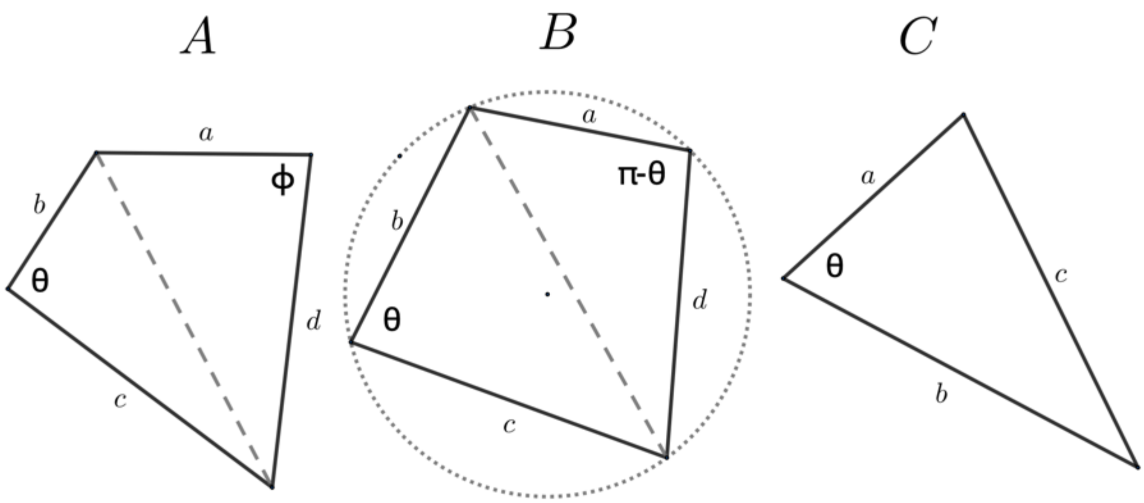
4.2 From Heron to von Staudt

Mathematics advances by showing that, at each stage, a result is a special case of a more general one. We illustrate this process for the area formula for a quadrilateral.



A C19, Bretschneider & von Staudt: $\sqrt{(s-a)(s-b)(s-c)(s-d) - abcd\left(\cos\frac{\theta+\varphi}{2}\right)^2}$

B C7, Brahmagupta: $\sqrt{(s-a)(s-b)(s-c)(s-d)}$

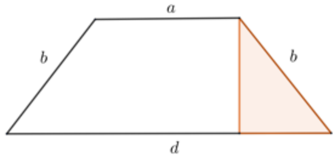
C C1, Heron: $\sqrt{(s-a)(s-b)(s-c)s}$

s is the semi-perimeter.

Because the labelling of the sides is arbitrary, algebraic symmetry must be preserved in these formulas.

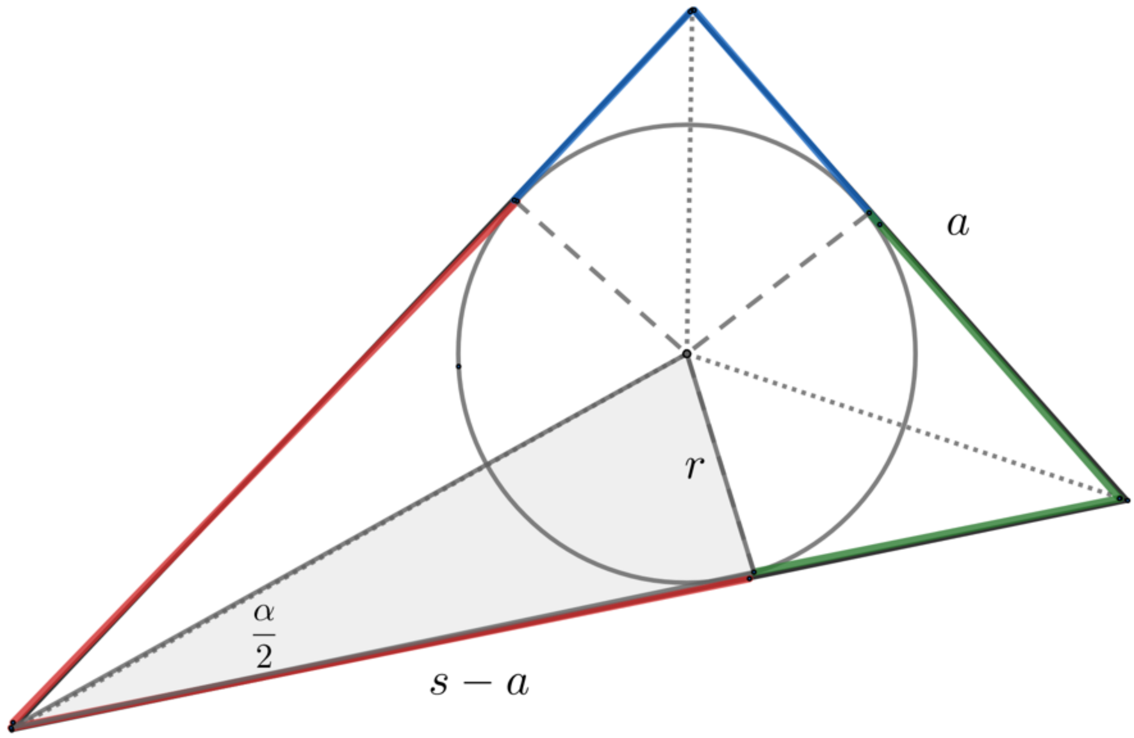
A It appears to be violated in the first but that is because we could equally have chosen to label the other pair of angles. The reason we must specify angles is that, otherwise, the quadrilateral would not be rigid: think of it as a linkage we can bend around. (Only the triangle is a rigid polygon.) And, if it is not rigid, it does not have a specified area.
B Because of the supplementary angles in a cyclic quadrilateral, the red term contains $\cos\frac{\pi}{2} = 0$ and vanishes.
C All triangles are cyclic. All we have done here is to say that a triangle is a cyclic quadrilateral where one side has zero length.

*Test the formulas on some special polygons. For example, try **A** on a parallelogram, **B** on a regular trapezium and a rectangle,* **C** *on a right triangle and an isosceles triangle. Do you obtain more familiar expressions?* You have to work a bit at the regular trapezium: you should find you get a difference of two squares inside the bracket which enables you to use Pythagoras in the red triangle:



The isosceles triangle is two red triangles back-to-back.

On seeing Heron's formula, the first question students ask is, "Why does it talk about the semi-perimeter, *s* ?" It arises from this geometry:



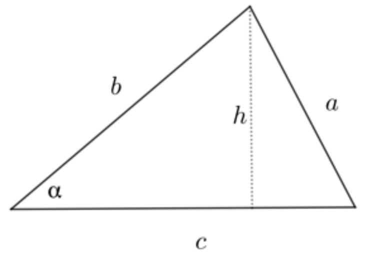
The perimeter comprises two each of the coloured lengths, the semi-perimeter *s* comprises one each therefore. The red length, for example, is then equal to *s* - *a*.

The all-important dimension in the figure is *r*, the radius of the incircle.
The triangle comprises the three dotted triangles, total area
$\boldsymbol{A} = \frac{ar}{2} + \frac{br}{2} + \frac{cr}{2} = r\frac{(a+b+c)}{2} = \boldsymbol{rs}$. [**Equation 1**]
It also comprises three pairs of triangles like the grey one, total area
$A = r[(s-a) + (s-b) + (s-c)]$.
In the grey triangle, $(s-a) = rcot\left(\frac{\alpha}{2}\right)$. Substituting the three expressions like this, we have:
$A = r^2\left[cot\left(\frac{\alpha}{2}\right) + cot\left(\frac{\beta}{2}\right) + cot\left(\frac{\gamma}{2}\right)\right]$.
Since $\frac{\alpha}{2} + \frac{\beta}{2} + \frac{\gamma}{2} = \frac{\pi}{2}$, we can use the special relation that the sum of the cotangents equals their product, so that
$A = r^2 cot\left(\frac{\alpha}{2}\right) cot\left(\frac{\beta}{2}\right) cot\left(\frac{\gamma}{2}\right)$. Changing the subject of the grey triangle formula,
$cot\left(\frac{\alpha}{2}\right) = \frac{s-a}{r}$, etc., so that, after cancelling,

$\boldsymbol{A} = \frac{(\boldsymbol{s}-\boldsymbol{a})(\boldsymbol{s}-\boldsymbol{b})(\boldsymbol{s}-\boldsymbol{c})}{\boldsymbol{r}}$. [**Equation 2**].
Multiplying [**Equation 1**] by [**Equation 2**], we get
$A^2 = s(s-a)(s-b)(s-c)$.
And finally, taking the square root of each side, we obtain the Heron formula,
$A = \sqrt{s(s-a)(s-b)(s-c)}$.
Label the red length x, the blue length y, the green length z and rewrite a in terms of x, y and z.

Visit the Wikipedia entry 'Heron's formula'. See if you prefer one of the other derivations given there.

From this figure we can derive an alternative expression for the area of a triangle.

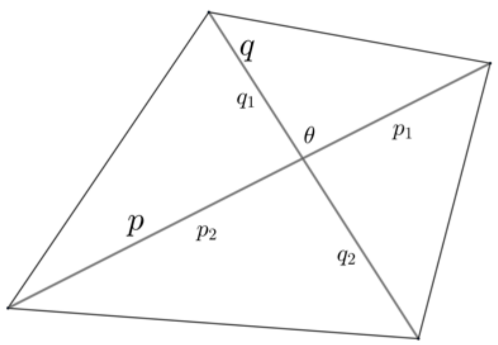


Area = ' half height times base'
$= \frac{1}{2}hc = \frac{1}{2}.b\sin(\alpha).c = \frac{1}{2}\boldsymbol{bc}\ \mathbf{sin}\ (\boldsymbol{\alpha})$.

And we can write corresponding expressions for the other sides and angles.

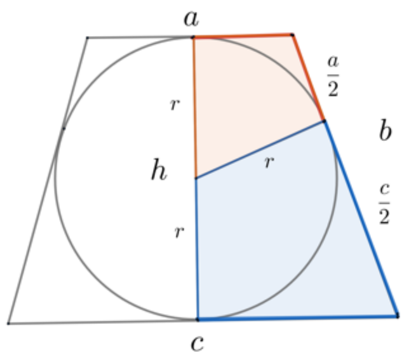
So the area of triangle ***C*** can also be written $\frac{1}{2}\boldsymbol{absin\theta}$. I've split **A** and **B** into triangle pairs. *Write down corresponding expressions for the areas of **A** and **B**. Check the new formulas against the same special polygons you tried before.*

If we know the lengths of the diagonals, *p*, *q*, mutually inclined at the angle θ, we have a simple equation for the area of quadrilateral **A**: $\frac{1}{2}\boldsymbol{pq\ sin\theta}$. To see why this is so, we label the four segments into which the diagonals are divided and apply our triangle formula to each region into which they divide the quadrilateral.



Area = $\frac{1}{2}\sin\theta\,(p_1q_1 + q_1p_2 + p_2q_2 + q_2p_1)$.
Simplify to confirm the formula.

The important thing about [**Equation 1**] is that it applies to *any* polygon with an incircle (any *tangential* polygon). All regular trapezia are cyclic. Here is a regular trapezium which is also tangential (being both, we say it is *bicentric*):



We see that $b = \frac{a+c}{2}$, the arithmetic mean of a and c.

Comparing corresponding sides in the two similar kites, the red and the blue, we have $\frac{a/2}{r} = \frac{r}{c/2}$, so $\frac{a/2}{h/2} = \frac{h/2}{c/2}$ and
$h = \sqrt{ac}$, the geometric mean of a and c.

Since $b > h$ unless $a = b = c$, the arithmetic mean for two unequal quantities exceeds their geometric mean.

From the formula for the area of a triangle $\frac{1}{2}ab\sin\theta$ we can derive the formula for the area of a circle. The triangle shown below has area $\frac{1}{2}r^2 sin\left(\frac{2\pi}{n}\right)$. n of these gives us the area of a regular n-gon, $\frac{1}{2}nr^2 sin\left(\frac{2\pi}{n}\right)$. As n tends to infinity, and the polygon approximates a circle, the sine of an angle approaches the measure of the angle itself in radians.

$\frac{1}{2}nr^2 sin\left(\frac{2\pi}{n}\right) \to \frac{2\pi n r^2}{2n} = \pi r^2$ is therefore the area of a circle radius r.