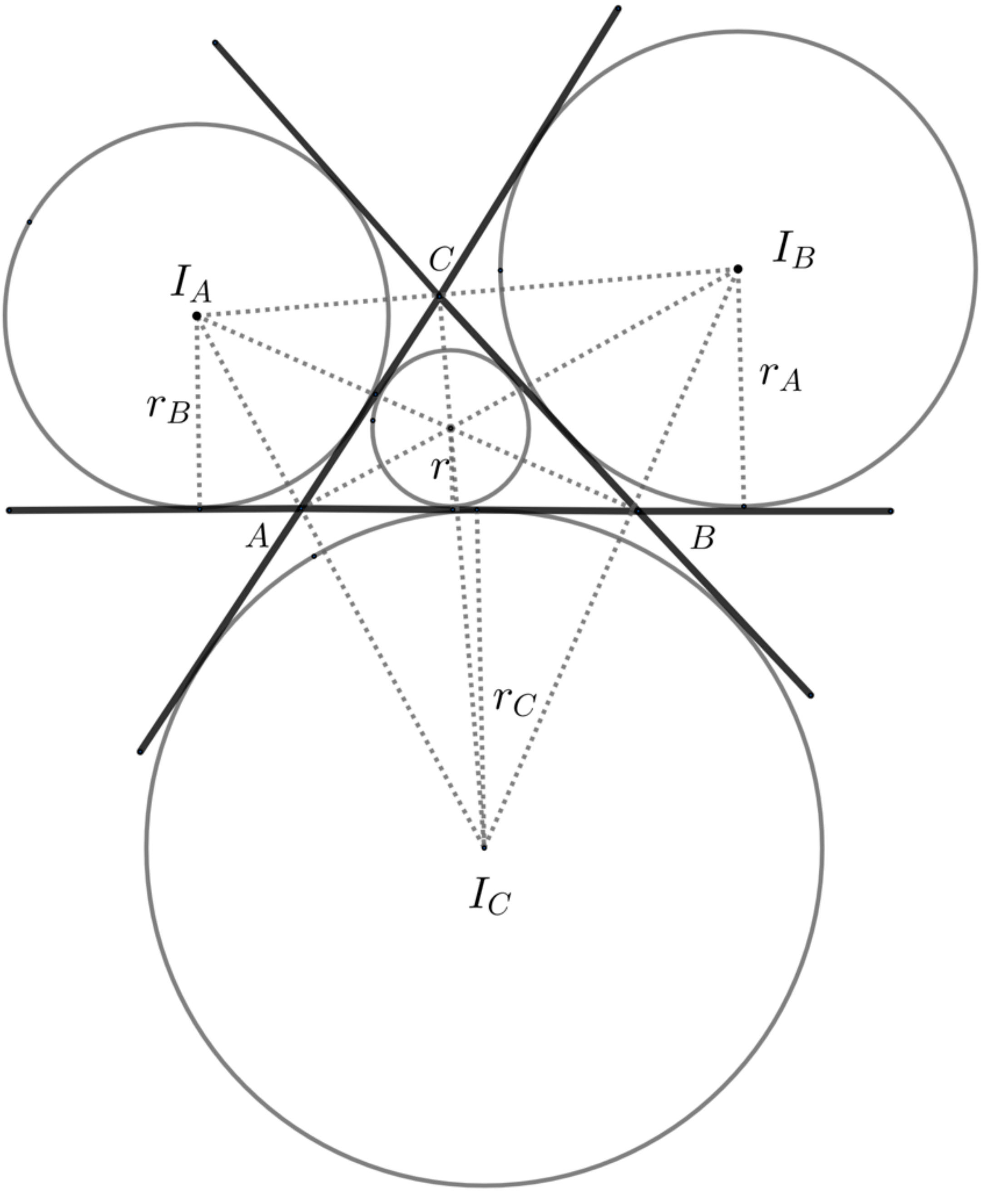
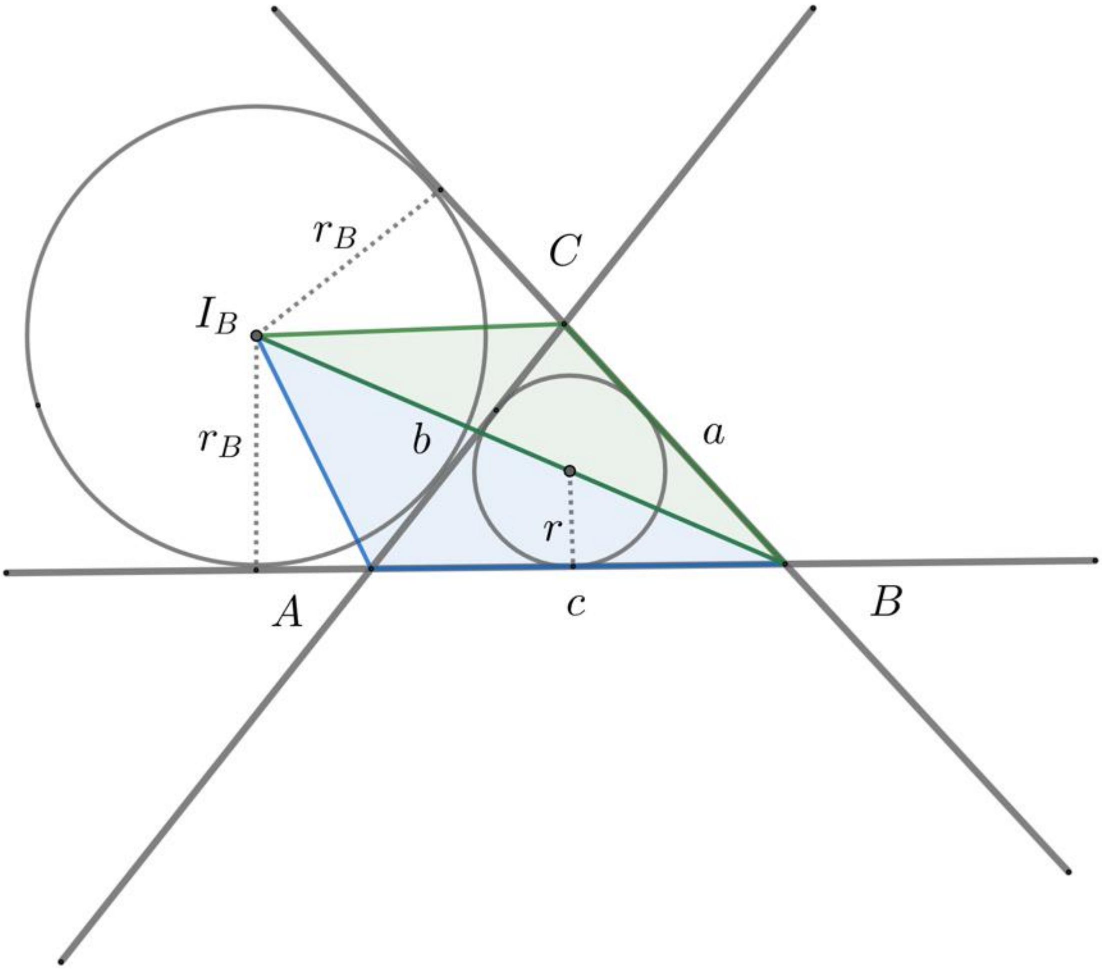
3.2 The e-circles of a triangle

If we see the sides of a triangle as lines of unlimited length, and we obey the instruction 'Draw the circles which touch the three sides of the triangle', we shall have to draw four circles, the incircle and three e-circles:



The centres of all four circles lie where the angle bisectors meet. In the case of the e-circles these are the external bisectors, which are perpendicular to the internal ones.



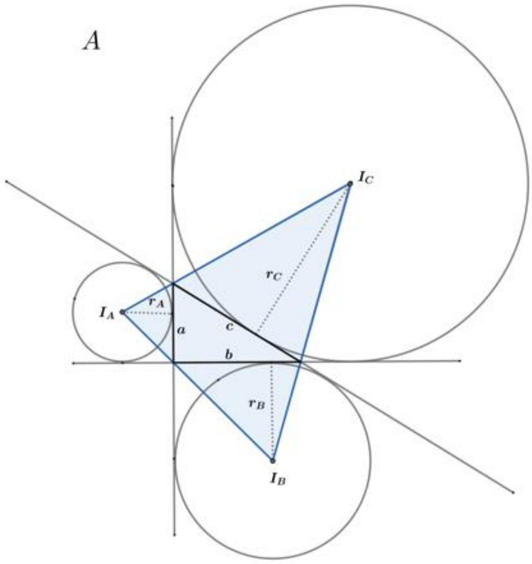
We can make the area, *A*, of triangle *ABC* by adding the blue and green triangles and subtracting that on the outside of side *b*, giving us

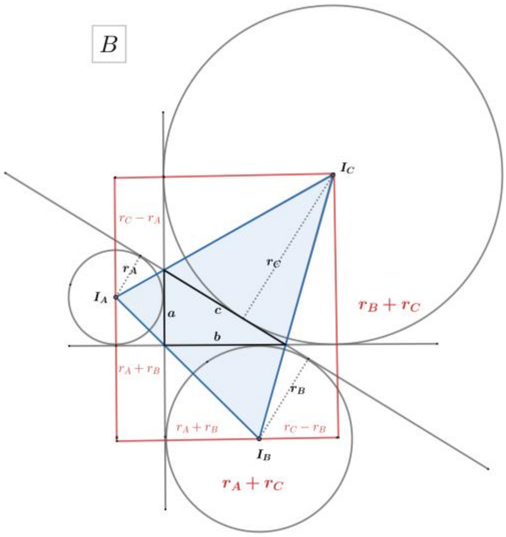
$A = \frac{(c+a-b)r_B}{2}$ = $(s-b)r_B$, which we know from the section **From Heron to von Staudt** is also = rs, where $s = \frac{a+b+c}{2}$.
So $r_B = \frac{rs}{(s-b)}$. *Write equivalent expressions for* r_C, r_A. Notice that these are all bigger than *r*.

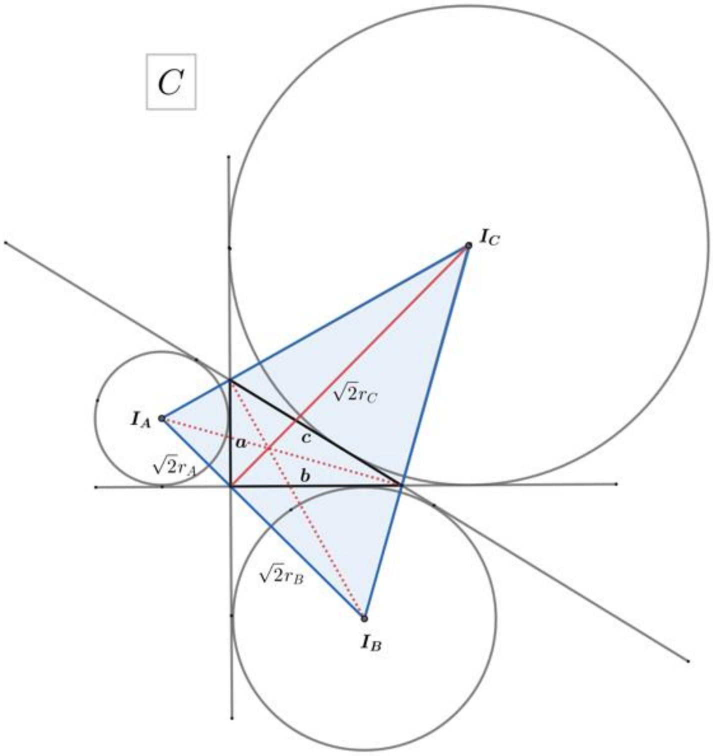
Returning to the equations like $A = (s-b)r_B$, again we can write similar equations for r_C, r_A. Recall the Heron equation from the section **From Heron to von Staudt**:
$A = \sqrt{s(s-a)(s-b)(s-c)}$.
By multiplying the short equations together to give A^4, *and squaring the Heron equation to give* A^2, *show that* $A = \sqrt{r r_A r_B r_C}$.

An interesting result concerns the area of triangle $I_A I_B I_C$ where the central triangle is right-angled.

There are many ways in which we might calculate the area. We could for example add the four blue triangles in ***A*** or subtract the right triangles from the red rectangle in ***B***. Though a number of methods will furnish a check, the best will be the one which requires the least algebra. A good candidate is ***C***.







The interior and exterior bisectors of an angle are perpendicular. Take the solid red line which bisects the right angle. The right angled isosceles triangles give the length of the three segments labelled. The area of the triangle $I_A I_B I_C$ is then ‘half base x height’

$$=\frac{1}{2}\sqrt{2}(r_A + r_B)\sqrt{2}r_C$$

$$=\left(\frac{rs}{s-a}+\frac{rs}{s-b}\right)\frac{rs}{s-c}$$

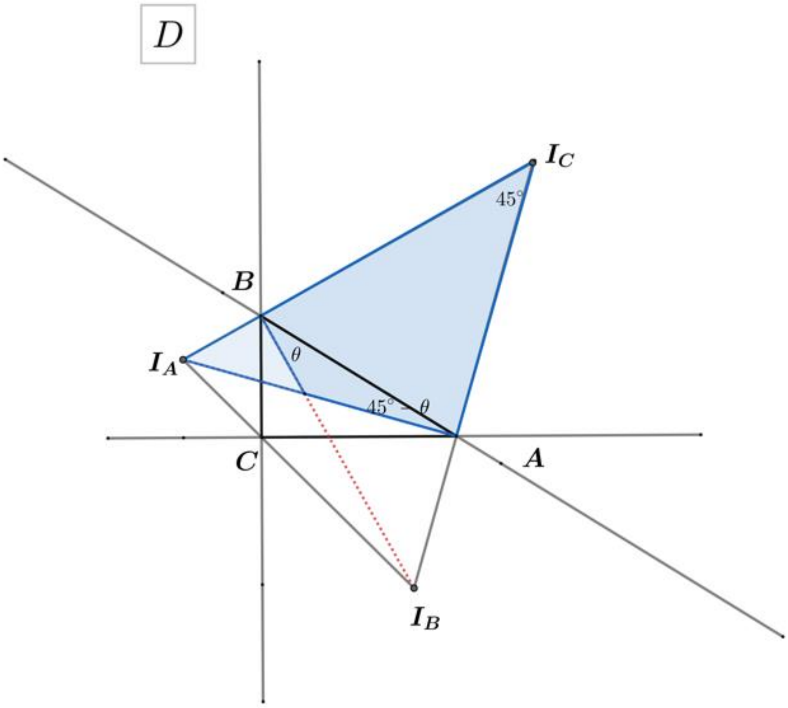
$$=\frac{(rs)^2(s-b+s-a)}{(s-a)(s-b)(s-c)}$$

$$\frac{(rs)^2 c}{(s-a)(s-b)(s-c)}.$$

Multiply top and bottom by s and we have $A^2 sc$ in the numerator and A^2 in the denominator, so the required area is $\frac{A^2 sc}{A^2} = sc$.

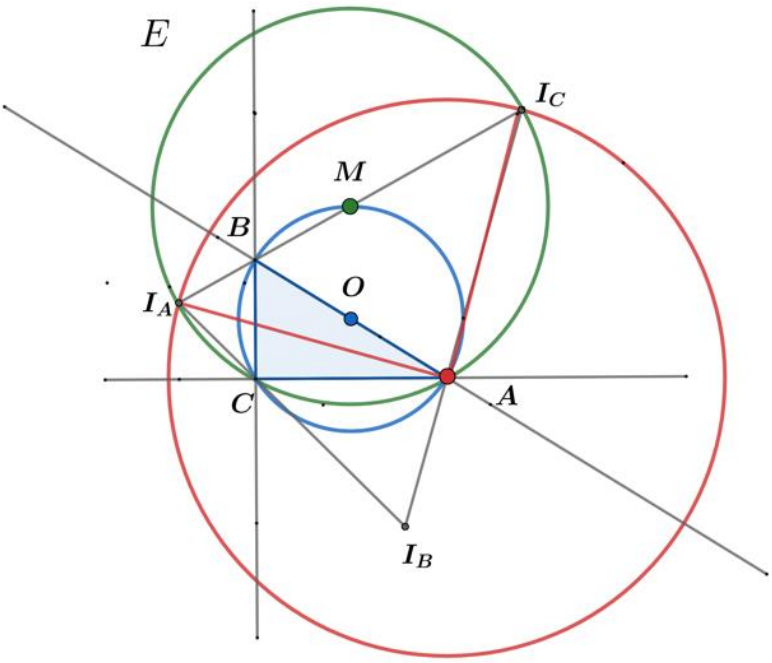
Before leaving this case, it is worth looking for other geometry of interest.

You will need to do a bit more algebra for this one: *Show that the hypotenuse bisects the area of triangle $I_A I_B I_C$.*



An angle θ is marked.
Why is angle $I_A AB$ $45° - \theta$?
Why, as a consequence, is angle $AI_C B$ $45°$?
Why then is $I_A A = I_C A$?

We could make a similar argument for $I_B B$ and $I_C B$.



The equal lines found in ***D*** are shown as radii of the red triangle in ***E***.

Why does the green circle pass through A?

Why does the blue circle pass through M?