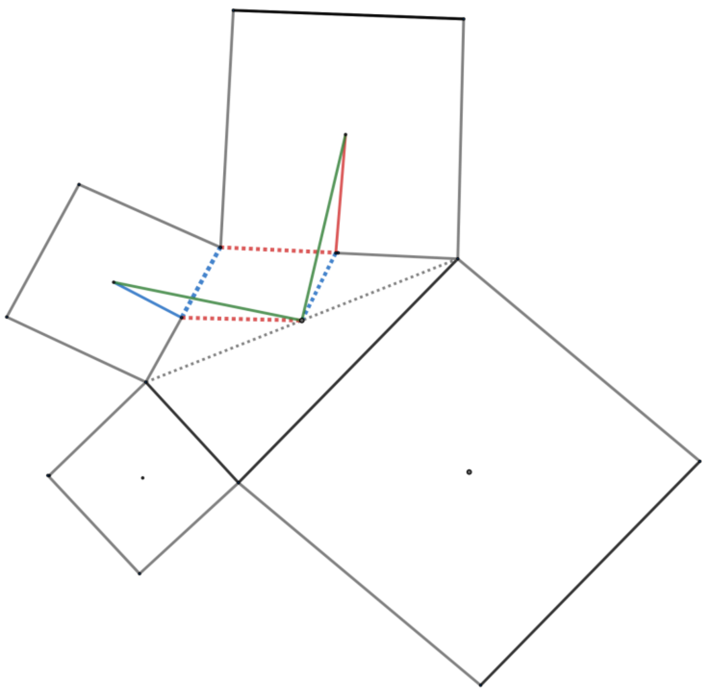
4.4 From Finsler-Hadwiger to van Aubel

4.4.1: Finsler-Hadwiger

The Finsler-Hadwiger theorem concerns a pair of squares hinged at a vertex. van Aubel's theorem concerns a chain of four squares hinged so as to enclose a quadrilateral. We shall use the same figure to show both.

We have chosen a diagonal of the central quadrilateral and marked the midpoint. The solid lines join the square centres to the midpoints of sides. In each colour the solid and dotted lines are equal and perpendicular. *Why?* The quadrilateral formed from the dotted lines is a parallelogram. *Why?*

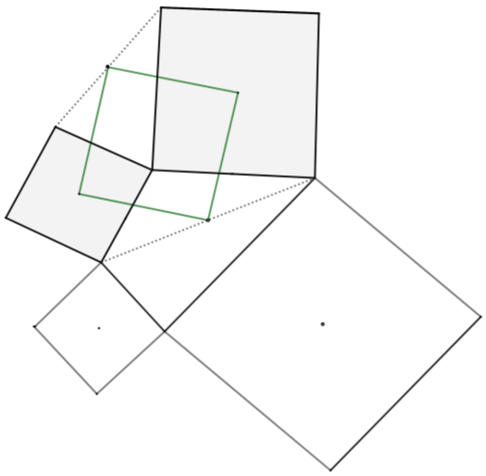
One triangle has sides: red - dotted blue - green.
Another has sides: dotted red - blue - green.



The angles between
the red and dotted blue sides
must be equal to the angle between
the blue and dotted red sides.
Why?

Therefore the two triangles are congruent side-angle-side.

Since the corresponding red sides, and the corresponding blue sides, are equal and perpendicular, so are the green sides.

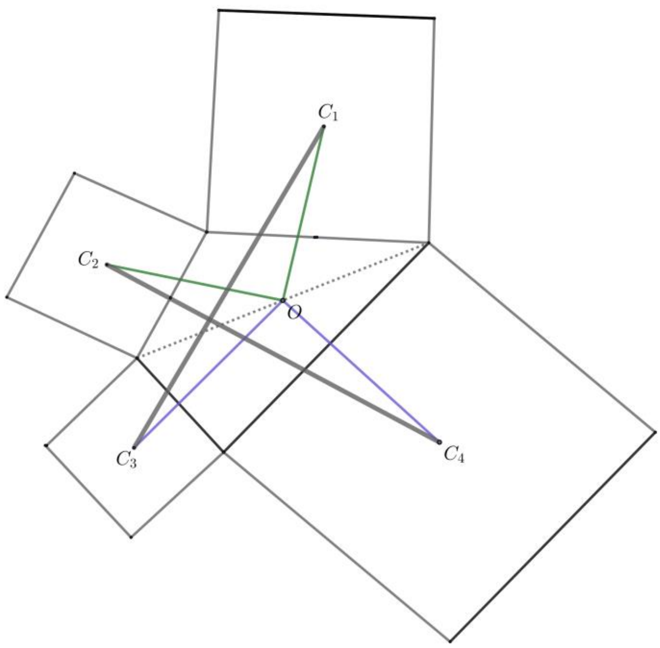


That the green lines are equal and perpendicular is all we need to prove the Finsler-Hadwiger theorem because, as the left hand figure shows, we could equally well have chosen the midpoint of the upper dotted line and arrived at the same result.

The green quadrilateral is therefore a square. This is the Finsler-Hadwiger theorem.

4.4.2: van Aubel

We now use what we have found about the green lines to prove van Aubel's theorem.

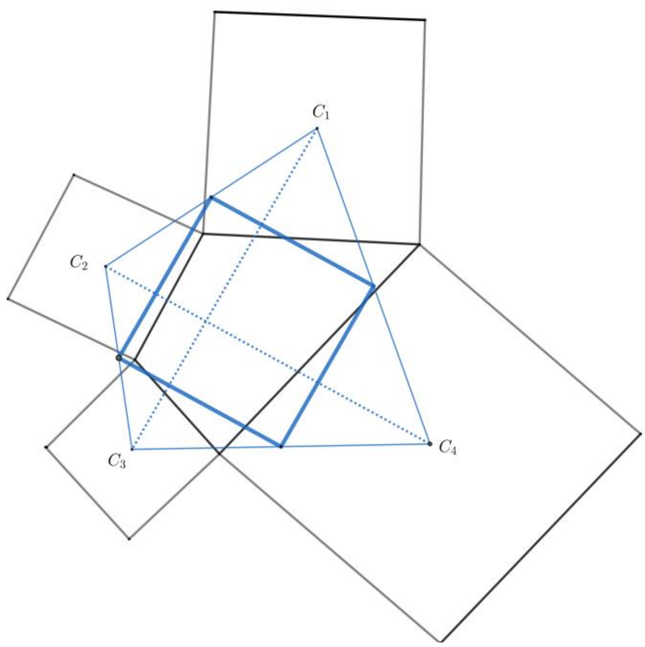


Triangles C_1OC_3 and C_2OC_4 Each have a pair of green and lilac sides, but they are perpendicular. So, just as before, we have a pair of equal angles between equal, perpendicular sides and the triangles are congruent and mutually rotated by a right angle.

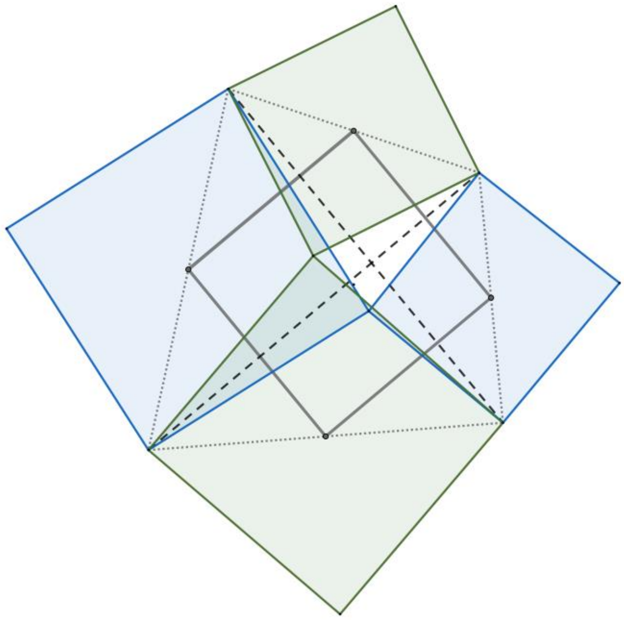
The black sides are therefore equal and perpendicular. This is van Aubel's theorem.

4.4.3 Corollaries

van Aubel's quadrilateral $C_1C_2C_3C_4$ is both *orthodiagonal and equidiagonal*. Join the side midpoints of any quadrilateral and you get a parallelogram. In the case of the van Aubel quadrilateral the result is a square.



Returning to our Finsler-Hadwiger diagram, we can find more squares in it.



Begin with the hinged pair of green squares. By the Finsler-Hadwiger construction we produce the black square.

Begin instead with the hinged pair of blue squares. By the Finsler-Hadwiger construction we again produce the black square.

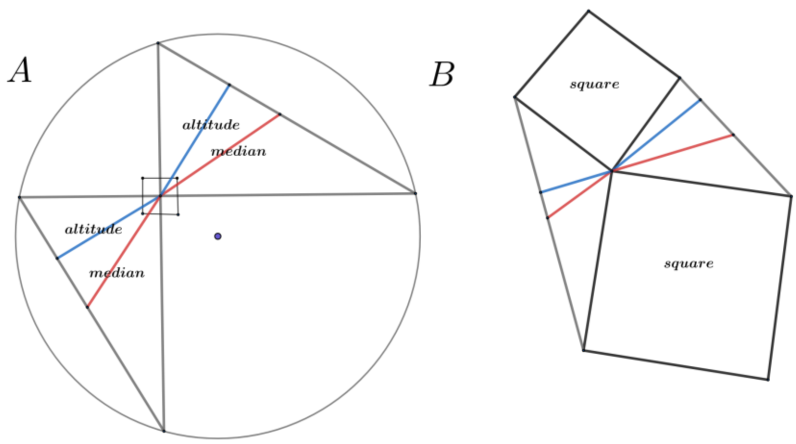
The green pair and the blue pair are thus complementary.

Comparing the black dotted figure with the thin blue figure above, we see that the black dotted figure is an equidiagonal, orthodiagonal quadrilateral.

Indeed we still have the van Aubel figure. The blue and green squares are still built on the four sides of a quadrilateral, the thin one with two adjacent blue sides and two adjacent green sides. Follow clockwise, starting at the top. We have a green square on a green side, then a blue square on a blue side, then again a blue square on a blue side, *but turned over*, and lastly a green square on a green side, *but turned over*.

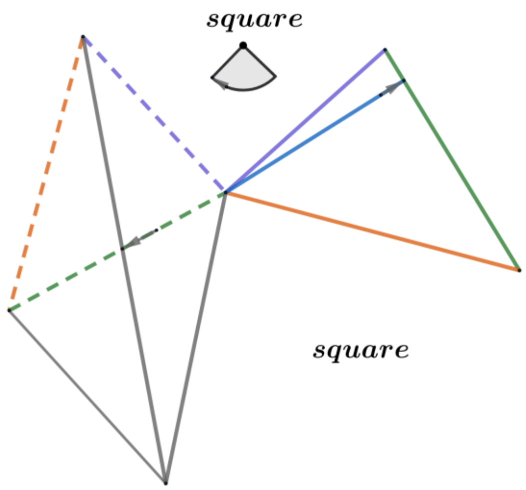
4.4 Finsler-Hadwiger and Brahmagupta

In section **4.3: Orthodiagonal quadrilaterals**, subsection **Further properties of a *CO***, we meet what we may call the Brahmagupta property from the theorem illustrated there (***A***). The same dual property holds in the Finsler-Hadwiger figure (***B***). In each we have a pair of triangles sharing a vertex from which *the altitude to one is a median of the other*.



A In 4.3, subsection **Further properties of a *CO*** we establish the relationship by identifying equal angles and thence isosceles triangles.

B Here we take the upper square and the attached triangle on the right and rotate the figure clockwise by a right angle. The solid line segments map to dashed segments of the same colour



By this transformation:
1.) If we extend the dashed green line up and to the right, it meets the solid green line at right angles.
2.) The dashed orange segment is equal and parallel to the solid black segment. The dashed green line is therefore one diagonal of a parallelogram and bisects the other. So half the dashed green segment is a median of the triangle containing it.
The same line, extended as necessary, is thus both an altitude of the right hand triangle and a median of the left hand one.

We could equally well take the lower square and the attached triangle to the left, and argue in the same way to show the dual relation.