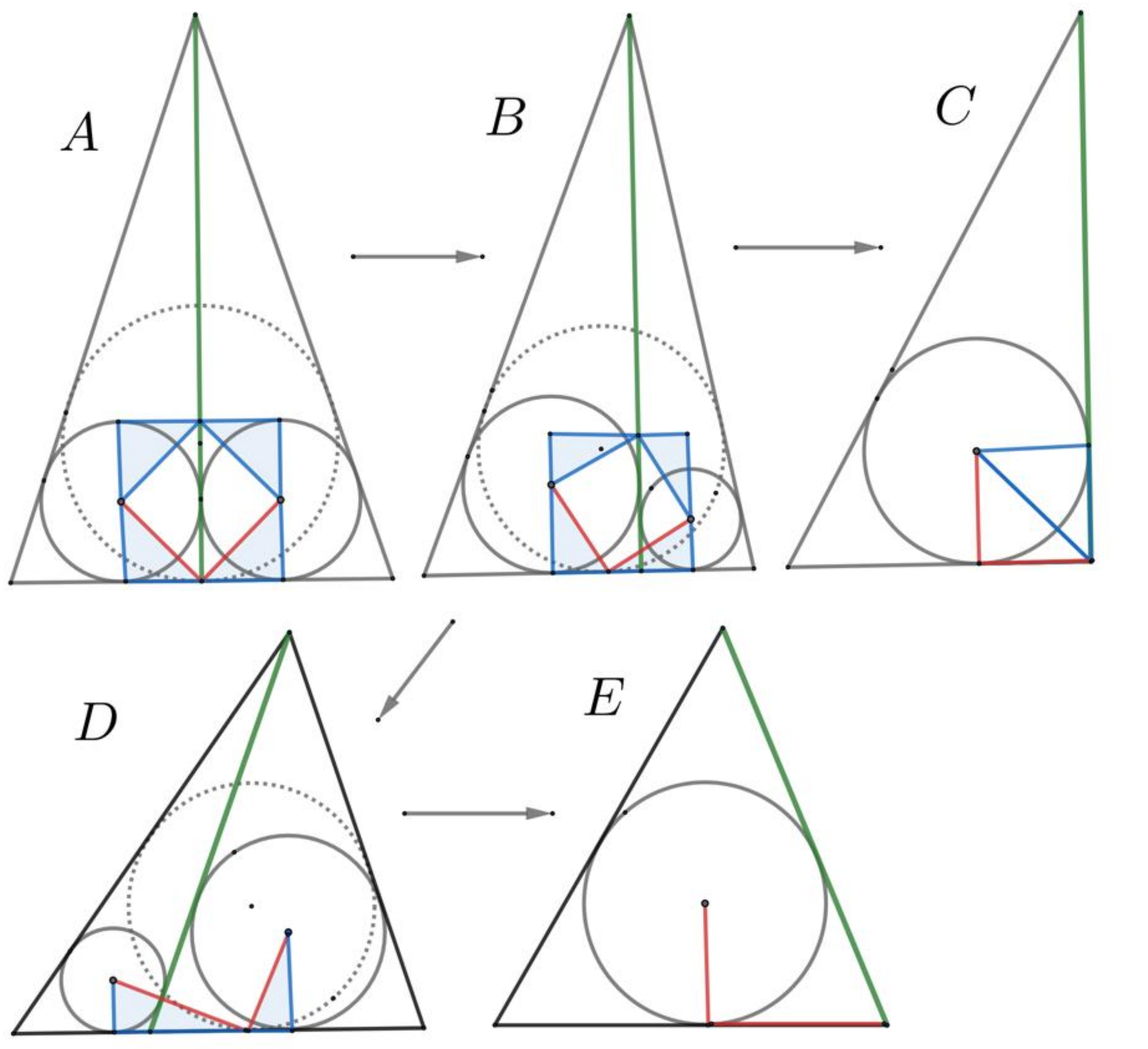
3.1 Shvetsov's right angle

A preliminary survey

We begin with an isosceles triangle. ***A*** therefore has a symmetry axis. The red lines bisect right angles. Thus the red lines are equal and perpendicular and define a square. The blue right triangles are isosceles. The dotted circle touches the triangle base at the foot of the altitude.

In ***B*** we see that we can interpret the green line in ***A*** as an altitude. The dotted circle no longer touches the base at the foot of the altitude. But, to our surprise, we find that the red lines are again equal and perpendicular and define a square which has one vertex on the altitude. The right triangles are no longer isosceles but are congruent.

In ***C*** we have a limiting case: the altitude coincides with a triangle side. The circle right of the altitude is of zero size; the circle left of the altitude coincides with the dotted circle.



In ***D*** we interpret the green line as any cevian, not necessarily one perpendicular to a triangle side. Again we find we get the right angle between the red lines, but the lines are no longer equal. The blue right triangles are no longer congruent but remain similar.

E again is a degenerate case. We retain the right angle but, as in case ***C***, the right triangles are reduced to their hypotenuses.

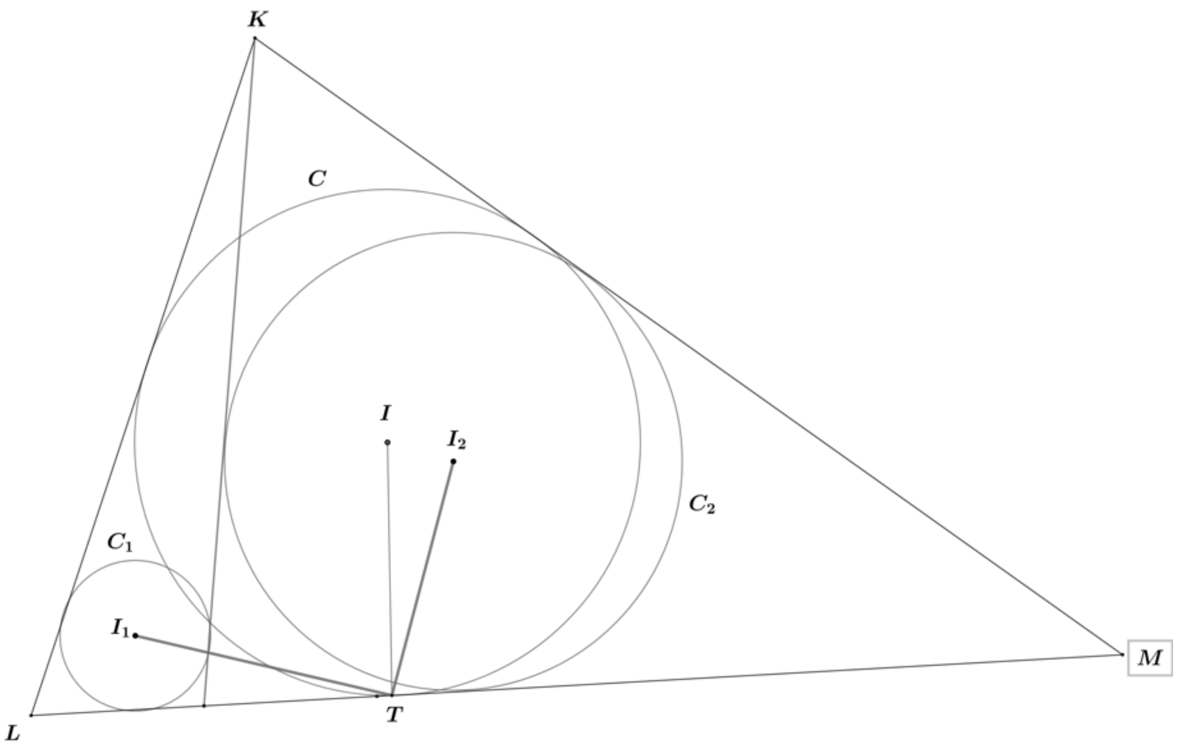
What happens if we have an obtuse-angled triangle so that an altitude can fall outside the base or we take a cevian to meet the opposite side outside the triangle? In neither case are the red lines perpendicular. It seems that the solid black circles must lie on opposite sides of the green line for this to be the case. The point is that here the altitude is not a cevian, and all five cases ***A*** to ***E*** illustrate a theorem about cevians.

Our theorem is: *Given a triangle KLM divided by a cevian KD, the centres of the incircles in the sub-triangles subtend a right angle at the point where the incircle of KLM touches side LM.*

Source: Arseniy Akopyan, *Geometry in Figures*, 4.5.34, 4.5.35, the latter due to D. V. Shvetsov and used in the Sixth Geometrical Olympiad in honour of I. F. Sharygin, 2010, Correspondence round, Problem 8.

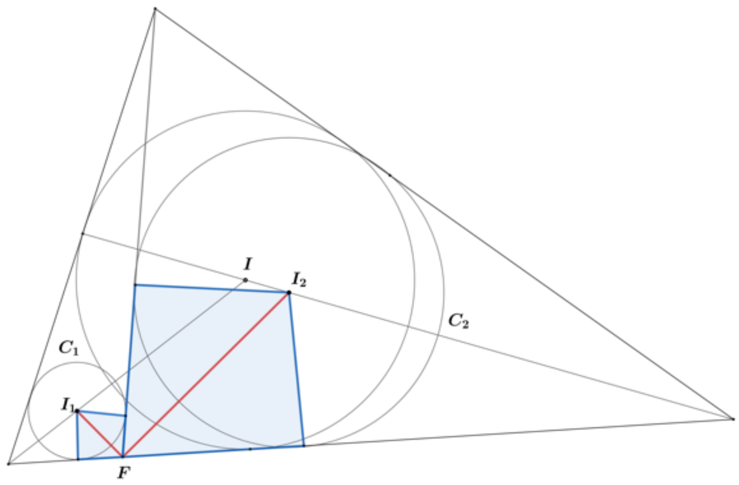
We shall prove this. ***A*** is our figure. We must show that angle I_1TI_2 is a right angle.

A



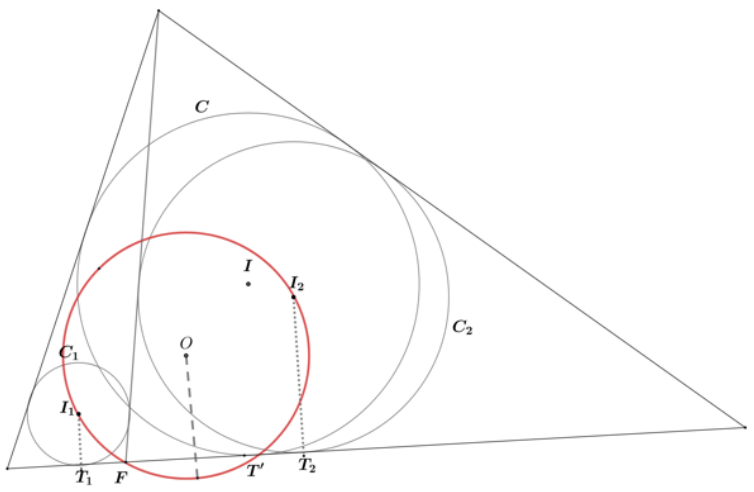
In ***B*** we have drawn radii from the smaller circles to the cevian and the base. We have two congruent right kites. The red lines therefore bisect the angle to the left at *F* and to the right and are perpendicular.

B

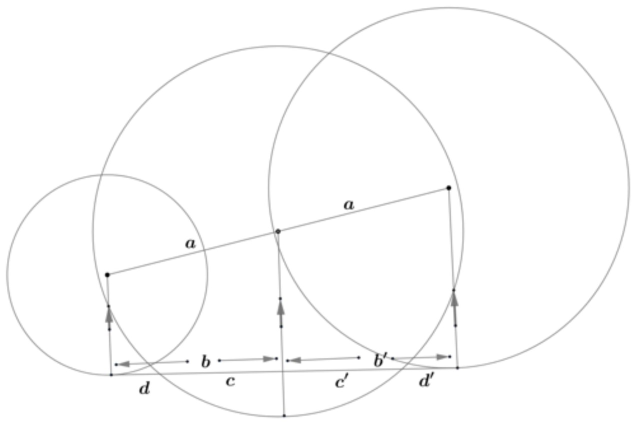


By the converse of Thales' theorem F lies on the circle with diameter I_1I_2.

C



In ***C*** we have shown this circle, which cuts the base in two points, F and T'. We have projected the diameter on to the base and drawn an axis of symmetry for the circle perpendicular to the base. One consequence of this symmetry is that $|T_1F| = |T'T_2|$. The full argument follows.



$b = b'$ by the parallel intercept theorem, (which in turn identifies equal ratios in similar triangles). **(A)**

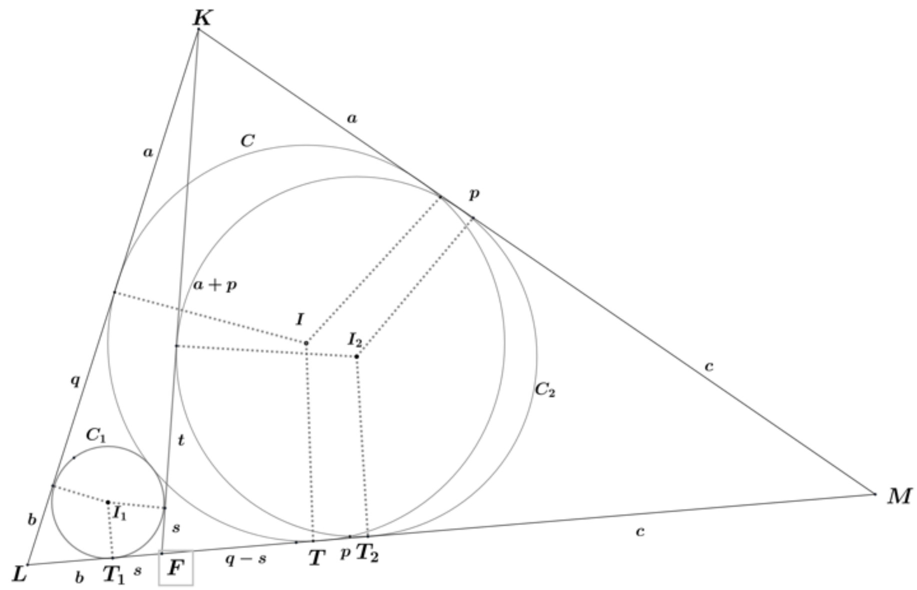
$c = c'$ because a circle radius perpendicular to a chord bisects it. **(B)**

Given **(A)**, **(B)**, we have by subtraction $d = d'$.

D

In ***D*** we have used the tangents-from-a-point theorem to find equal distances. We have also done a little arithmetic on the figure to simplify our final algebra, which reduces to this:

$s + t = p + q - s, 2s = p + q - t$. **(1)**
$a + p + t = a + q, q = p + t$. **(2)**



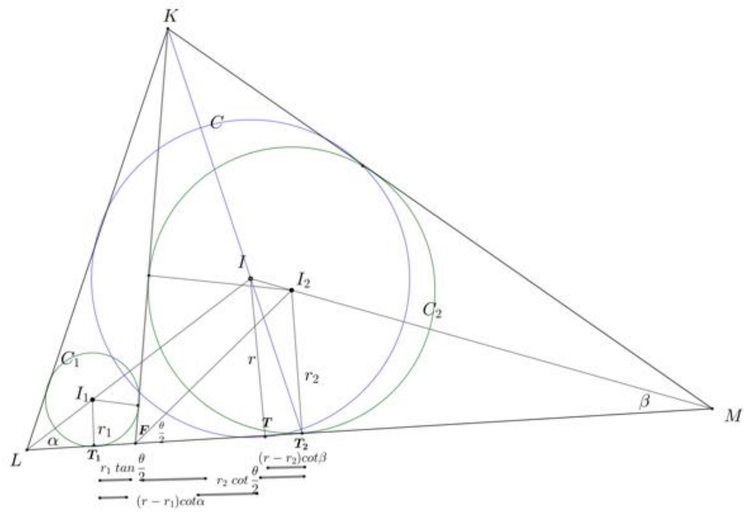
So $|T_1F| = |TT_2|$.
This means that *T'* and *T* are one and the same point.
That in turn means that *T* lies on the red circle
and, by Thales' theorem, I_1TI_2 is a right angle, as required.

Corollaries

1. Look again at ***D*** . Substituting back into equation **(1)** we find $q - s = t$.

2. In ***E*** we have used trigonometry and similar triangles to work out lengths.

E



Because $|T_1F| = |TT_2|$, we have:

$$r_1 \tan\frac{\theta}{2} = (r - r_2)\cot\beta \qquad \textbf{(3)}$$

and therefore

$$r_2 \cot\frac{\theta}{2} = (r - r_1)\cot\alpha. \qquad \textbf{(4)}$$

Multiplying **(3)** by **(4)**:

$$r_1 r_2 = (r - r_1)(r - r_2)\cot\alpha\cot\beta, \quad \textbf{(5)}$$

an equation independent of the angle
θ the cevian makes with *LM*.

The paradox is resolved when we realise that, having chosen our triangle, and therefore $r, \alpha, \beta,$ though we are free to choose θ, our choice will determine the sizes of r_1, r_2.

Take an equilateral triangle with unit inradius. This reduces equation **(5)** to: $r_1 r_2 = 3(1 - r_1)(1 - r_2)$. Taking $r_1 = \frac{1}{2}$, gives $r_2 = \frac{3}{4}$, a ratio $r_1 : r_2 : r :: 2 : 3 : 4$.