Work through subsection **4.1: The circle and its quadrilaterals**. This establishes the properties we need to tackle the special constructions explored here.

4.3.1 Arc bisection: turning a *C* into a *CO*

We next show how to make a *CO* from a *C*.

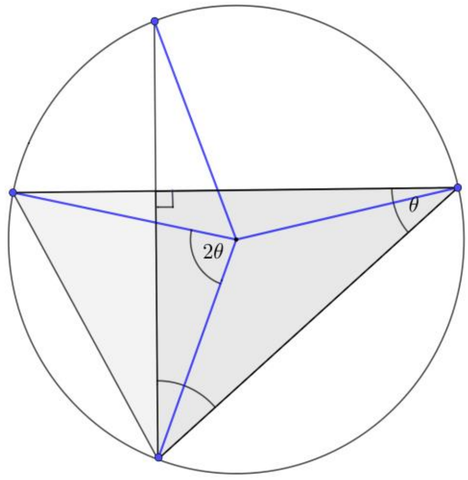
Figures ***A*** and ***B*** are incomplete. *In the course of completing them you will learn and justify the method.*

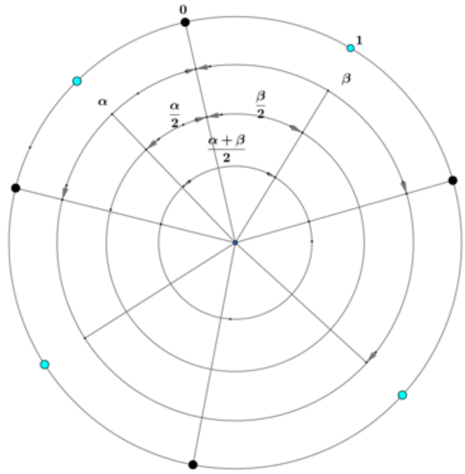
A In the big grey triangle we use the fact that the angle at the centre is twice that at the circumference. So the blue arc bottom left subtends 2θ at the centre. *Label the unmarked angle to produce the corresponding answer top right.* The dark grey triangle is a right triangle. So the angles we marked are complementary. *What can you say therefore about the angles at the centre?*

So we can say:
If the diagonals of a cyclic quadrilateral are perpendicular, then the sums of the central angles subtended by the pairs of opposite sides are equal to each other and to π.
That is our theorem, but we need the converse:
If the sums of the central angles subtended by the pairs of opposite sides of a cyclic quadrilateral are equal, the diagonals are perpendicular.
Is this true? *Reverse the process in the previous paragraph by which you allocated angles.*

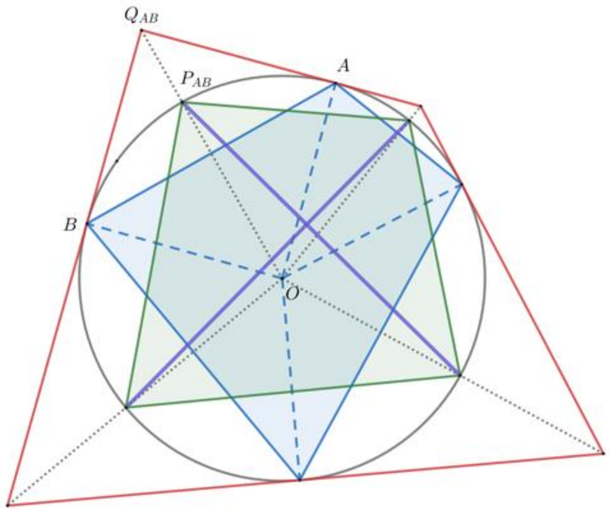
B The black points are the vertices of our original, generation 0, cyclic quadrilateral, spaced round the figure at respective angles of $\alpha, \beta, \gamma, \delta$. The turquoise points belong to the one we have constructed, of generation 1. Beginning with the angles α, β and working inwards, we have found the angle between the upper two ‘turquoise’ radii. *Starting with* γ, δ *and, working inwards from the bottom, work out the angle between the lower turquoise radii. What is the sum* $\alpha_1 + \beta_1 + \gamma_1 + \delta_1$*? What therefore do the lower and upper blue angles sum to? What do you conclude?*

A *B*





In the section **Dual polygons** we make a dual by taking a *C* and drawing tangents at the vertices. From the blue cyclic quadrilateral below we have made the red *T*, so a dual pair. But, as you see, we have also made the green *CO*. We see that a green vertex (e.g. P_{AB}) and a red vertex (e.g. Q_{AB}) are in line with the circumcentre of the blue quadrilateral, *O*.



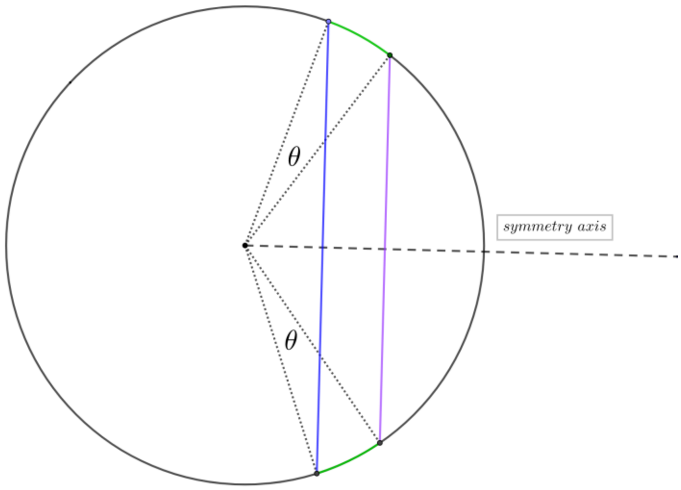
Sharing a symmetry axis with the kite $OAP_{AB}B$ is the 90° kite $OAQ_{AB}B$.

To summarise, to every *C* there corresponds a *CO* and a *T* (the dual).

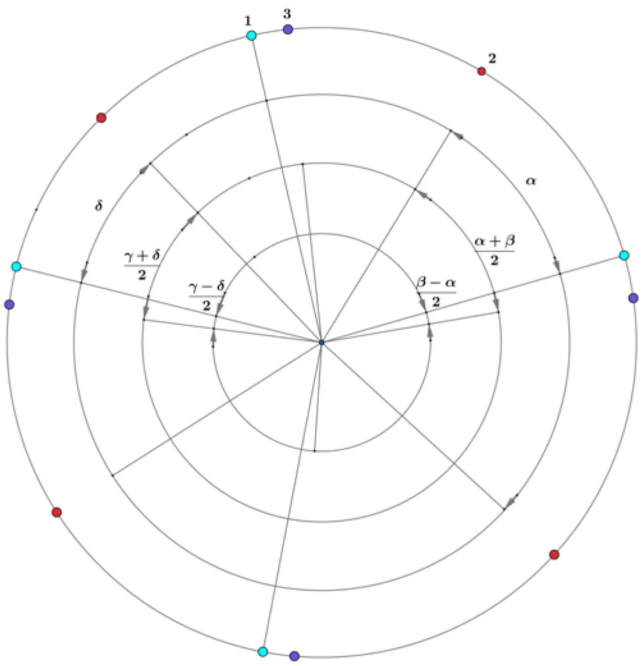
Iterating arc bisection

We take a *C*, make an *CO* from it, take *that*, and make a new *CO* from that, and continue in this way. We find an interesting result: *In alternate generations the diagonals have the same orientation.*

To show this, we first need a property which tells us when two chords are parallel. The two green arcs subtend equal angles at the centre of the circle and are therefore equal. Without the chords the figure has the symmetry axis shown. If we draw in the chords, they must preserve the symmetry. They are therefore both perpendicular to the symmetry axis and so parallel to each other.



Applying arc bisection repeatedly, we know that, beginning with generation 1, the diagonals are perpendicular in every subsequent generation. we therefore need only compare the orientation of one diagonal from each of a pair of quadrilaterals under discussion. Further, we only need prove the assertion for generations 1 and 3 in order to proceed by induction to the general case. In the next figure, the turquoise vertices denote generation 1; the red, 2; the lilac, 3.



On the figure we have worked out the angles subtended at the centre by the turquoise-lilac arcs between the ends of a turquois diagonal and a lilac diagonal. *Show that, since by our CO theorem* $\alpha + \gamma = \beta + \delta$ *for all generations from 1 on, these are equal.*

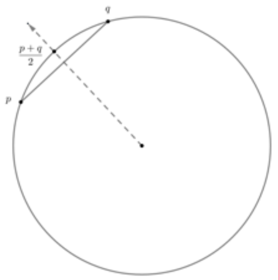
From the little lemma above about parallels, we infer that the turquoise and lilac diagonals are parallel.

This is enough to prove the claim.

The problem tackled algebraically

We mark the position of a point on the circumference by an angle, measured in convenient units and in a given sense, from some arbitrary reference direction. For generation 0 these are *a, b, c, d*.

Working to modulus 2π, we work out means of pairs of values. A mean represents two things: **(a)** a midpoint on the arc between two points, **(b)** the orientation of the chord between them, as shown in the figure.



1. We write the means of consecutive generation 0 points, taking a as $(a + 2\pi)$ where needed.
This gives the generation 1 points.
2. We write the means of alternate generation 1 points.
This gives the orientations of the generation 1 diagonals.
We find the difference is $\frac{\pi}{2}$, showing that the generation 1 diagonals are perpendicular.

$$a \qquad b \qquad c \qquad d \qquad a + 2\pi$$

$$\frac{a+b}{2} \qquad \frac{b+c}{2} \qquad \frac{c+d}{2} \qquad \frac{d+a}{2} + \pi$$

$$\frac{a+b+c+d}{4} \qquad \frac{a+b+c+d}{4} + \frac{\pi}{2}$$

3. We take the generation 1 points and repeat the process.

$$\frac{a+b}{2} \qquad \frac{b+c}{2} \qquad \frac{c+d}{2} \qquad \frac{d+a}{2} + \pi \qquad \frac{a+b}{2} + 2\pi$$

$$\frac{a+2b+c}{4} \qquad \frac{b+2c+d}{4} \qquad \frac{c+2d+a}{4} + \frac{\pi}{2} \qquad \frac{d+2a+b}{4} + \frac{3\pi}{2}$$

$$\frac{a+b+c+d}{4} + \frac{\pi}{4} \qquad \frac{a+b+c+d}{4} + \frac{3\pi}{4}$$

4. We take the generation 2 points and repeat the process.

$$\frac{a+2b+c}{4} \qquad \frac{b+2c+d}{4} \qquad \frac{c+2d+a}{4} + \frac{\pi}{2} \qquad \frac{d+2a+b}{4} + \frac{3\pi}{2} \qquad \frac{a+2b+c}{4} + 2\pi$$

$$\frac{a+3b+3c+d}{8} \qquad \frac{b+3c+3d+a}{8} + \frac{\pi}{4} \qquad \frac{c+3d+3a+b}{8} + \pi \qquad \frac{d+3a+3b+c}{8} + \frac{7\pi}{4}$$

$$\frac{a+b+c+d}{4} + \frac{\pi}{2} \qquad \frac{a+b+c+d}{4} + \pi$$

Comparing the orientations of the diagonals of generations 1 and 3, we see that they are mutually rotated $\frac{\pi}{2}$. Since the angle between the diagonals of a pair is also $\frac{\pi}{2}$, the orientation of the pair of generation 3 diagonals is the same as that of the pair of generation 1diagonals. We then argue by induction to the general case.