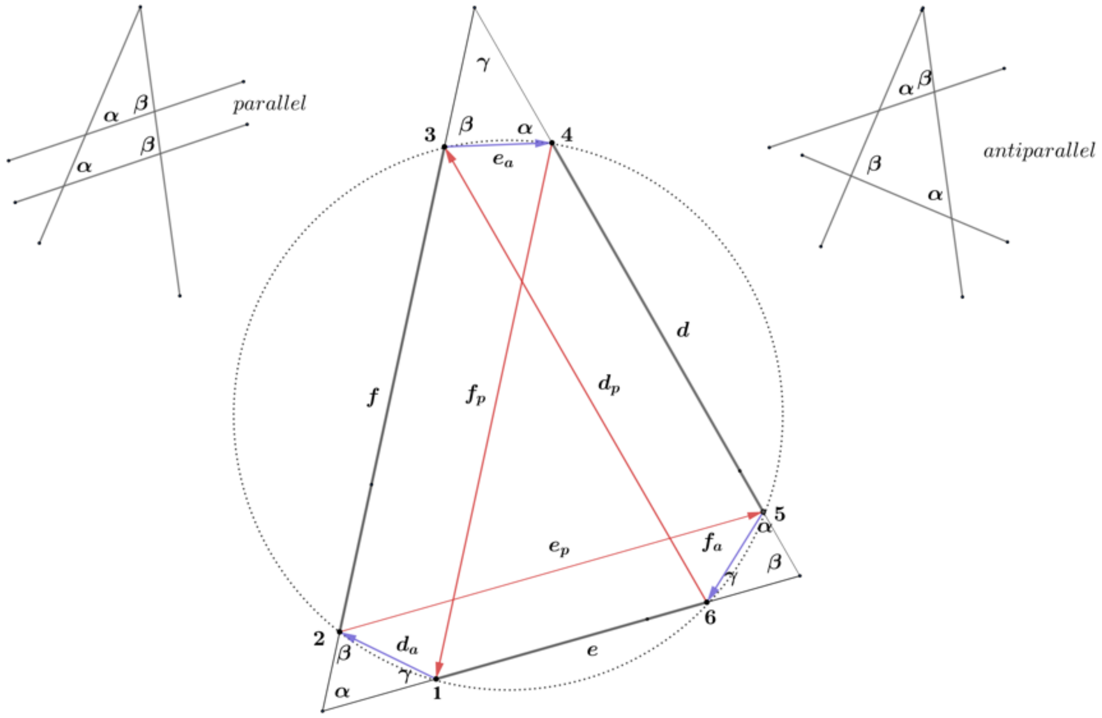
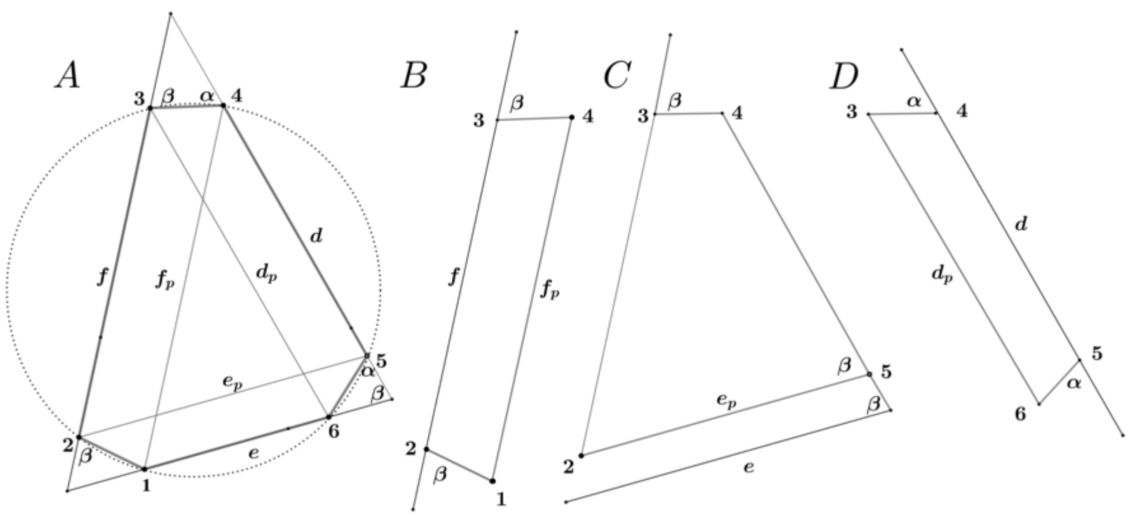
3.8 Tucker circles and the orthic triangle

Tucker circles

The upper figure contrasts two pairs of lines, one parallel, the other *anti*parallel. In the main figure we start at some point P on a triangle and follow the path shown by the alternating lilac and red lines. Correspondingly, we move alternately antiparallel and parallel to the side opposite. This is indicated by the suffix to the letter. Thus line d_a is antiparallel to side d, e_p is parallel to side e. What we find is that, after six moves, we arrive back at our starting point.



We also show that the 6 vertices of the hexagon we have traced lie on a circle. We shall prove that this is indeed so.



From ***A*** we extract the facts we need.

From the parallel lines and equal angles in ***B*** we infer that **1234** is a regular trapezium. Regular trapezia are cyclic, so points **1**, **2**, **3**, **4** share a circle.

From the equal angles in ***C*** we infer that the quadrilateral **2345** is cyclic, so points **2**, **3**, **4**, **5** share a circle.

From the parallel lines and equal angles in ***D*** we infer that **3456** is a regular trapezium, so points **3**, **4**, **5**, **6** share a circle.

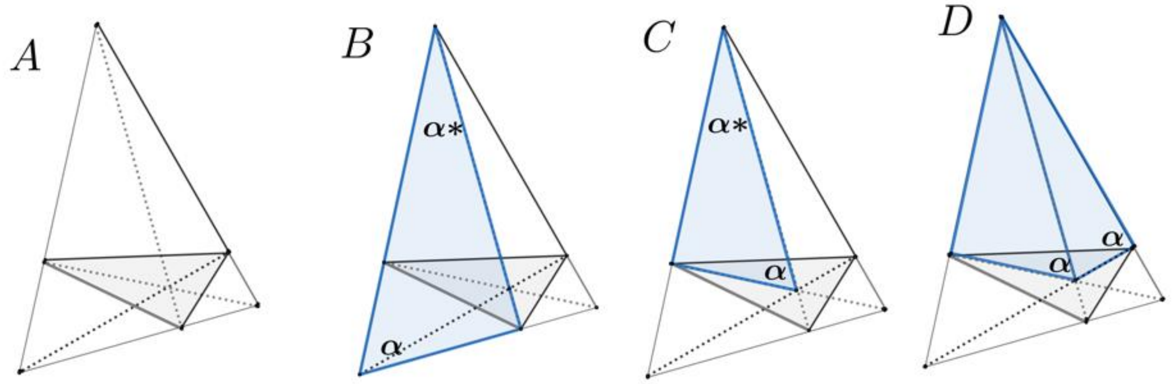
With the sets of concyclic points side by side:

1 2 3 4
  2 3 4 5
    3 4 5 6

we see that each set shares three points with each of the others. Therefore all six are concyclic. [We use this argument also in our discussion of **The nine-point circle**.]
We call the resulting circles *Tucker* circles. (Notice that the small triangles are congruent, and similar to the main triangle.)

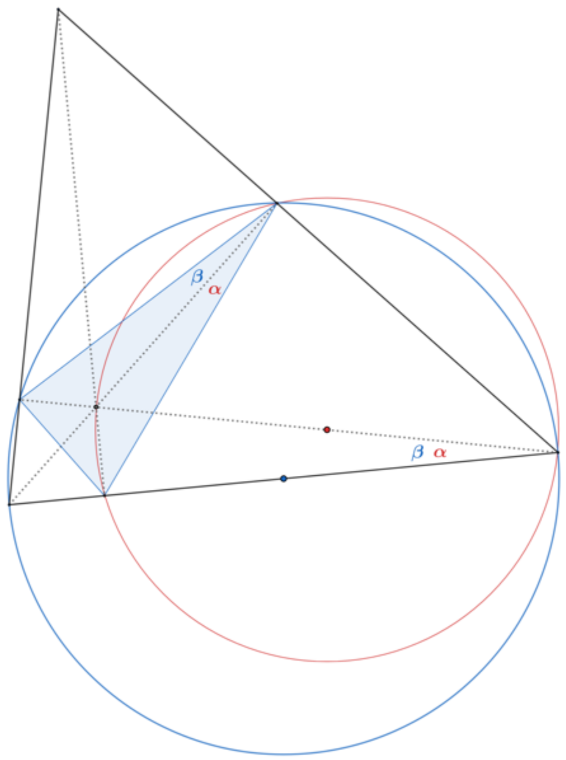
The orthic triangle

Imagine that we had begun our path round the triangle at the foot of an altitude and only taken the lilac steps. The result would have been the triangle joining the feet of all three altitudes, the *orthic* triangle (shaded in ***A***). In this sequence of figures we track a particular angle round the figure, showing how the result is three sides, each antiparallel to the opposite side of the main triangle. (The small triangles isolated by the orthic triangle are in general similar rather than congruent.)



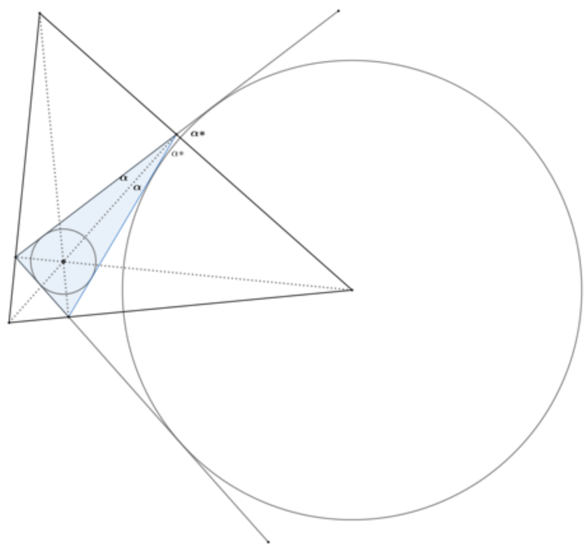
In ***B*** we identify in a right triangle the complementary angle to $\boldsymbol{\alpha}$, $\boldsymbol{\alpha}^*$.
In ***C*** we identify in a second right triangle the complement-of-the complement: $(\boldsymbol{\alpha}^*)^* = \boldsymbol{\alpha}$.
In ***D*** we identify a second angle on the same base in a cyclic quadrilateral, and thus equal to the first by the same segment theorem.

We shall prove that the altitudes bisect the angles of the orthic triangle internally, and the sides of the original triangle bisect them externally.
By the converse of Thales' theorem we can construct the red and blue circles.

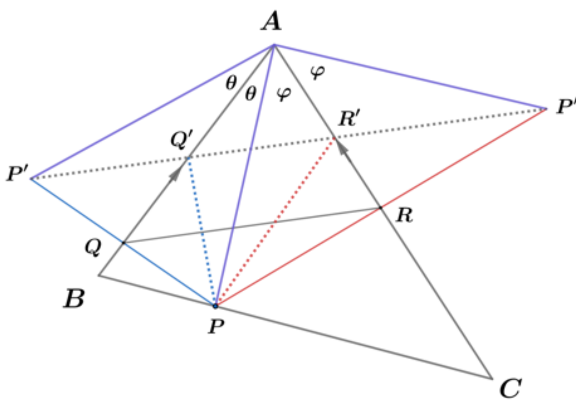


Applying the same segment theorem:
in the red circle, we have the two equal angles α ;
in the blue circle, we have the two equal angles β.
But we see that these two angle sizes label the same angle.
Therefore $\alpha = \beta$ and, generalising from the particular altitude chosen, each altitude bisects an angle of the orthic triangle internally. Since an altitude is perpendicular to a side, the side of the main triangle passing through the foot does so externally.

The interior angle bisectors give us the centre of the incircle of the orthic triangle, and each interior angle bisector and a pair of exterior angle bisectors give us the centre of one of its three e-circles:

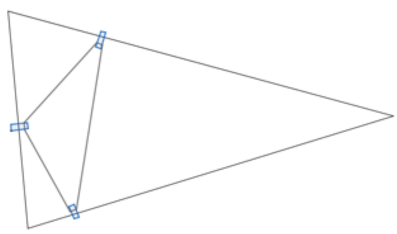


It turns out that the orthic triangle solves the following problem posed in 1775 by Giovanni Fagnano and solved by Lipót Fejér as we shall describe: *Given a triangle, what is the smallest triangle within it with a vertex on each side?*



We choose a position for vertex P on side BC and reflect it in two sides of the triangle. The perimeter of triangle PQR is also the length of the path $P'QRP''$. We can shorten the path by straightening it out, so Q goes to Q' and R goes to R'. We cannot reduce the angle $\angle P'AP''$, which is twice $\angle A$, but we can reduce $|P'P''|$ by reducing the three equal lilac lengths, and we can do that by moving P till it is the foot of the altitude from A. By symmetry, the optimal positions for Q' and R' must also be the feet of altitudes. So the solution to Fagnano's problem is the orthic triangle.

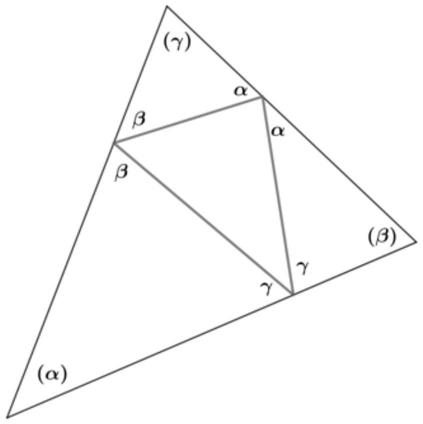
We can also use a mechanical computer of the sort employed by Mark Levi to show several things. Though our set-up is idealised, we shall specify the elements to make the properties more believable. The outer triangle is formed from three steel rods. The inner triangle is an elasticated cord. This runs without friction through nylon rings through which the steel rods pass. (You can think of the elements as deck furniture on a yacht.) We imagine the set-up has come to equilibrium.



1. Because the tension is constant throughout, the resultant force on each ring is a vector bisecting the angle there. Since the forces are in equilibrium, they all pass through the same point, thus proving that *the angle bisectors of a triangle are coincident.*

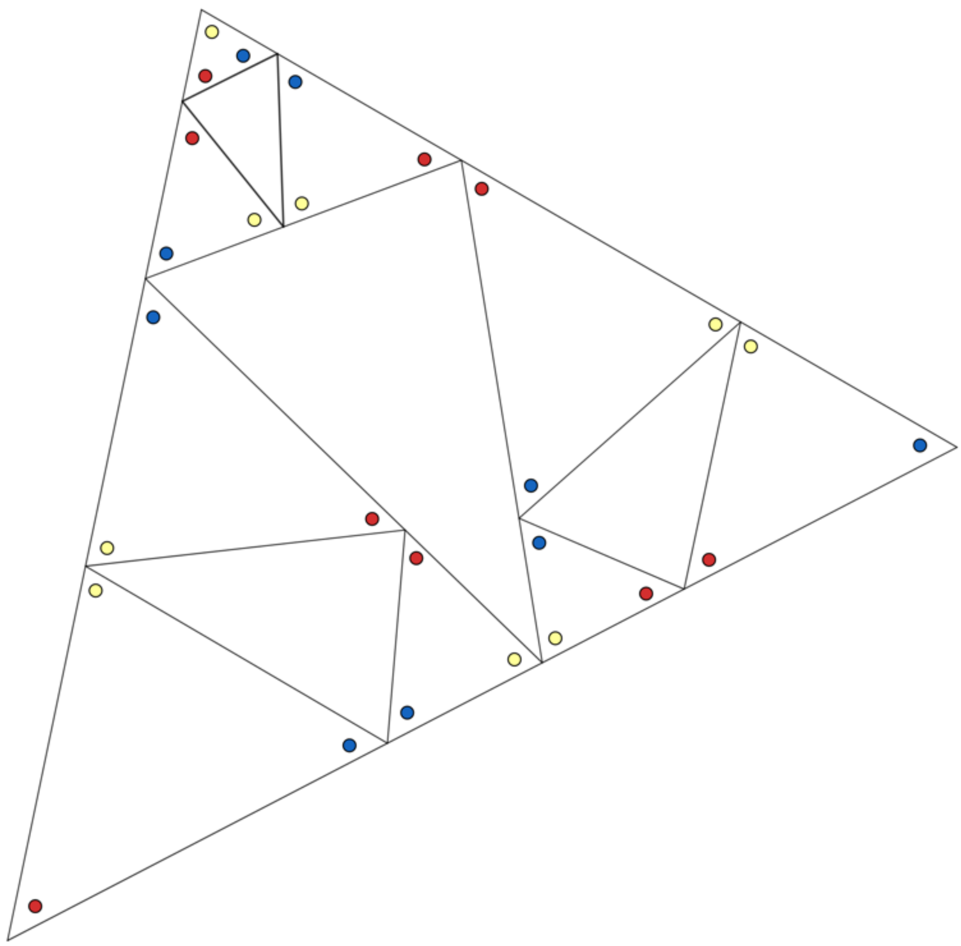
2. In equilibrium, the potential energy is least, which means that the tension is minimal, which in turn means that the perimeter is least. We have thus found the triangle required by Fagnano.

3. Since there is no net force along a rod, the cord must make equal angles on either side.



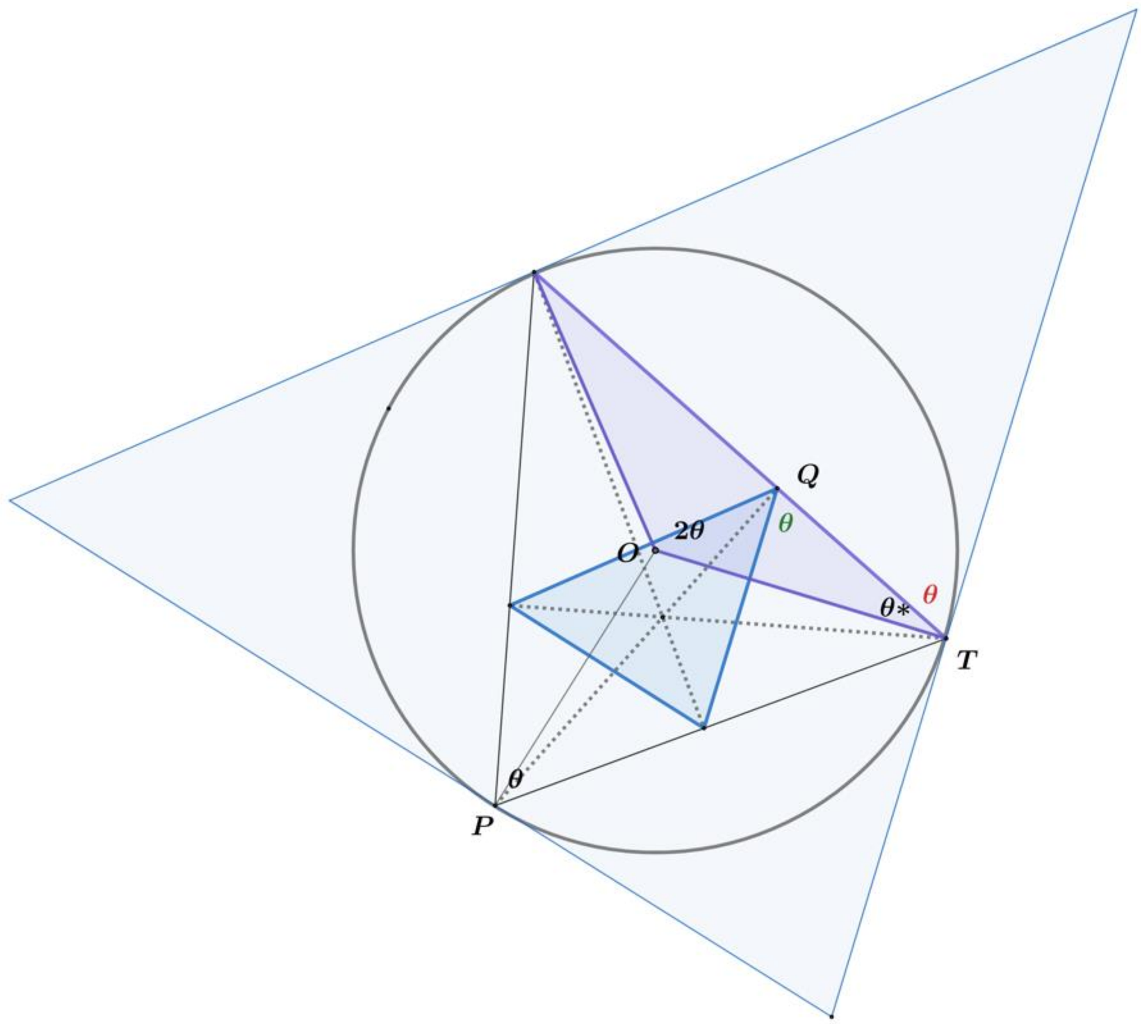
Working out the supplementary angles in brackets, we see we have specified the orthic triangle. We have thus found the solution to Fagnano's problem by a mechanical method.

We can continue to subdivide the three similar triangles around the orthic triangle indefinitely:



Triangles sharing a vertex have opposite orientation. The orthic triangles of different orders are also similar to each other.

One interesting property of the orthic triangle is the following. We draw the circumcircle of the original triangle, and tangents at the vertices. This gives us the *tangential* triangle, the dual of the original. We shall show that the orthic triangle is similar to this.
In this figure an asterisk indicates the complementary angle: $\alpha^* = \left(\frac{\pi}{2} - \alpha\right)$.



We follow two routes, starting at P:

1.
At P we have θ.
At O, 2θ (angle at circle centre = twice angle at circumference).
At T, θ^* (base angle in lilac isosceles triangle = $\frac{\pi-\theta}{2}$).
At T, θ (tangent meets radius at right angles, so we have $(\theta^*)^* = \theta$).

2.
At P we have θ.
At Q, θ (because, as proved above, a side of the orthic triangle is antiparallel to the corresponding side of the main triangle).

So the green and red angles on alternate sides of the transversal between a side of the orthic triangle and a side of the main triangle are equal. The two sides are therefore parallel, and the triangles similar, as required.