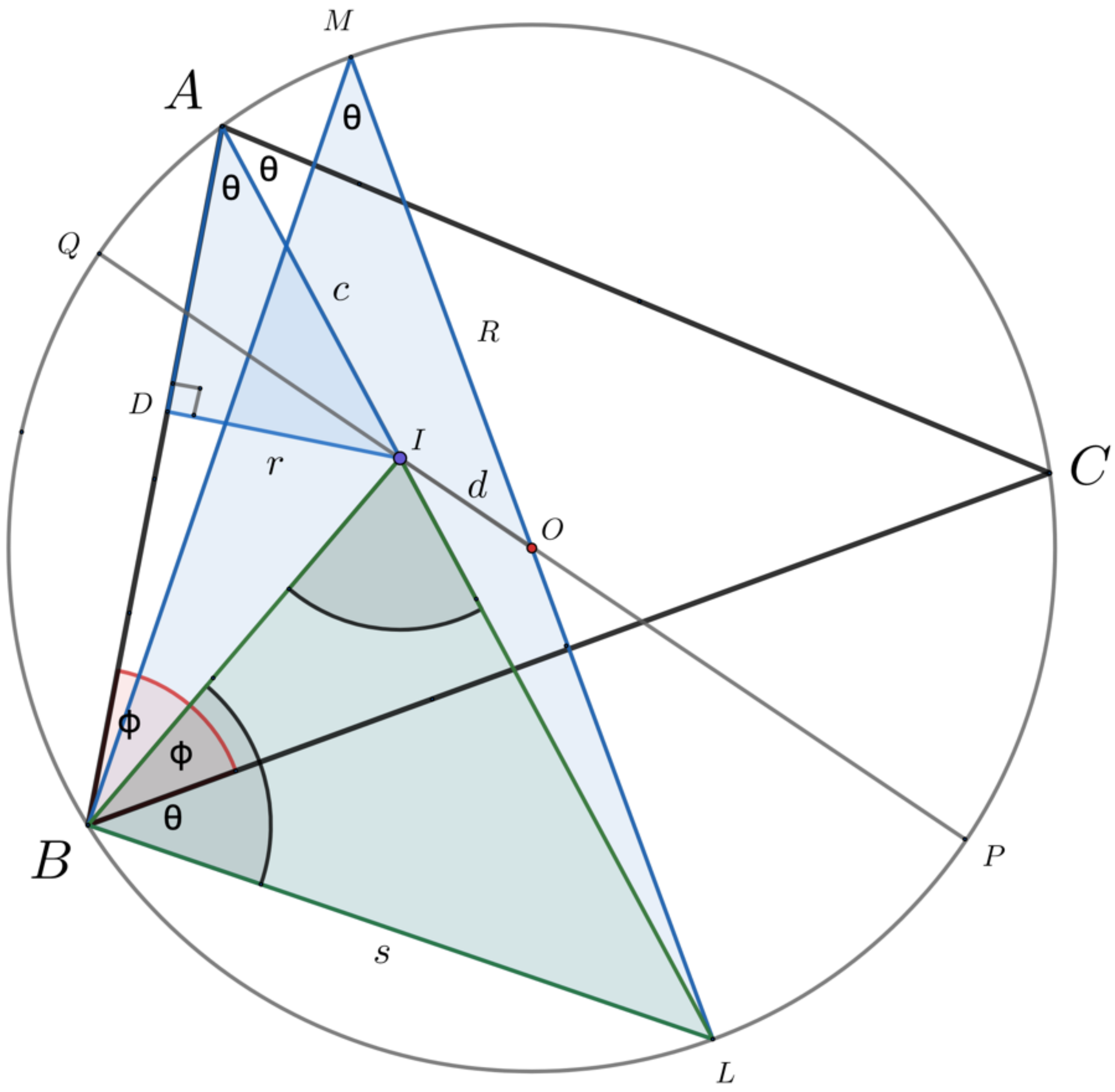
3.3 From Euler to Poncelet

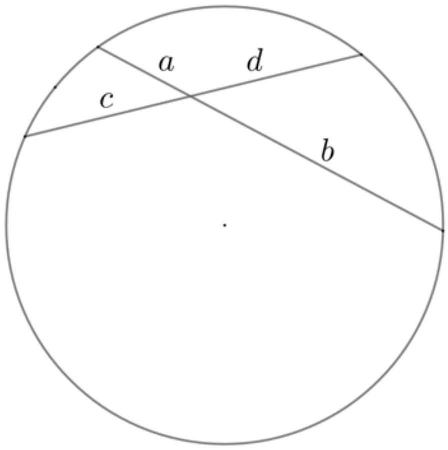
This story tells how an C18 discovery was generalised in C19.

I is the incentre of the triangle, *O* the circumcentre, *d* the distance between them.
r is the inradius, *R* the circumradius.



We shall prove a result of Euler using these theorems and properties:

1. Angles in the same segment are equal.
2. The angle in a semicircle is a right angle.
3. If two right triangles share a second angle, they are similar.
4. the exterior angle of a triangle is the sum of the interior, opposite angles.
5. In this figure *ab* = *cd* (the intersecting chord theorem):



6. The incentre lies on an angle bisector.
7. If a triangle has two equal angles, the two sides which do not lie between them are equal.

*Use **1.**, **2.**, **3.** to show that the blue right triangles are similar.*

Comparing corresponding sides, $\frac{r}{s} = \frac{c}{2R}$, $cs = 2Rr$. [**Equation 1**]

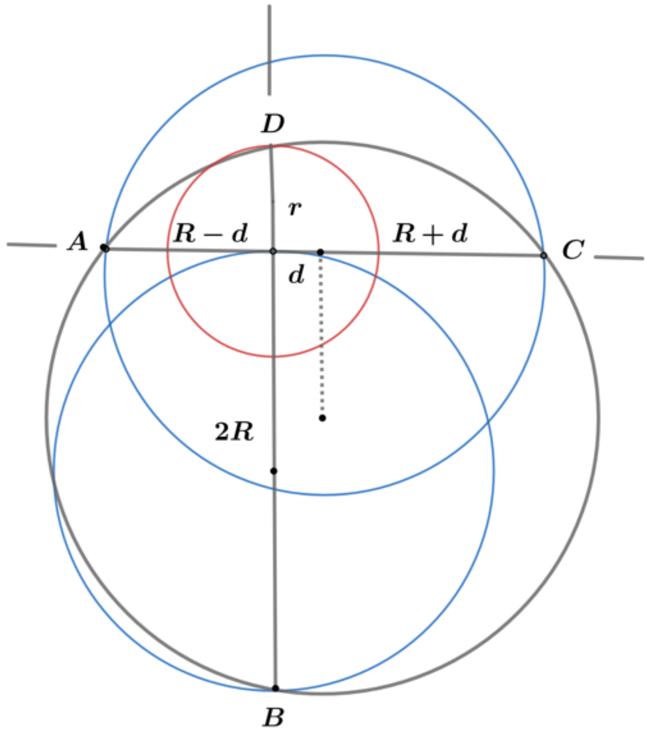
*Use **1.**, **4.**, **6.** to show that the unmarked green angle is equal to angle LBI.*

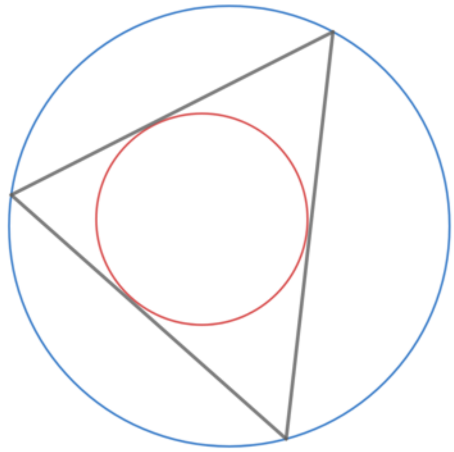
*Use **7.** to show that $|IL|$ = s.*

*Use **5.** to show that $cs = (R + d)(R - d)$.* [**Equation 2**]

Combining [**Equation 1**] and [**Equation 2**], we find Euler's result, $\boldsymbol{2Rr = (R + d)(R - d)}$**.**

A geometric representation of the Euler formula

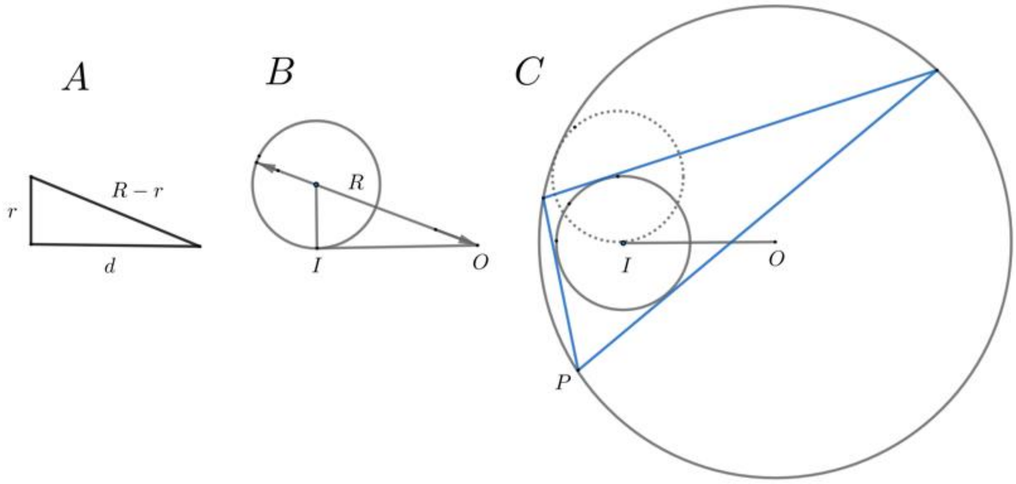




This depends on the intersecting chord property.

Draw the blue circle, radius *R*. Draw an extended vertical diameter. Draw a horizontal tangent. Centre a point *d* to the right of the point of tangency, draw a second blue circle. Draw the circle through *A, B, C*. This cuts the vertical line in *D*, distant *r* from the point of tangency.

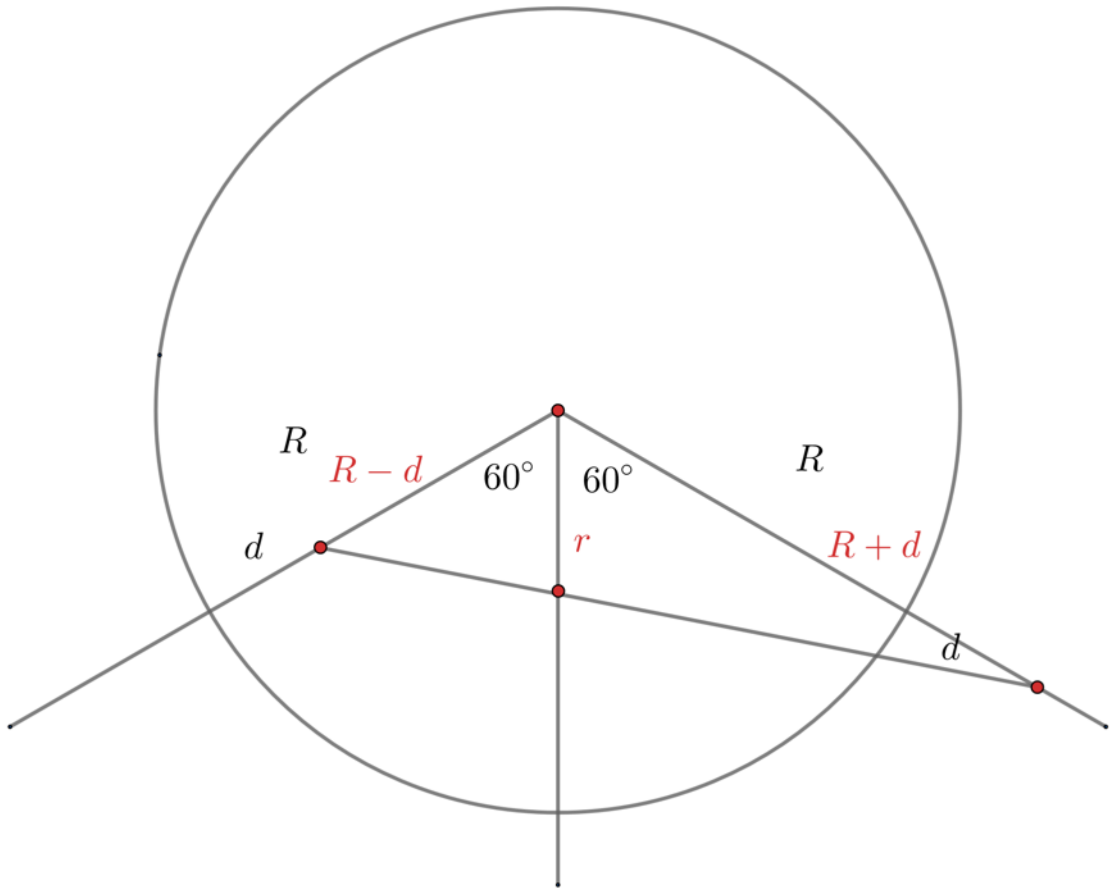
We can rearrange the equation like this:
$d^2 = R^2 - 2Rr$,
add r^2 to both sides:
$d^2 + r^2 = R^2 - 2Rr + r^2 = (R - r)^2$,
and we see we have the right triangle ***A***, which we can set in figure ***B*** and again in figure ***C***:



This gives us a way to construct the incircle and the circumcircle *independently of the triangle.* We choose r and draw a circle that size. We choose d and draw a tangent that length. This gives us the position of O and the size of R.

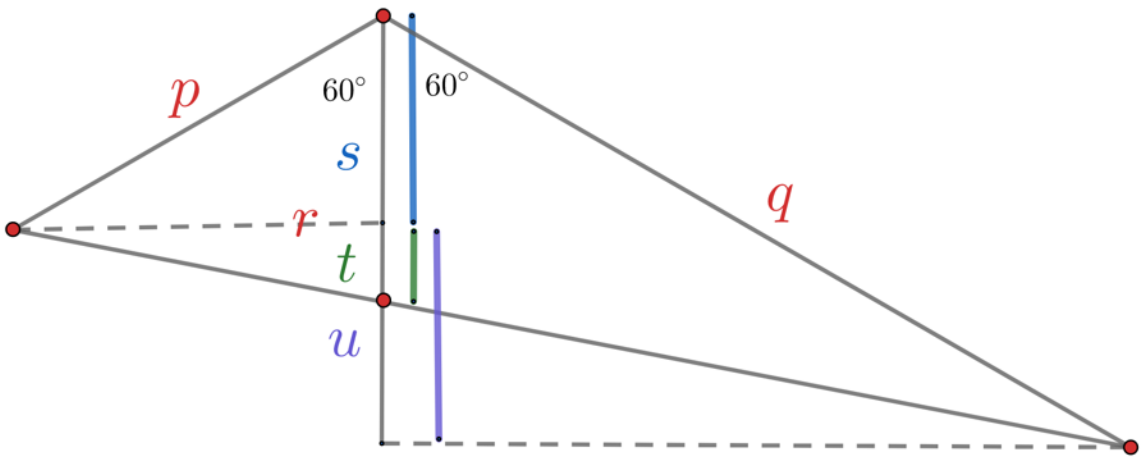
Poncelet's insight was this. We can choose any point P on the circumcircle and draw three lines in sequence, each of which is tangent to the incircle and meets the circumcircle at the starting point of the next line, and know that the final point Q will coincide with the first point P. That is to say, we can draw an infinite number of triangles which share the same incircle and circumcircle. (He showed further that there is a construction which ensures this for every bicentric polygon.)

We can write Euler's formula like this: $\frac{1}{r} = \frac{1}{R-d} + \frac{1}{R+d}$. If we draw 3 lines at 60° like this ...



... and record the value of $R - d$ on the left and $R + d$ on the right, we can read off the value of r on the middle scale by laying a ruler between them. Why does this work? *Try the problem yourself before reading on.*

One approach is to project onto the centre line. The similar triangles obtained give us the ratios we want. Write $R - d = p, R + d = q$.



$$r = s + t = p\cos(60°) + \frac{p}{p+q}(q-p)\cos(60°)$$
$$= \frac{p}{2} \qquad\qquad + \frac{p(q-p)}{2(p+q)}$$
$$= \frac{p}{2}\left(1 + \frac{q-p}{p+q}\right) = \frac{p}{2}\left(\frac{p+q+q-p}{p+q}\right) = \frac{pq}{p+q}.$$
$$\frac{1}{r} = \frac{p+q}{pq} = \frac{1}{p} + \frac{1}{q} = \frac{1}{R-d} + \frac{1}{R+d}.$$