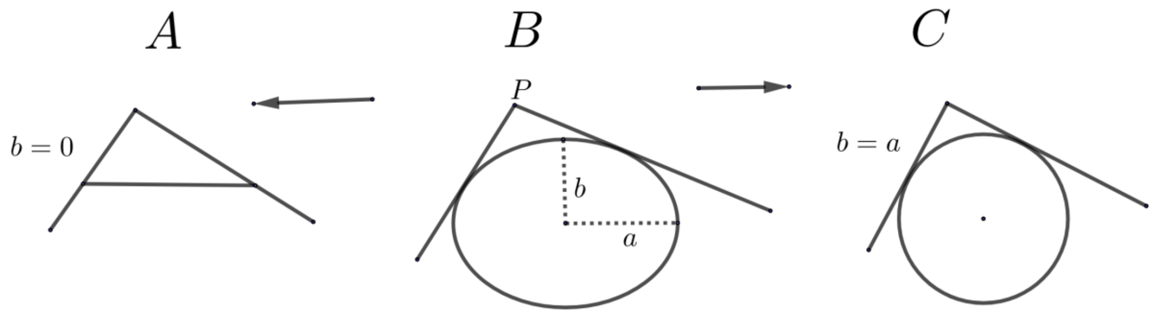
2.1 The orthoptic circle

B shows an ellipse. *P* lies at the intersection of two perpendicular tangents. What is its locus? Since we are not told the values of *a* and *b*, we take the limiting cases *A* and *C*.



In *A* we take the flat ellipse to be the diameter of a circle. A right angle is the angle it subtends at the circumference. The locus is therefore a circle. In *C* the locus is a circle by symmetry. We can imagine a continuous transformation taking the flat ellipse to the circle and would not expect the locus of *P* to change between the limits we have chosen. It remains then to prove that the locus in the general case is indeed a circle.

We need the condition for a line to be tangent to an ellipse. We solve $y = mx + c$ with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ to give a quadratic in *x*, then put in the condition for equal roots, obtaining $y = mx \pm \sqrt{b^2 + a^2m^2}$. We solve this to give a quadratic in *m*, representing a pair of tangents: $(x^2 - a^2)m^2 - 2xym + (y^2 - b^2) = 0$, then impose the condition for the tangents to be perpendicular, i.e. for the product of the roots to be -1, and we have $x^2 + y^2 = a^2 + b^2$, the equation of a circle, centre the origin, radius $\sqrt{a^2 + b^2}$.

We can check this value for the radius in our three figures. *A* is clear. In *B* and *C* we use Pythagoras. To do so in *B* we first set the ‘frame’ perpendicular to the axes.