolemy's theorem $ef = ac + bd$. We can substitute this expression into either of our two new equations to give these further results:
$(a^2 + b^2)f_1 + (c^2 + d^2)f_2 = e(ac + bd)$. **[Equation 3]**
$(b^2 + c^2)e_2 + (d^2 + a^2)e_1 = f(ac + bd)$. **[Equation 4]**