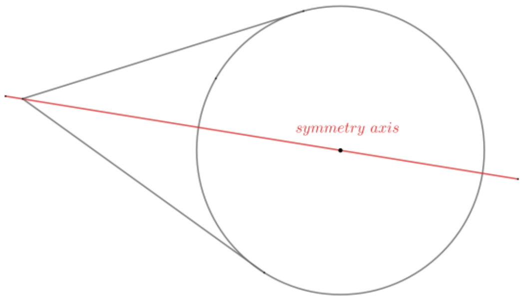
# 1.2 Closed circle chains

## 1.2.1 Rings of tangential circles

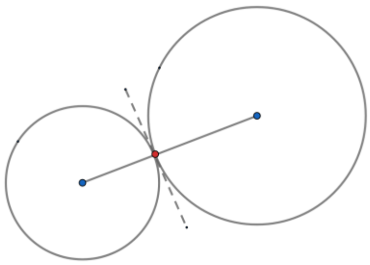
In this section we look at sets of circles where each circle is tangent to two others, thus forming a closed chain. We shall need three properties. (**(2)** and **(3)** you will also find under the heading **Circles & quadrilaterals**.)

**(1)** The first is that through any 3 points not in line, just one circle can be drawn.

**(2)** The second is that tangents from a point are equal. We see that this must be true by symmetry:

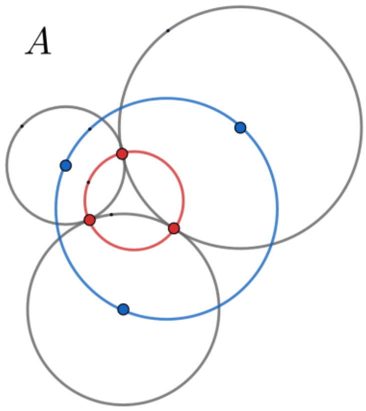



**(3)** The third is that the point of tangency of two circles lies on a line through their centres, and the common tangent is perpendicular to that line. This again is true by symmetry:

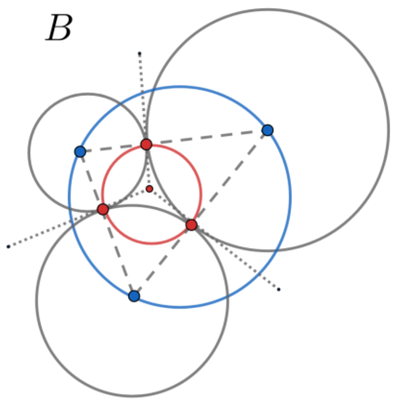


In the unadorned figure ***A*** we have two circles, each passing through and defined by three points, in accordance with **(1)**. But the blue points are the centres of the black circles and the red points their points of contact. We see from ***B*** how much hidden structure **(2)** and **(3)** induce.

*A*

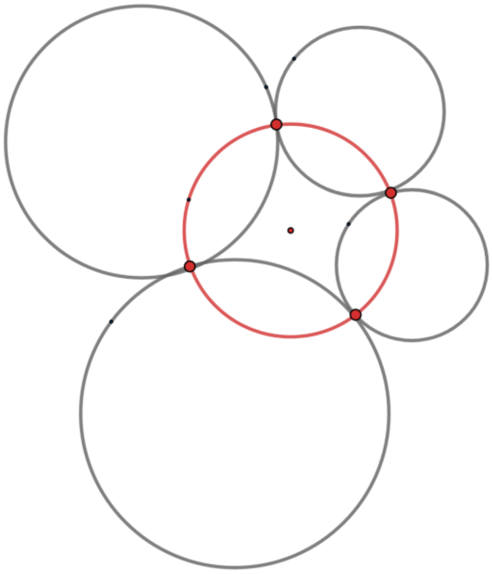


*B*

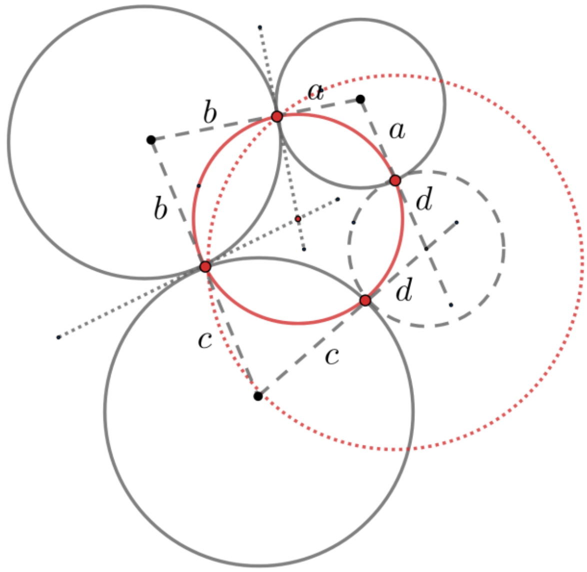


In ***C*** we have another unadorned figure, this time a chain of four circles. We see that a red circle passes through their four points of contact. We know that any three points define the circle so there must be something special about the fourth point.

$C$



$D$

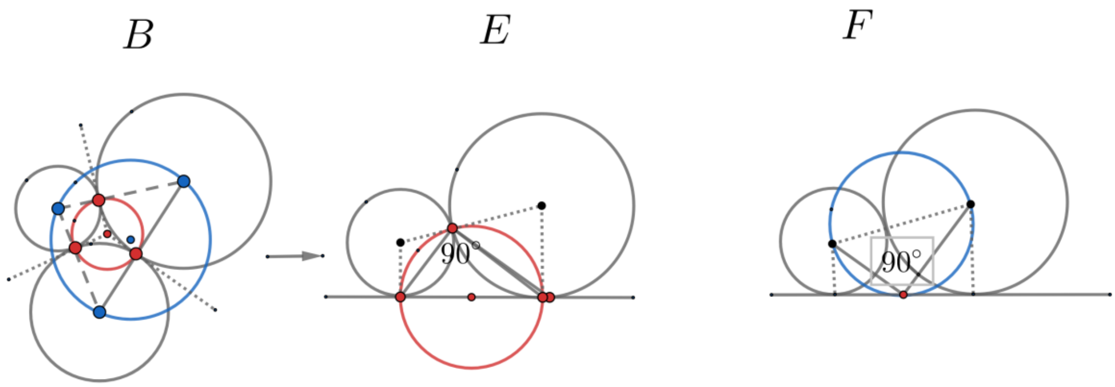



In ***D*** the three solid black circles are in place. Through their two contact points we could draw an infinite number of circles like the red dotted one. If, however, we draw a circle tangent to their two lines of centres, so that its centre lies at the intersection of the two dotted tangents, we create the site for a fourth circle. Its centre lies at the intersection of the two dashed lines of centres. The lines of centres form a quadrilateral tangential to the red circle. *By summing the labelled lengths, what do you discover about tangential quadrilaterals?*

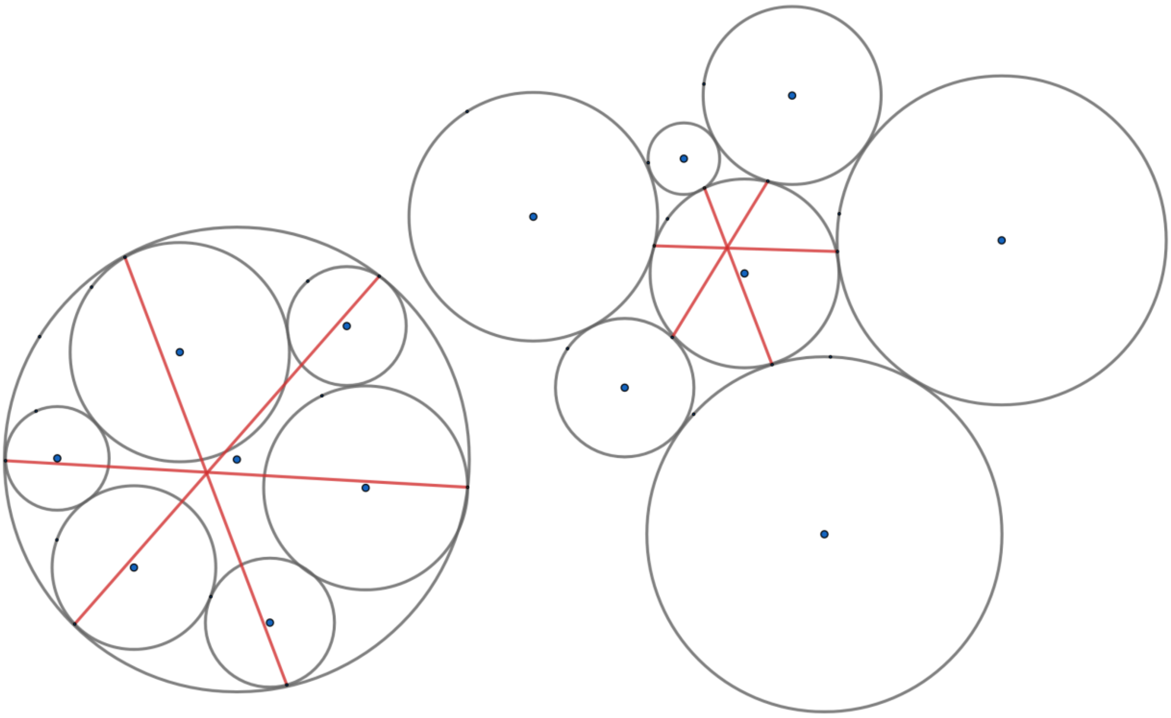
In a similar way we can construct chains of any number of circles whose contact points lie on a circle inscribed in a polygon whose vertices are the centres of the circles in the chain.

*If there are six circles in the chain, how are the side lengths related? Generalise for a tangential polygon with* $2n$ *sides.*

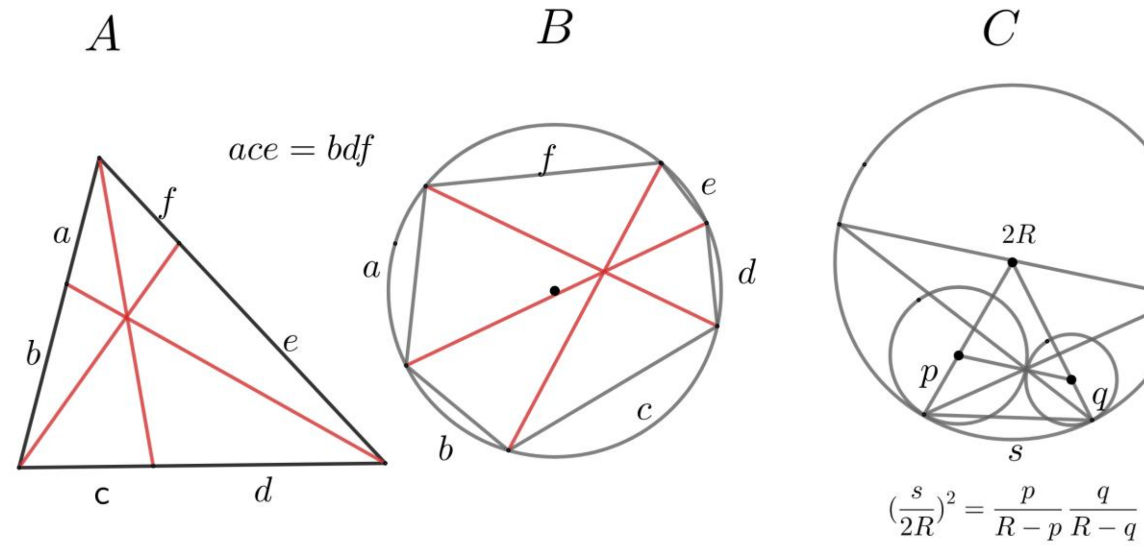
If we draw ***B*** so that the bottom circle is of infinite size, as shown in ***E***, two of the dotted radii form a diameter of the red circle and, since the angle in a semicircle is a right angle, we can mark the angle of 90° where shown. As also shown by the dotted lines, ***E*** contains half of a tangential, and therefore bicentric, trapezium. ***F*** shows where another right angle occurs. To see why, go to the section **Dual polygons**.




If a chain of six circles is inscribed in a circle or circumscribed about it, and we join opposite contact points, the lines coincide. This is the ‘seven circles theorem’.



To prove this, we need results ***B*** and ***C***, stated on the figures below. Each can be proved by means of similar triangles but we shall not do so. ***B*** is the analogue for circles of the better known ***A*** for triangles, Ceva’s theorem.



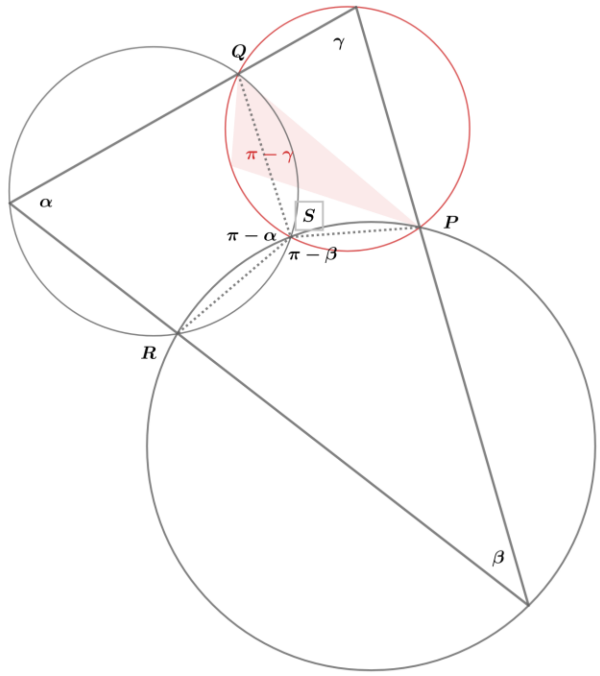

Let the radii of the six circles $C_1, C_2, C_3, \ldots$ be $r_1, r_2, r_3, \ldots$ respectively. Let the chords between the contact points of the circles with the seventh circle have lengths as follows:

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $C_1$ | | $C_2$ | | $C_3$ | | $C_4$ | | $C_5$ | | $C_6$ | | $C_1$ |
| | $a$ | | $b$ | | $c$ | | $d$ | | $e$ | | $f$ | |
| $r_1$ | | $r_2$ | | $r_3$ | | $r_4$ | | $r_5$ | | $r_6$ | | $r_1$ |

Taking our ***C*** formula, we have for example $a = 2R\sqrt{\frac{r_1}{R-r_1}}\sqrt{\frac{r_2}{R-r_2}}$, symbolised by the first blue bar. We see that the product *ace* contains exactly the same square root terms as the product *bdf*. Thus *ace* = *bdf* and from ***B*** the red lines must coincide.

## 1.2.2 Rings of intersecting circles

We now move from mutually tangential circles to circles which overlap. We shall need as lemma Miquel's three circles theorem, which we now prove.




Each of three circles passes through the vertex of a triangle and the points $P, Q, R$ on the adjacent sides through which the other circles pass.

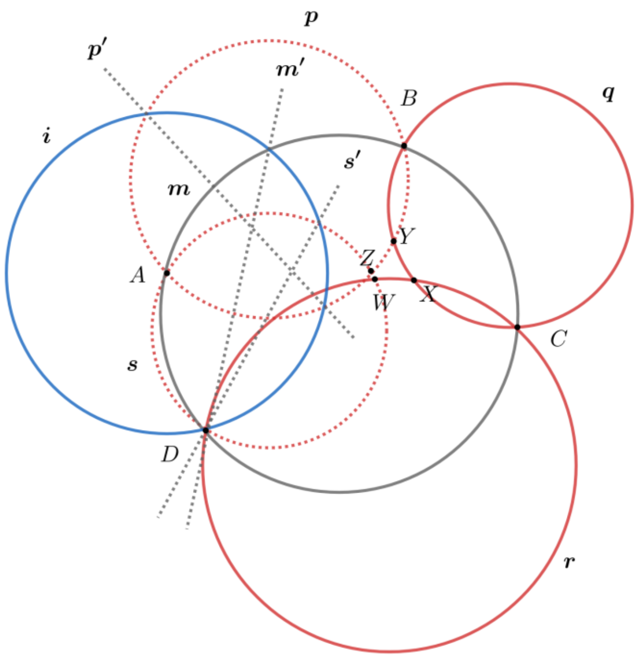
They will intersect in pairs again within the triangle. Let the two black circles intersect in *S*.

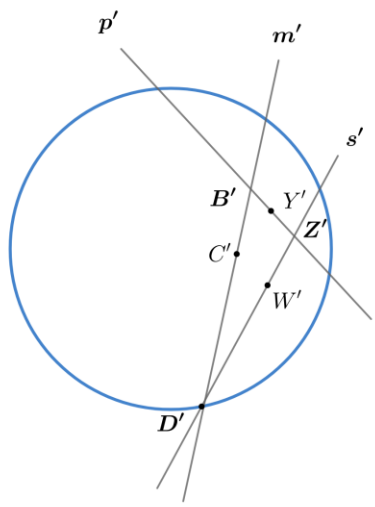
As interior angles of a triangle, we have:
$\alpha + \beta + \gamma = \pi$, $\alpha + \beta = \pi - \gamma$.

Since opposite angles of a cyclic quadrilateral are supplementary (theorem *X*),
angle *QSP*
$= 2\pi - (\pi - \alpha) - (\pi - \beta)$
$= \alpha + \beta = \pi - \gamma$.

By the converse of theorem *X*, any point at which *QP* subtends an angle of $\pi - \gamma$ lies on the red circle.

But *S* is such a point.
Therefore all three circles cut in a single point. This is Miquel's three circles theorem. We now proceed to the one we're really interested in, Miquel's *six* circles theorem.




The figure on the left shows four red circles, *p*, *q*, *r*, *s*. (We shall see why two are dotted in a moment.) Each cuts its two neighbours. The four outer points in which this happens, *A*, *B*, *C*, *D*, are concyclic, (they lie on the black circle, *m*). The theorem claims that the four inner points, *W*, *X*, *Y*, *Z* are concyclic too.

(We shall prove this particular result only. The theorem is in fact more powerful: we can take any five of the six sets of four points and show the other four are concyclic.) The following proof is due to ‘MvG’, on the internet news group Math Stackexchange, 17.11.17.

We centre on *A* a circle of inversion, (the blue circle, *i*.) The circles *m, p, s* all pass through *A* and therefore map to the straight lines *m', p', s'* shown on the right. Accordingly, under this inversion, *B, D, Z,* which are intersection points of the circles concerned, map to the corresponding intersection points on the straight lines, *B', D', Z'*. *C* maps to a point *C'* on *m'* between *B'* and *D'*. And we can locate *W'* and *Y'* in a similar way at the positions shown. Since circles map to circles, we know that the circle through *B',C',Y'* (*q'*) cuts the circle through *D',C',W'* (*r'*) in *X'*. But, by the three circles theorem, *X'* must lie on the circle through *W',Y',Z'*. Inverting the transformation to restore the original figure, *W,X,Y,Z* must therefore be concyclic, as required.

A corollary:

Another transformation which takes circles to circles is stereographic projection, used by crystallographers. Here we project from the north pole of a sphere onto a plane tangent to the south pole. Say we have four small circles drawn on the sphere, each of which cuts a neighbour in two points, and the upper of each of those four points lies on a circle. Projecting onto the plane we have our 6-circle figure. Inverting the transformation, we know that the lower of each of those four points also lies on a circle.