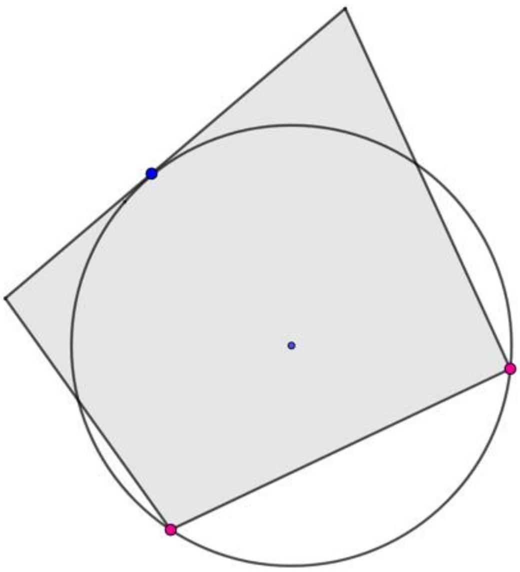
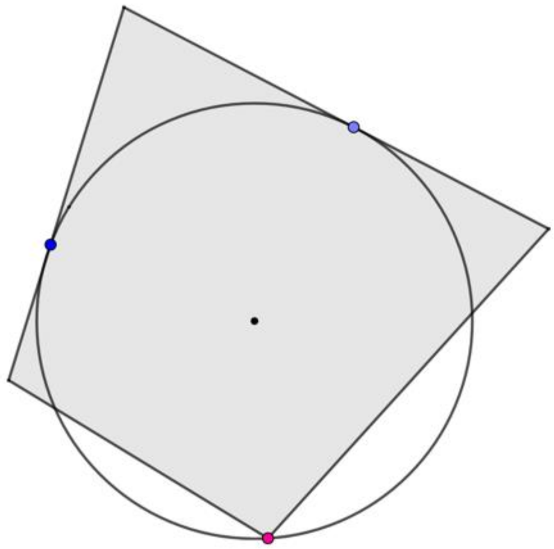
4.1 The circle and its quadrilaterals

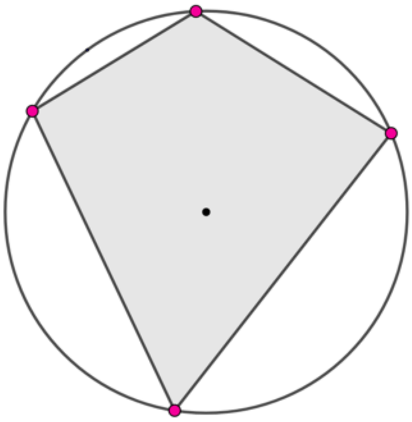
This is about quadrilaterals associated with the circle. Our quadrilaterals will be *simple* (not self-intersecting) and *convex* (containing both diagonals). What we shall find is that, however asymmetric the quadrilateral, the symmetry of the circle will assert itself in giving the associated quadrilateral a special property.

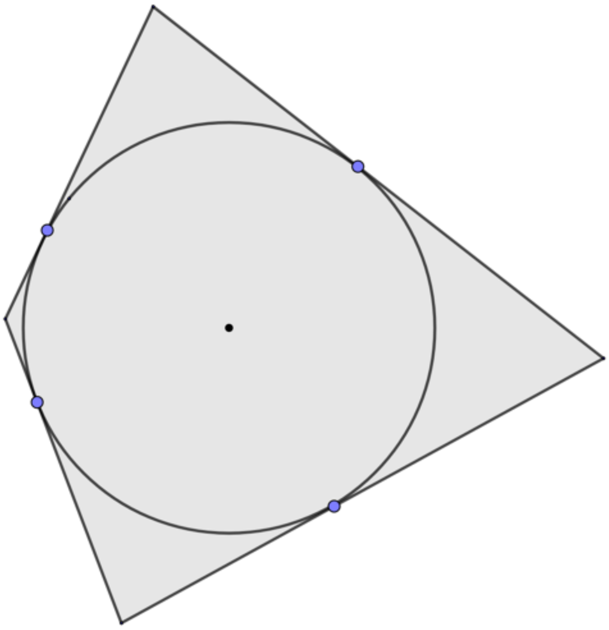
Any quadrilateral may be drawn so that either two vertices lie on the circle and the other two lie on a tangent to it, or one vertex lies on the circle and the other three lie on tangents to it:



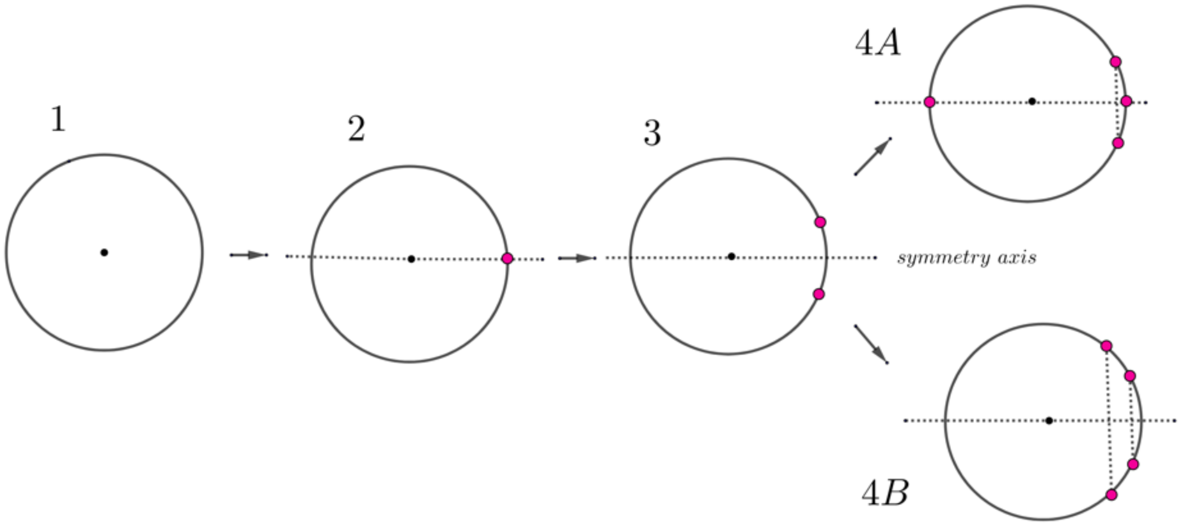


These will be of interest to us in subsection **5.2.8**. But here we shall be concerned with quadrilaterals whose four vertices lie on the circle - *cyclic* quadrilaterals - or whose four sides are tangent to it - *tangential* quadrilaterals:





We begin with a circle. The number of symmetry axes is infinite. Every line through the centre is a symmetry axis. We add points progressively. How can we add them so as to preserve a symmetry axis for the whole figure?



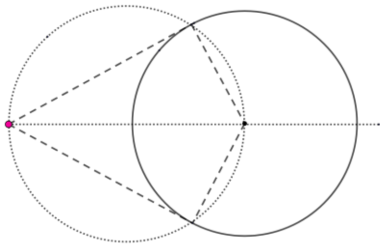
1: The circle itself.

2: Adding any single point preserves symmetry.

3: Adding any second point preserves symmetry.

4: A third point would upset symmetry so we must add two. There are two ways to do this (**4A** and **4B)**.

2: The single point may lie anywhere. Here the symmetrical picture has been completed by adding two equal tangents. They meet the radii at right angles, forming a *tangent* or *right* kite, so called. The kite in **4A** is of this type. We have



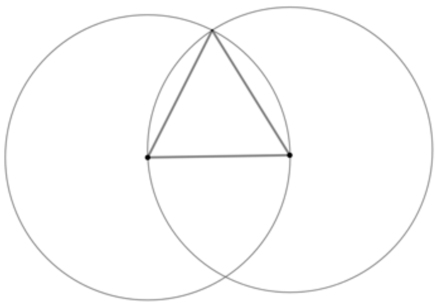
added the dotted circle for comparison.
If we remove the kite, we see that any
pair of circles has a symmetry axis.

4A: The quadrilateral is a kite. The diagonals are perpendicular (it is *orthodiagonal*). The angle between the sides of different length is a right angle (Thales' theorem). Since all kites are tangential, this special one is both cyclic and tangential (*bicentric*). It is the only quadrilateral apart from the square, (a special case), which is both bicentric and orthodiagonal.

4B: The quadrilateral is an isosceles trapezium. The parallel sides are chords bisected perpendicularly by the symmetry axis. The other pair are of equal length and subtend the same angle at the circumcentre.

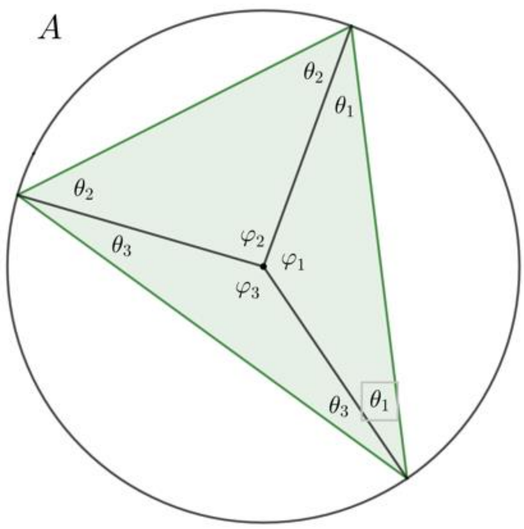
Say we are told that a cyclic quadrilateral has diagonals of equal length. These are equal chords of a circle. By the rotation symmetry of the circle, the resulting figure must have a symmetry axis. (In **4B** imagine joining the top left point to the bottom right and the bottom left to the top right.) In other words the quadrilateral is an isosceles trapezium.

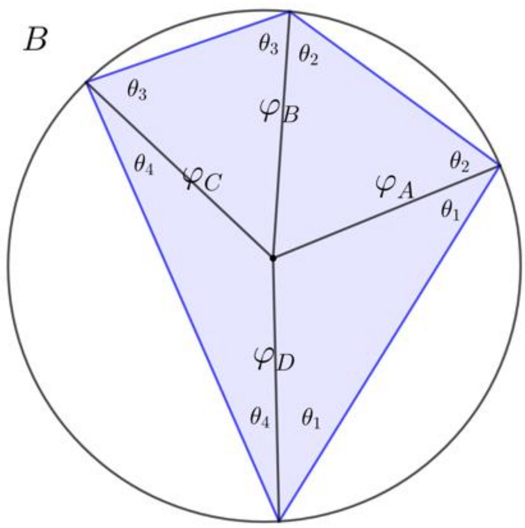
To construct an equilateral triangle, we draw two equal circles so that one passes through the centre of the other.



The three segments in bold are radii of equal circles, therefore equal themselves.

That little sequence offers a taste of how persistent the circle's symmetry is. We can derive most of the geometry of the circle from the fact that the radii are equal so that we immediately have isosceles triangles like those implicit in figures **3** and **4** above. *Use **A** and **B** to find out some.*



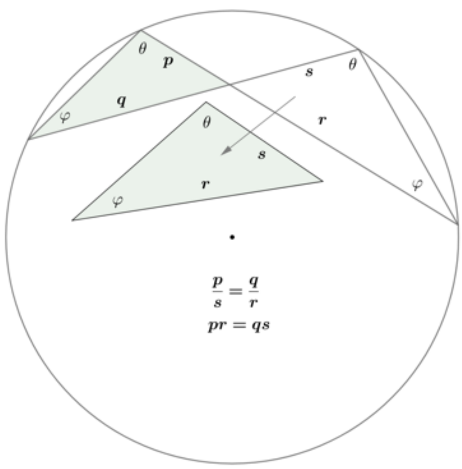


A *What is* $\theta_1 + \theta_2 + \theta_3$*?*
Write θ_1 *in terms of* θ_2, θ_3.
Write φ_1 *in terms of* θ_1.
Write φ_1 *in terms of* θ_2, θ_3.
You should find that, *subtended by a given chord, the angle at the centre is twice the angle at the circumference.*

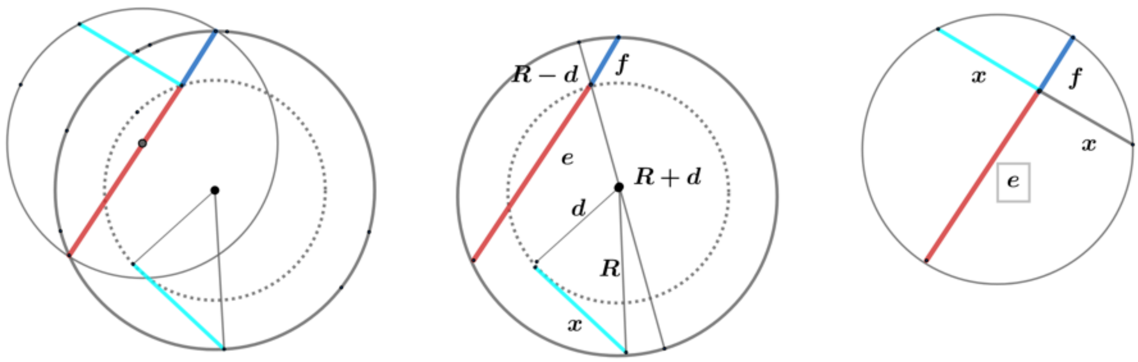
B *What is* $\theta_1 + \theta_2 + \theta_3 + \theta_4$*?*
$\varphi_A = \theta_1 + \theta_2$.
$\varphi_C = \theta_3 + \theta_4$.
Write φ_C *in terms of* φ_A.
You should find that *opposite angles are supplementary*.

If you kept one of the ***A*** chords fixed and moved the opposite vertex around, the same relation would pertain. *What do you conclude from this?* (the 'same segment' theorem).

Following straight on from that, we can identify two pairs of equal angles in the figure, below, thence two similar triangles, two equal ratios, and two equal products, the 'intersecting chord theorem'.

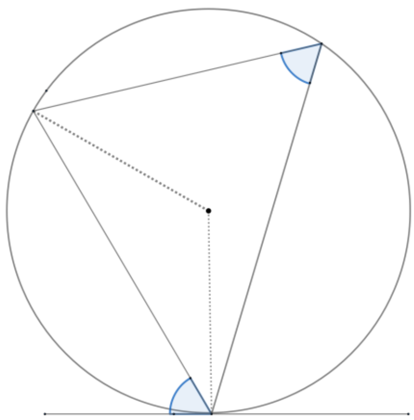


Here is an exercise on the intersecting chord theorem. *You may like to try it before reading on. Show that the two turquoise lengths in the left hand figure are equal.*

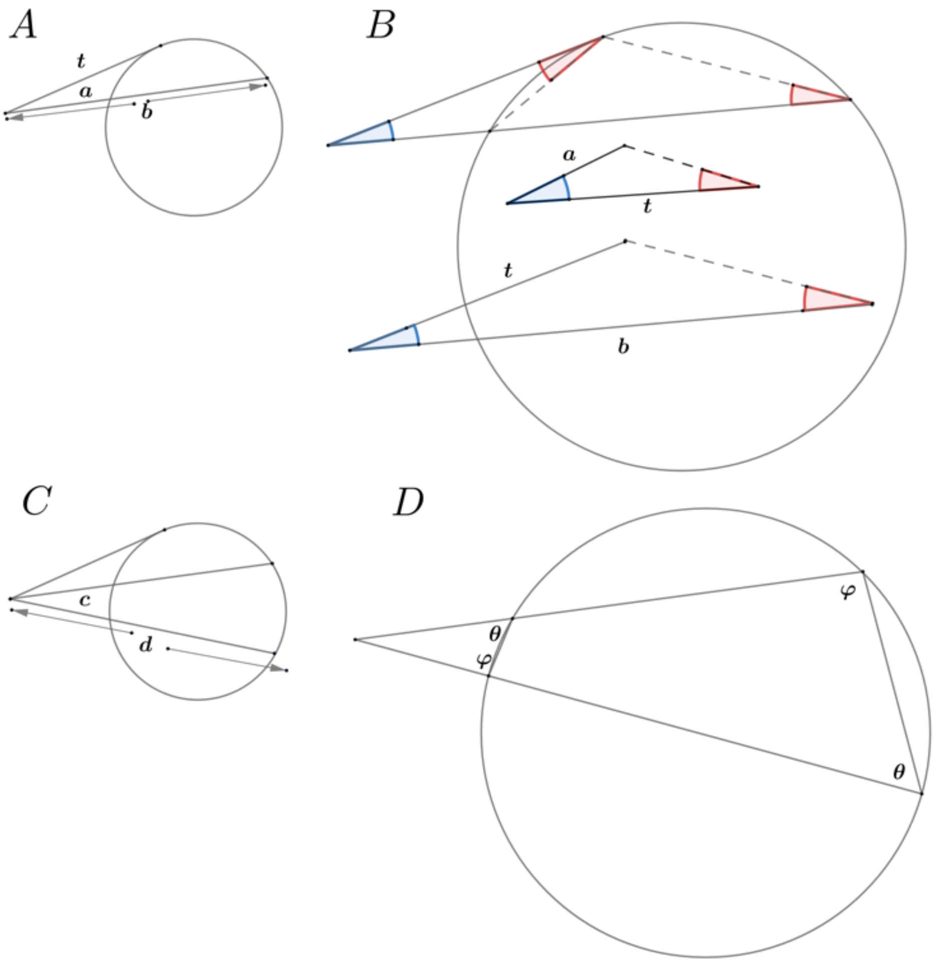


In the middle figure we have: $ef = (R + d)(R - d) = R^2 - d^2 = x^2$. And in the right hand figure we have the same product. Note that x is the geometric mean of e and f.
*Returning to figure **B**, what happens to the sum* $\theta_2 + \theta_3$ *when* $\theta_1 = 0$. *Express the relation in words* (Thales' theorem).

We need one more theorem. *Show that the blue angles are equal* (the 'alternate segment' theorem, in distinction to the same segment theorem stated above).



Much of Euclidean geometry consists in spotting similar triangles. Once identified, these can be compared most easily by setting them in the same orientation, as we did above in establishing the intersecting chord theorem. We can use similar triangles to prove a result involving secants and tangents.

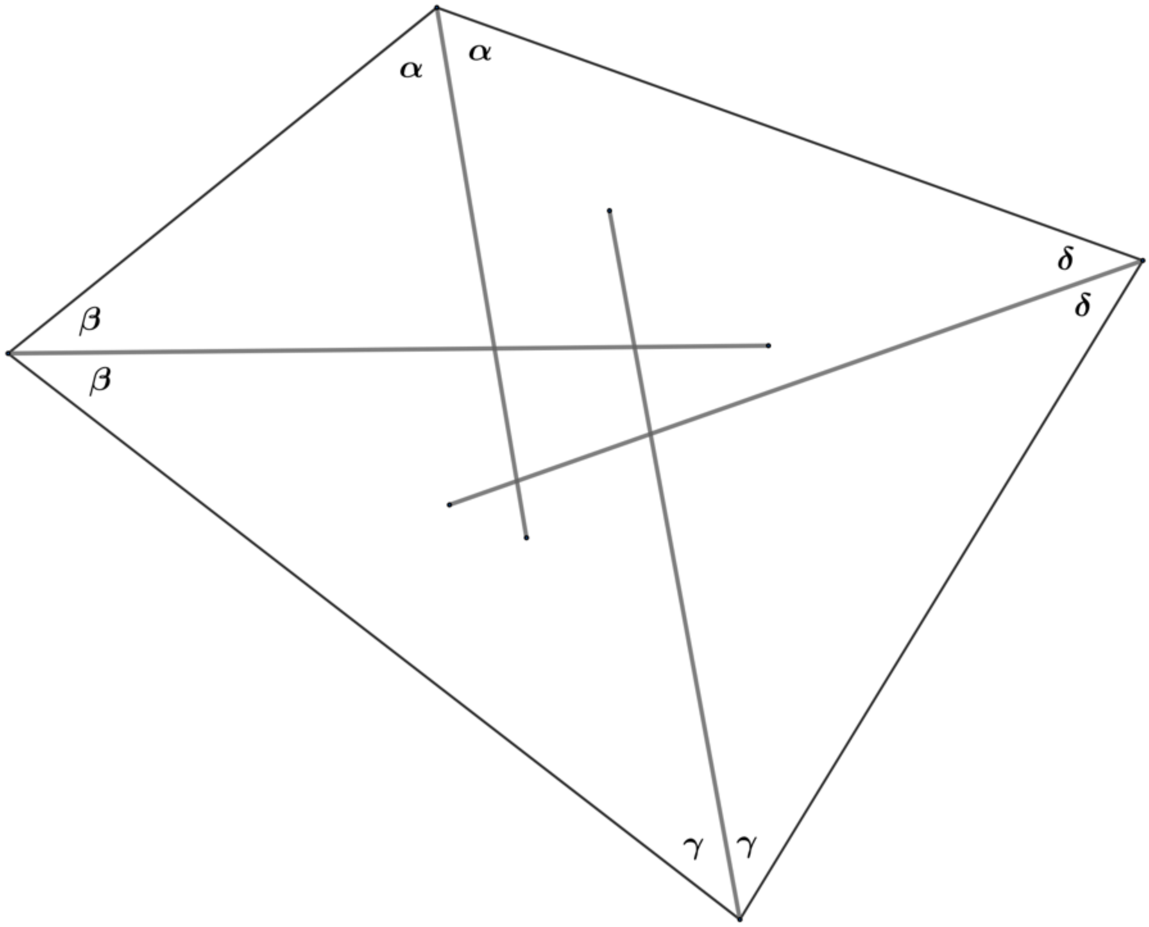


B reveals the similar triangles implicit in ***A***. Comparing corresponding sides, we have:
$\frac{a}{t} = \frac{t}{b} \Leftrightarrow ab = t^2$.

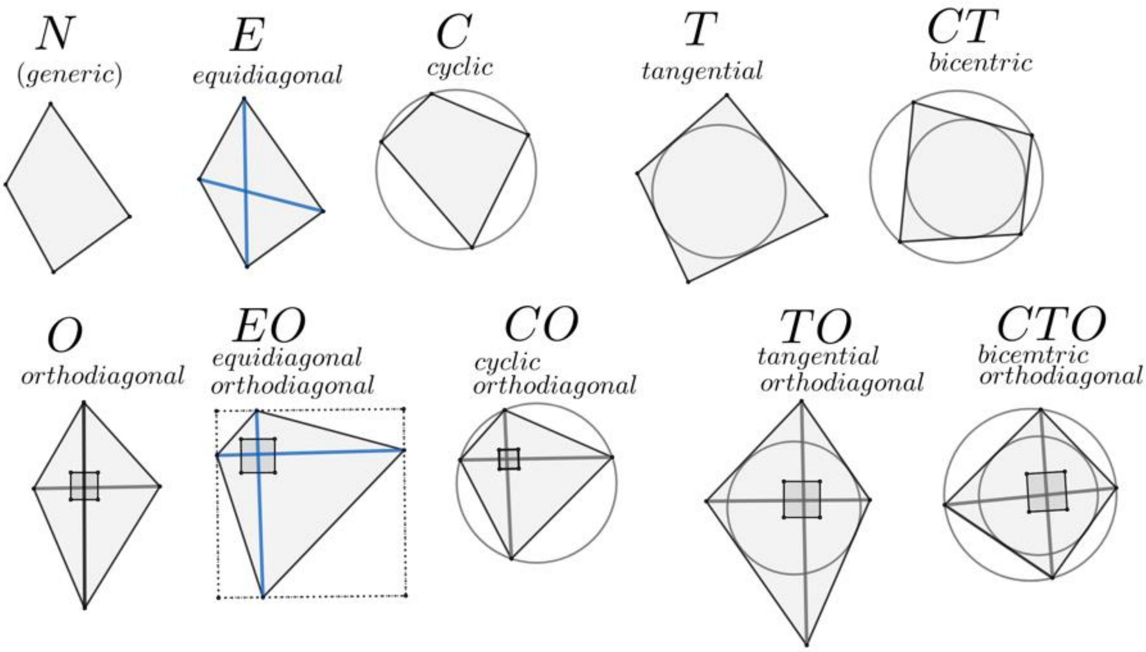
In ***C*** we have added a second secant. We now have:
$ab = t^2 = cd \Longrightarrow$
$ab = cd$.

In ***D*** we show the equal angles which arise from the cyclic quadrilateral. *Use the figure to show in a different way that* $ab = cd$.

Here is a theorem which is surprising because a special result emerges from a general construction. It is an instance of a global property not apparent when we examine a figure locally. We start with any convex quadrilateral and bisect the angles. The points where they cut in pairs are vertices of a new quadrilateral. *Label angles and discover the theorem.*



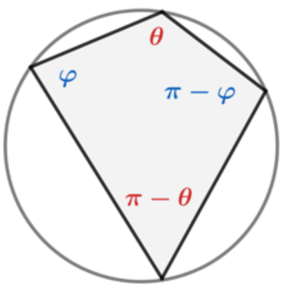
The following table contains the types of quadrilateral we are concerned with. We can use the code letters to refer to them. For example, instead of ‘a bicentric quadrilateral’ we can write simply ‘a *CT*’.

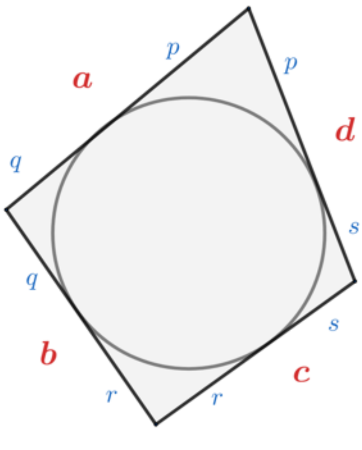


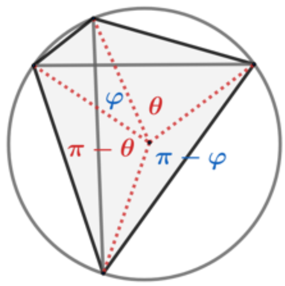
We need to complete this section by establishing some quantitative properties of these quadrilaterals.

First *C*, *T* and *CO*.

C | *T* | *CO*







Opposite vertical angles have the same sum (π).

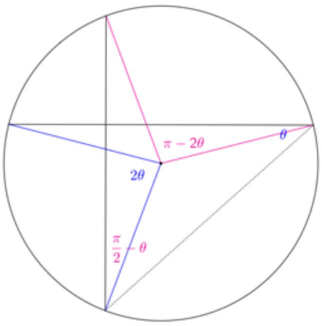
This was case ***B*** above.

Opposite sides have the same sum (s, the semi-perimeter).

This is a consequence of tangents from a point being equal. *Show it.*

Opposite central angles have the same sum (π).

This we explain below.



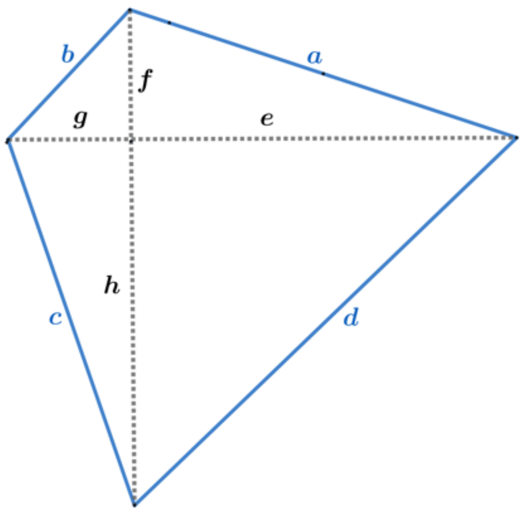
From ***A*** above we found that the angle at the centre is twice the angle at the circumference. Because of the right angle in the triangle lower right, the blue and red angles there sum to $\frac{\pi}{2}$. Therefore the blue and red angles at the centre sum to π.

We shall call this our *CO* theorem.

In the section **Theorems and their converses** we find that, not only are the above theorems true, but also their converses.

Now *O* and *TO*.

O



$$e^2 + f^2 = a^2$$
$$f^2 + g^2 = b^2$$
$$g^2 + h^2 = c^2$$
$$h^2 + e^2 = d^2$$

$$a^2 + c^2 = b^2 + d^2$$

This is a necessary condition for the blue quadrilateral to be orthodiagonal. It turns out that it is also sufficient. That is to say, you cannot draw a quadrilateral which satisfies that equation without its diagonals being perpendicular. We prove that in the section **Theorems and their converses**.

TO

We already know that, for *T*,
$a + c = b + d$, [**Equation 1**]
$\Longrightarrow (a + c)^2 = (b + d)^2$,
$\Leftrightarrow (a^2 + c^2) + 2ac = (b^2 + d^2) + 2bd$.
But we now know that for *O*,
$a^2 + c^2 = b^2 + d^2$.
Therefore, for *TO*,
$ac = bd$. [**Equation 2**]

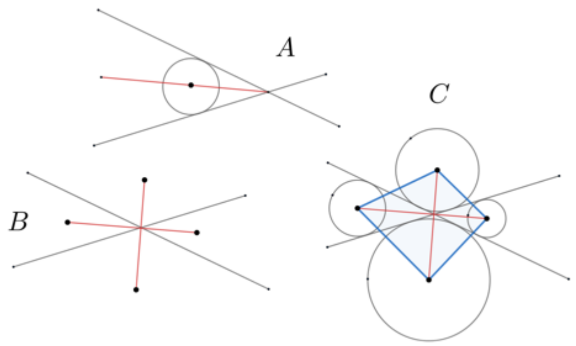
Manipulate **Equation 1** *and* **Equation 2**. You should find that c = *b* or *d*, and correspondingly, *a* = *d* or *b*. The quadrilateral has two pairs of equal, adjacent sides: it is a kite.

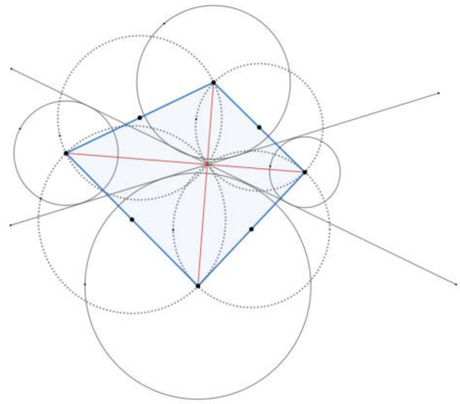
In the introduction to this section we showed how a tangent kite results from the circle's symmetry.

A If we extend the tangents, we see how the circle centre lies on the angle bisector.

B We show the internal and external bisectors of an angle. Working out the angles, we see that they must be perpendicular.

C Combining these two facts we deduce that the blue quadrilateral is an *O*. Alongside we have drawn the figure with the dotted circles resulting from the converse of Thales' theorem.





*With figure **B** in mind, explain this fact about an N and therefore all convex quadrilaterals. Divide the quadrilateral by its diagonals into four triangles and inscribe a circle in each. Then the lines joining opposite circle centres are perpendicular.*