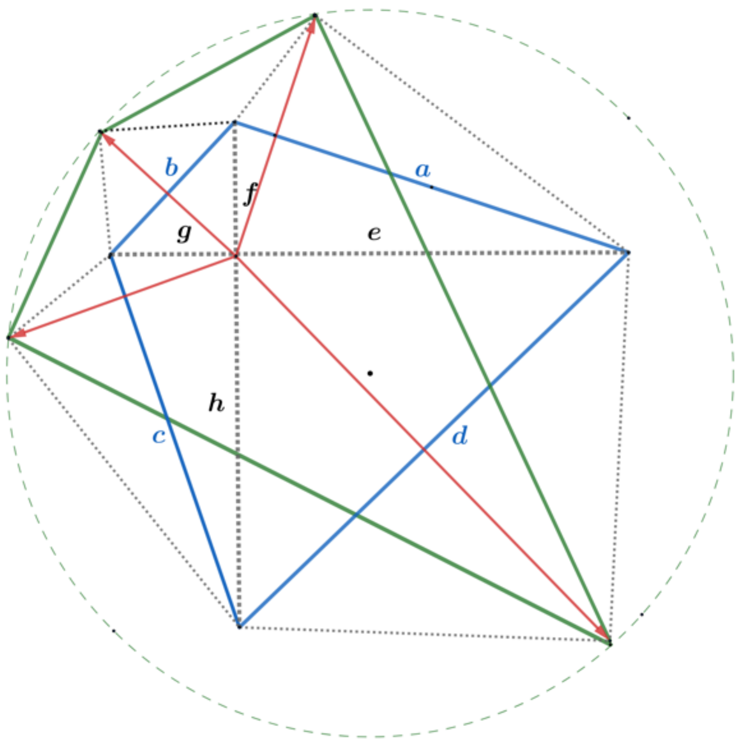
4.3.5 The 'flip' construction

Flipping an *O*

The green quadrilateral is obtained by taking the point of intersection of the diagonals and reflecting it in each side to produce four new vertices. We shall show that this quadrilateral is cyclic.



Begin by labelling equal sides in the 'flipped' right triangles.

Mark the right angles.

Identify the isosceles triangles created by the duplication of e, f, g and h, and label their base angles.

You can now write down the interior angle at each green vertex.

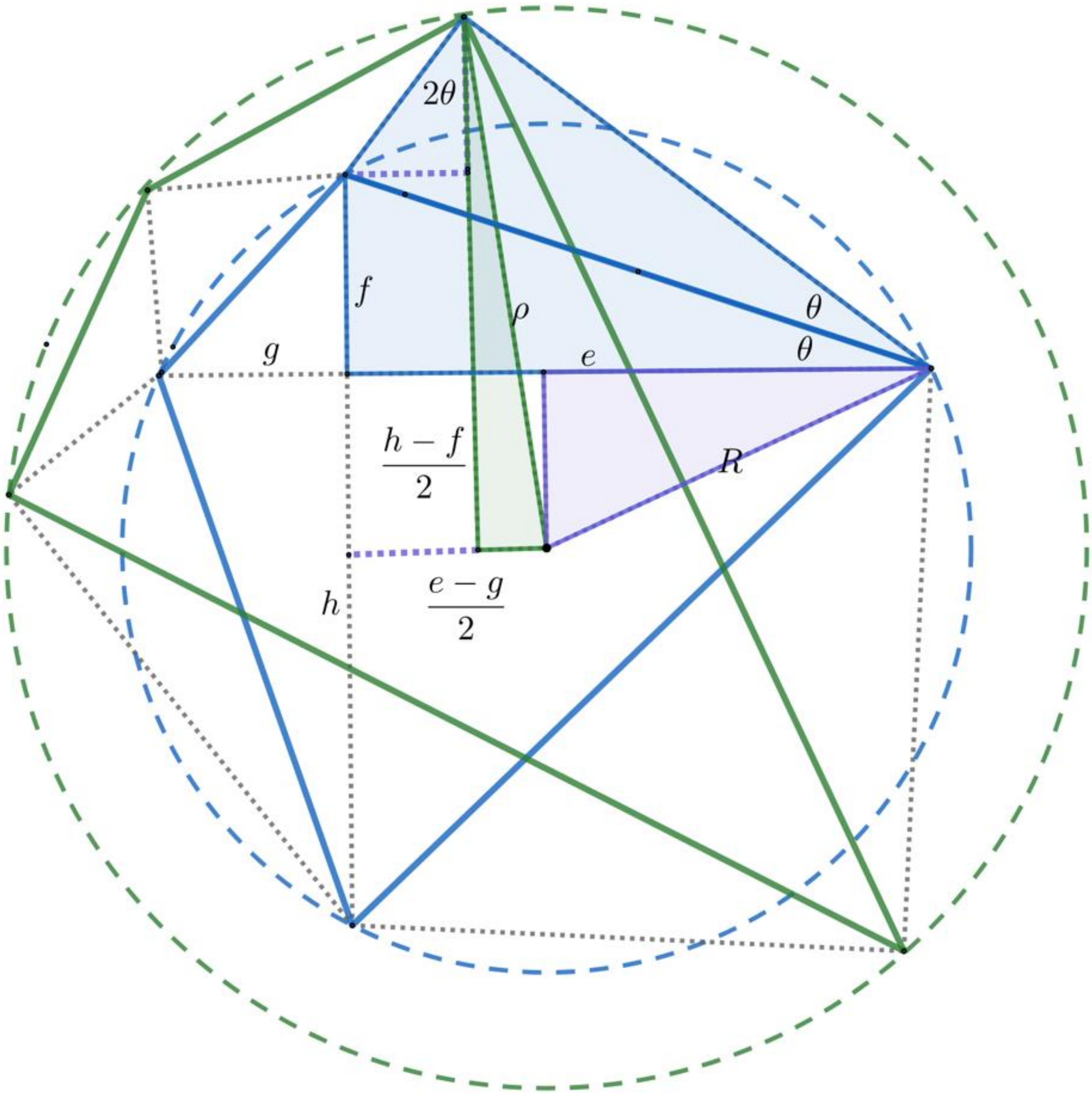
Sum the two pairs of opposite angles.

If the sums are equal, the quadrilateral must be cyclic.

As an additional task, join the feet of the perpendiculars from the diagonal intersection point on to the sides of the blue quadrilateral. Why is the quadrilateral which results similar to the green quadrilateral (and therefore also cyclic)? We show this by a second method below.

Flipping a *CO*

There is a further property. If the blue quadrilateral is itself cyclic, the circumcircles of the blue quadrilateral and the green quadrilateral are concentric. We shall now show this and find expressions for their diameters.



From the lilac triangle we can work out the diameter of the blue circle, $D_{blue} = 2R$.

By the intersecting chord property, $eg = fh$, therefore

$$D_{blue}{}^2 = 4\left[\left(\frac{h-f}{2}\right)^2 + \left(\frac{e+g}{2}\right)^2\right] = h^2 + f^2 - 2fh + e^2 + g^2 + 2eg = e^2 + f^2 + g^2 + h^2.$$

If our claim about the two circles is correct, the distances from the centre of the blue circle to each vertex of the green circle will be equal.

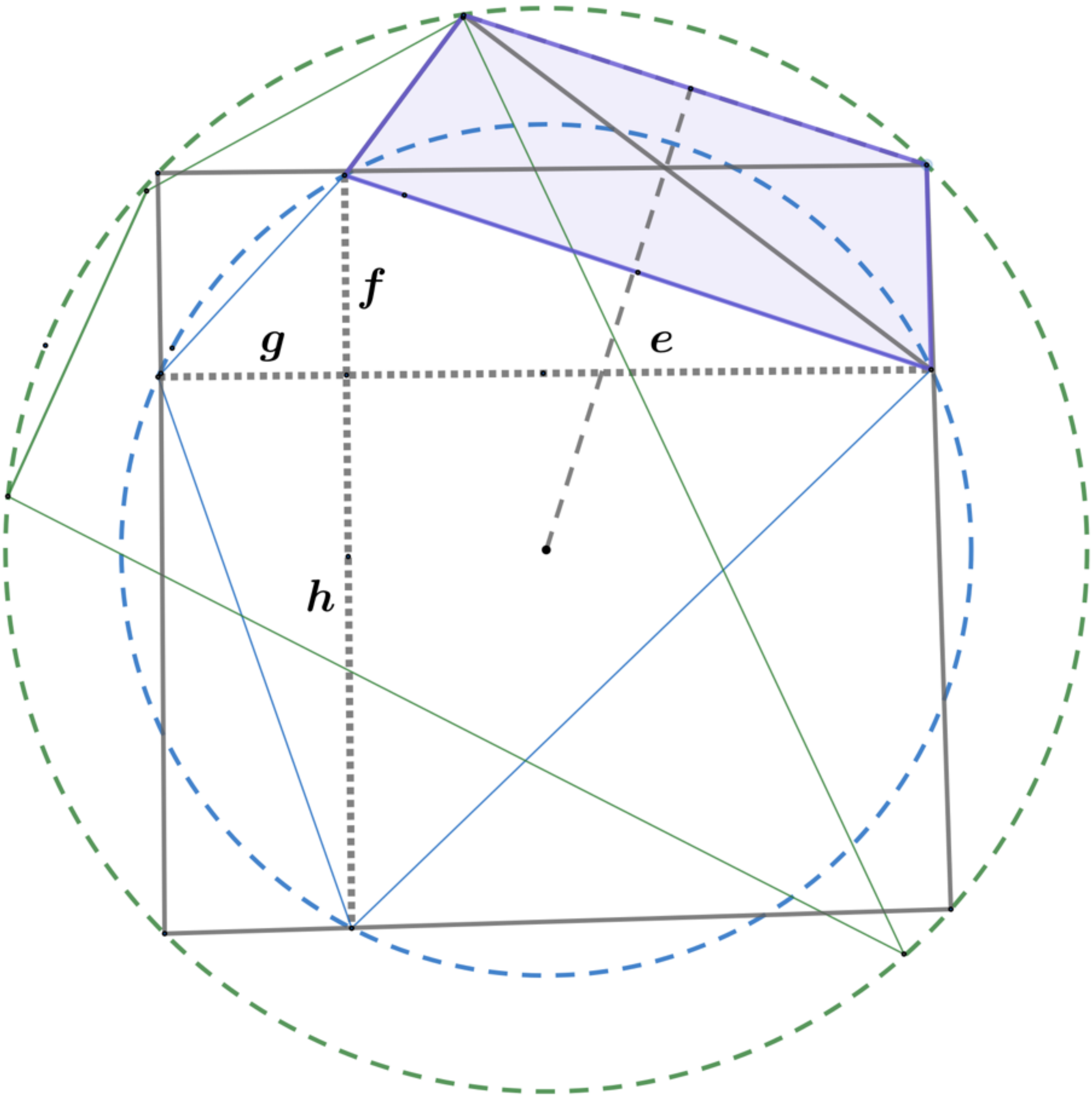
Trigonometry in the upper blue triangles gives us lengths we need to make up the shorter sides of the green right triangle. From Pythagoras we then find that indeed the required lengths are equal, giving this expression for the green diameter, $D_{green} = 2\rho$:

$$D_{green}{}^2 = e^2 + f^2 + g^2 + h^2 + 4eg.$$

Recalling that $eg = fh$, we have $4eg = 2eg + 2fh$. This enables us to rewrite the equation like this:

$D_{green}{}^2 = (e + g)^2 + (f + h)^2$.

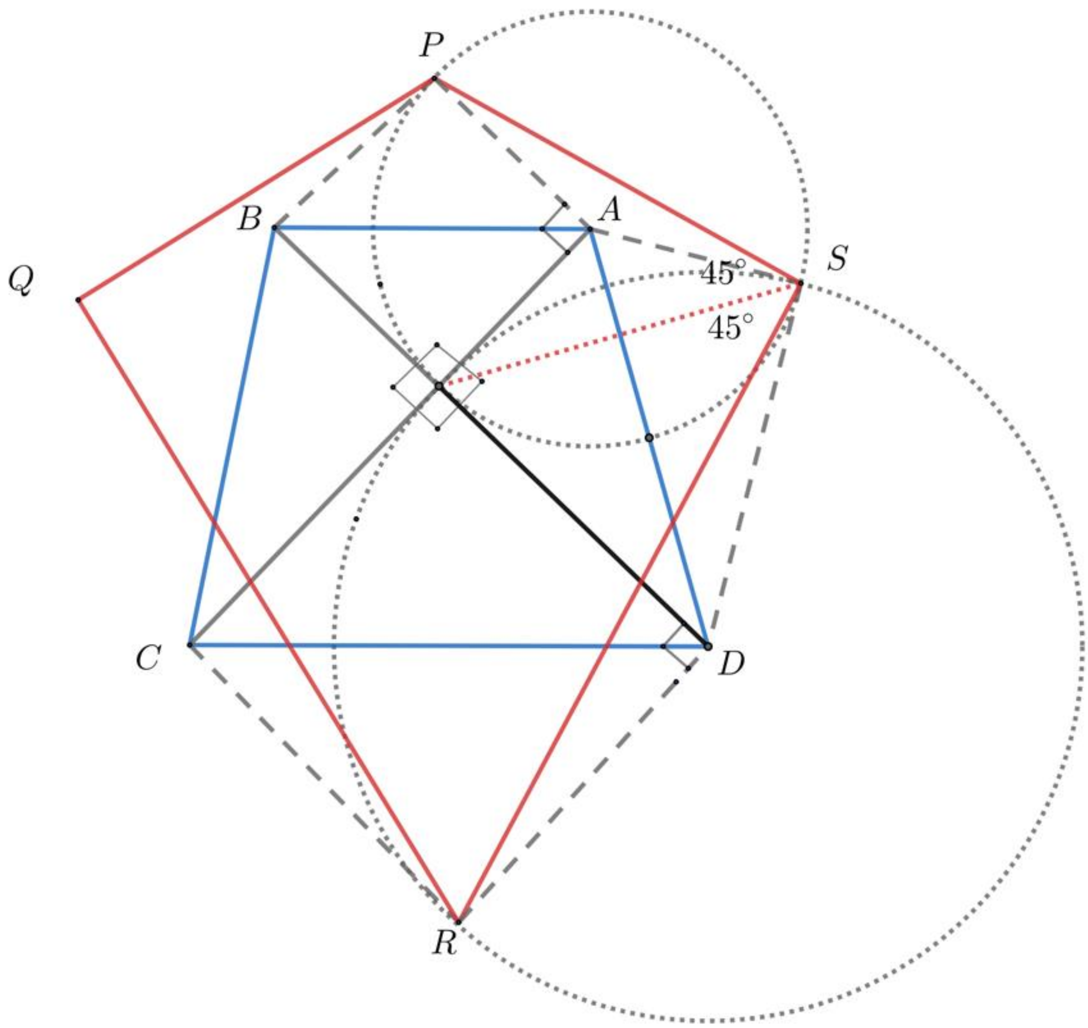
We can then represent the diameter of the green circle as a diagonal of the black rectangle here.



We see we have two congruent right triangles upon a side of the blue quadrilateral, completing a regular trapezium. The symmetry axis of this is a perpendicular bisector both of a chord of the circumcircle of the green quadrilateral and of a chord of the circumcircle of the blue quadrilateral. Their respective centres therefore lie on this line. But we can make the same construction on the other three sides of the blue quadrilateral with the same result. The two circles therefore share the same centre.

Flipping standard quadrilateral types

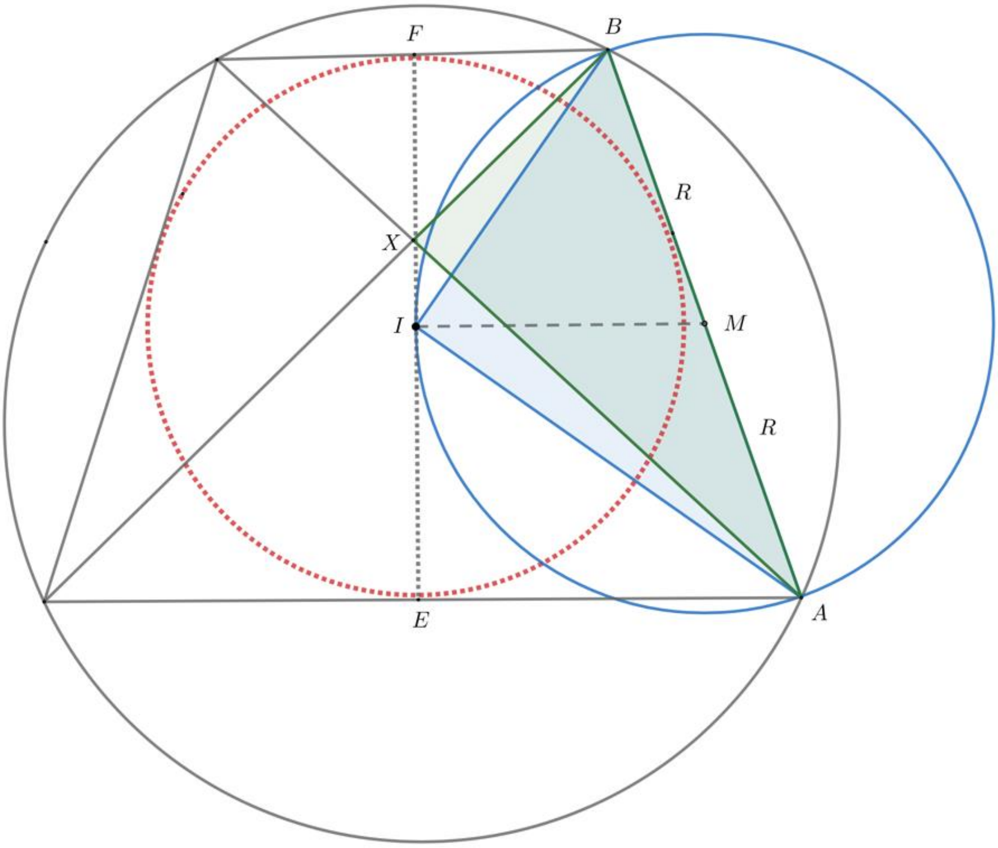
We now look at some special cases. *What quadrilaterals do we obtain by flipping a rhombus, a kite, a square?* What about the orthodiagonal isosceles trapezium?



Notice the right angles at the centres of the two circles. Because the angle at the centre is twice the angle at the circumference in the same segment, this results in two adjacent angles of 45° at *S*. *PQRS* is therefore a 90° kite, the same figure we obtained from the tangential isosceles trapezium by use of the common circle in the section **Dual polygons**. So, we have two transformations producing the same result:

Original figure	Transformation	Resulting figure
Tangential isosceles trapezium	Taking as vertices the points of tangency	90° kite
Orthodiagonal isosceles trapezium	Reflecting the intersection of the diagonals in the sides ('flipping')	90° kite

Can an isosceles trapezium be both a *T* and an *O*? Except in the limiting case of a square, No. We can see this from the following figure, which we analyse below.



We learn in the section **Dual polygons** that, if an isosceles trapezium is a *T*, angle *AIB* is a right angle.

We see that this angle is subtended by the diameter of the blue circle, *AB*. The line joining the centres of the red and blue circles is parallel to the trapezium's top and base sides. This is the radius of the blue circle, which meets the symmetry axis of the trapezium, *EF*, at *I*. *EF* is therefore a tangent to the blue circle. If *AB* is to subtend a 90° angle at a point, that point must lie on the blue circle.

By symmetry the point of intersection *X* of the diagonals of the trapezium must lie on the symmetry axis, *EF*. But *AB* subtends a right angle at only one point on *EF* and that is *I*.

Therefore the diagonals of the trapezium cannot be perpendicular.