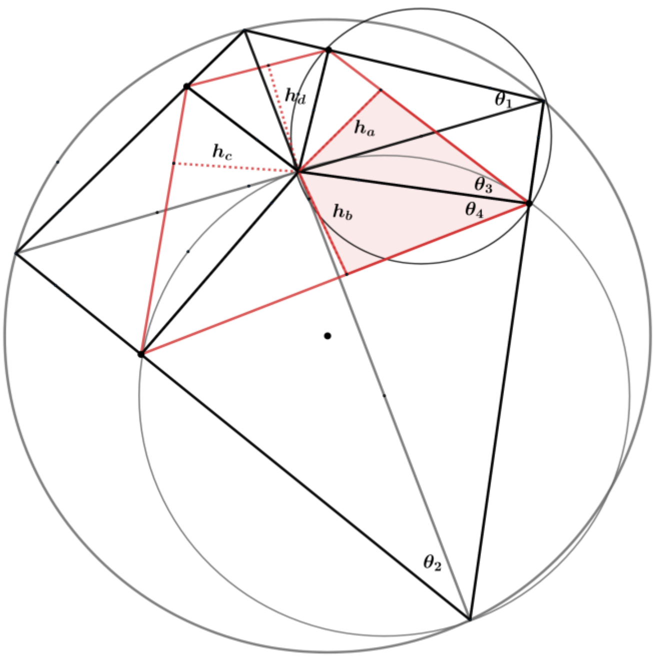
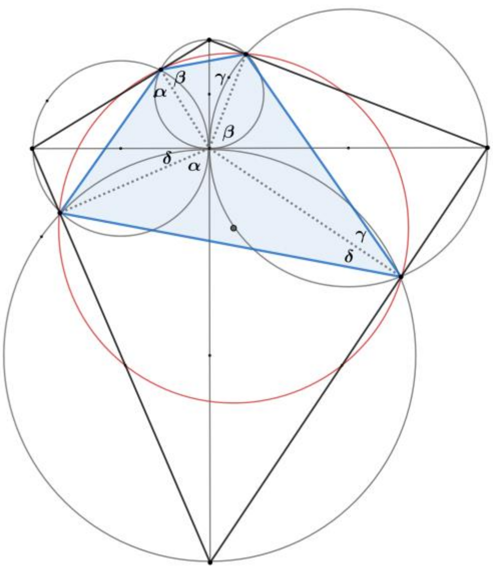
4.3.8 Pedal quadrilaterals from the intersection of diagonals in an *O*, a *C* and a *CO*

The figure formed by joining the feet of perpendiculars dropped from some special point on to the sides of a polygon is called a 'pedal' polygon. In the case of quadrilaterals, the results are particularly interesting if you choose as that special point the intersection of the diagonals.

We shall call the pedal quadrilateral from the diagonal intersection, P, and give the parent quadrilateral type as a suffix. So, for example, P_C will mean the pedal quadrilateral of a cyclic quadrilateral from the intersection of its diagonals.

Because of the right angles we are able to draw the small circles by the converse of Thales' theorem. Applying the same segment theorem first in the big circle, then in the small ones, we have:
$\theta_1 = \theta_2$;
$\theta_3 = \theta_1$, $\theta_4 = \theta_2$;
thus $\theta_3 = \theta_4$ and the two right triangles are congruent angle-side-angle.
Therefore $h_a = h_b$ and, working our way round the figure, we have
$h_a = h_b = h_c = h_d$, which are therefore radii of a circle inscribed in the red quadrilateral.



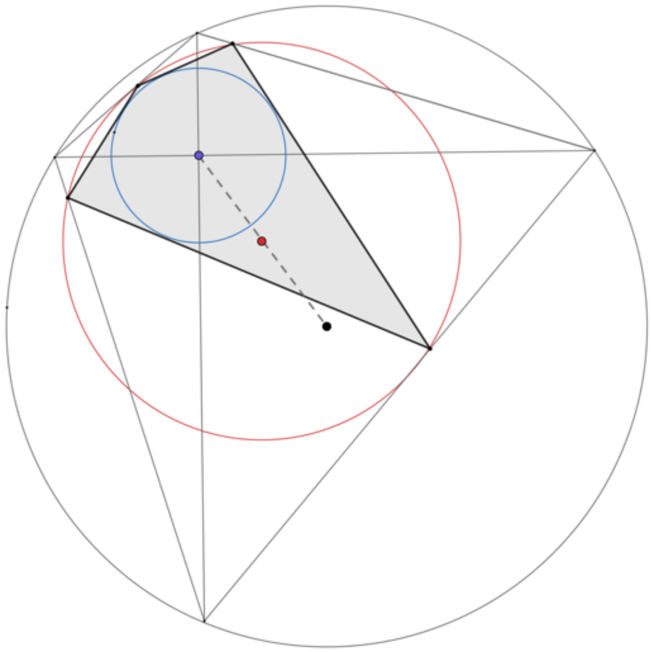


Using the alternate segment theorem, we are able to equate the angles marked. We see that $\delta + \alpha = \frac{\pi}{2}$, $\beta + \gamma = \frac{\pi}{2}$, so $\alpha + \beta + \gamma + \delta = \pi$.

We see that this is the sum of two opposite interior angles in the blue quadrilateral, which is therefore cyclic.

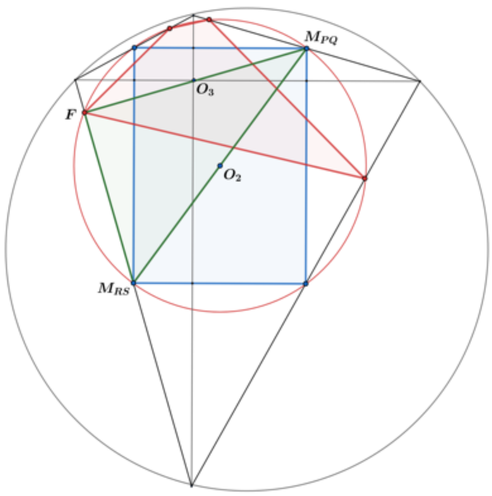
Combining those results

Since $P_c = T$ and $P_O = C$,
$\boldsymbol{P_{CO} = CT}$ **.**



It appears from the figure that the centres of the blue, red and black circles are aligned and equally spaced. We shall show that this is indeed the case.

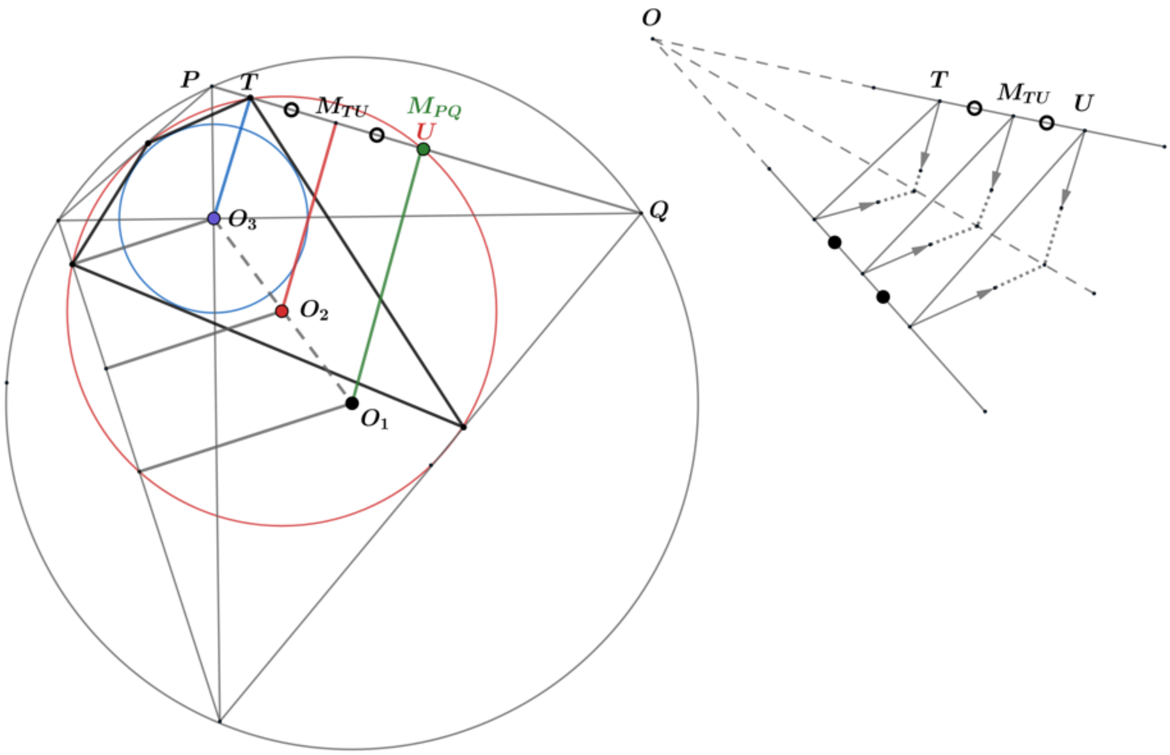
A In order to do so we first need to show that the blue and red quadrilaterals share the same circumcircle.



The blue quadrilateral is the Voronoi rectangle of the black one, that is, the figure formed by joining the side midpoints. The red quadrilateral is ours.

By Brahmagupta's theorem, $M_{PQ}O_3F$ is a straight line. $M_{PQ}M_{RS}$ is a diameter of the rectangle's circumcircle. By the converse of Thales' theorem the right angle at F means that F lies on the same circle. The same goes for the other red points so the red circle belongs to both quadrilaterals.

B We show three perpendiculars to *PQ*. *T*, *U* are the points in which the red circle cuts it. Therefore the midpoint M_{TU} lies on the perpendicular bisector of chord *TU*, which passes through O_2. Likewise M_{PQ}, the midpoint of *PQ*, a side of the grey quadrilateral, lies on the perpendicular bisector which passes through through O_1. But we know from ***A*** that M_{PQ} and *U* are the same point. We therefore have three parallel, equally spaced lines passing through the three centres. We show a matching set of lines extending from the opposite side of the grey quadrilateral. In the inset figure we show that we have three similar triangles which can be produced by enlargement from the centre *O*. By construction, their apices, our O_1, O_2, O_3, are collinear and equally spaced.



There is no end to the number of interesting theorems about quadrilaterals and circles. We end this section with another nested figure.

All the diagonals are concurrent.

The *CT* and the *T* are similar; the *C* and the *CO* are similar.

The *CT* and the *T* share the same pair of diagonals; the *C* and the *CO* share the same pair of perpendicular diagonals.

The figure is produced as follows. We begin with the *T* . Its diagonals divide it into four triangles. We insert the incircle in each. What we find is that their incentres are concyclic and, furthermore, the resulting quadrilateral is orthodiagonal. So that gives the *CO* and the *T*. We now use the converse of a result we proved above. We draw the dual of the *CO* which we know is a *CT*. Finally we draw the e-circles of the four triangles and find that their centres too are concyclic. Readers wishing to pursue these matters cannot do better than study this paper:

Josefsson, Martin, *More characterisations of tangential quadrilaterals*, Forum Geometricorum, volume 11 (2011), pp. 65 - 82.