3.7 Steiner tree constructions

Theorem: Consider the vertices of a triangle, all of whose angles are less than $\frac{2\pi}{3}$. Consider the point within it, the sum of whose distances from the vertices is least. Each side subtends an angle of $\frac{2\pi}{3}$ at the point.

Proof:

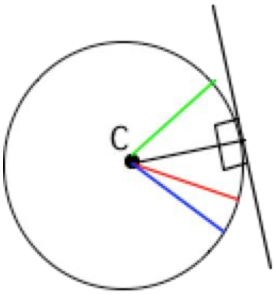


Figure 1

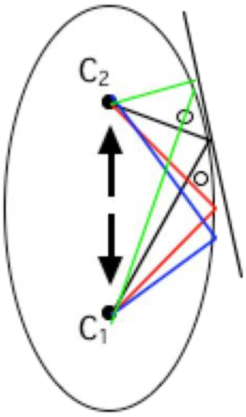


Figure 2

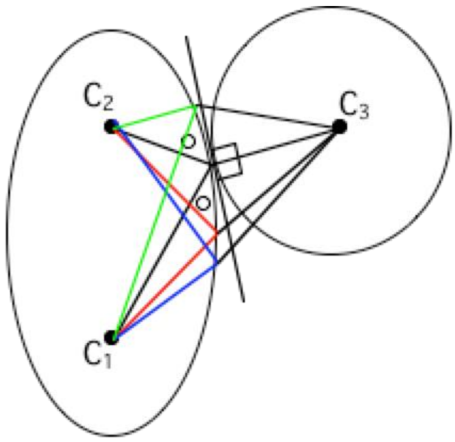


Figure 3

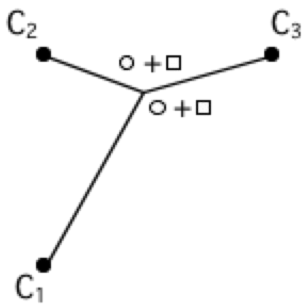


Figure 4

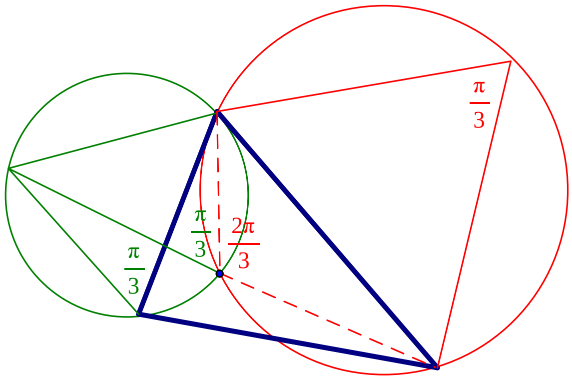
The argument runs as follows.

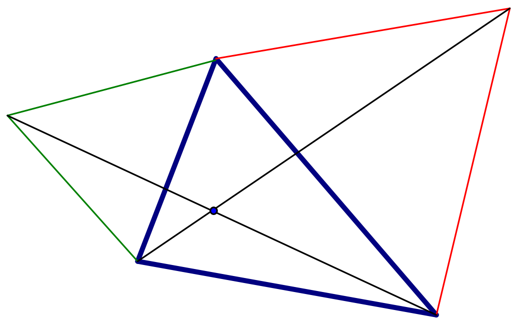
In Figure 3, , C_1, C_2, C_3 are our cities. Centred on C_3 we've drawn a circle which shares a tangent with the ellipse.
The combined length of the two black or two blue or two red or two green roads is the same.
However the road which comes from the junction to C_3 is shortest at the point of tangency.
At that point therefore the total length of the three roads is as small as possible.
Notice the pair of equal angles, both equal to 'circle' + 'square' (Figure 4).

How might the argument continue?

Given a triangle, how do we locate the Fermat point? On the left we have drawn equilateral triangles on two sides of our triangle and also their circumcircles. Because of the cyclic quadrilaterals like that shown in red, we know that the angle enclosed by the dashed red lines is the supplement of $\pi/3$, $2\pi/3$. The same would be true for all 3 such circles, therefore the point of intersection of any two, in particular the red and the green, is our Fermat point.

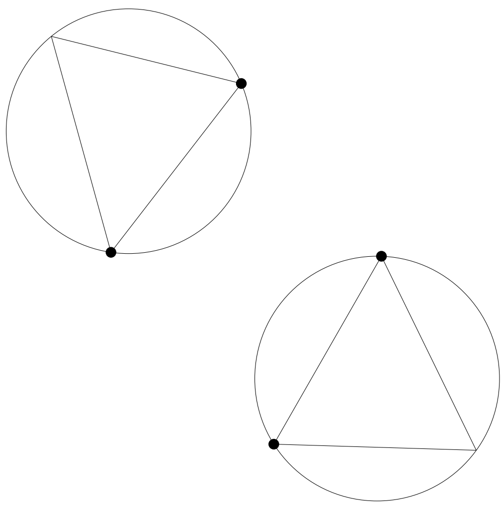
But we also note two angles in the same circle segments of the red and green circles respectively. These total π. So at the vertex marked by a dot the green line is extended by the red dashed line. And we could draw a similar line comprising a solid red segment on the right and a dashed green segment on the left. All we need draw therefore are the two lines shown in the figure on the right.



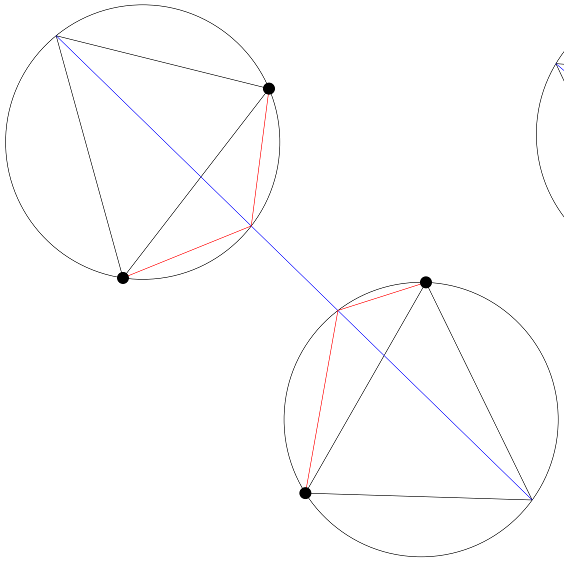


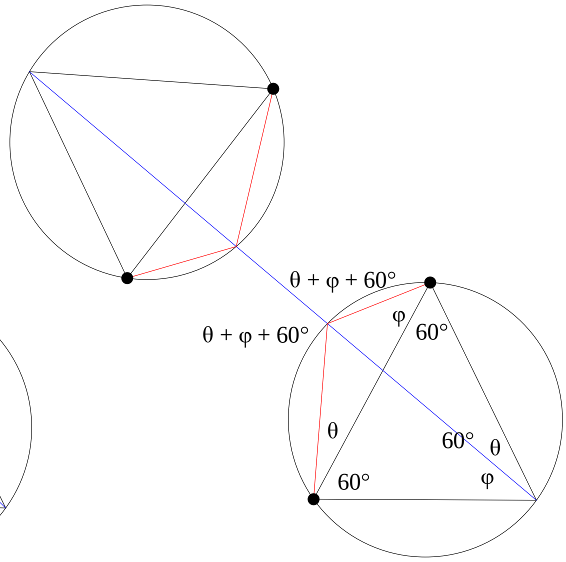
Given a quadrilateral, how do we construct the Steiner tree?

Draw equilateral triangles on two sides of the quadrilateral and draw circles through them:



Draw the blue line.
The red lines are your final figure.





To show why the construction works we've labelled angles in the right-hand diagram.
We've split a $'60°'$ into a θ and a φ.
The 'angles in the same segment' theorem repeats these angles where shown.
We can now use the 'exterior angle' theorem in the two triangles separated by the blue line to work out the sums shown.
But, since we know that $\theta + \varphi = 60°$, we know that this total is $120°$ in each case, and so, by subtraction, must our third angle be.