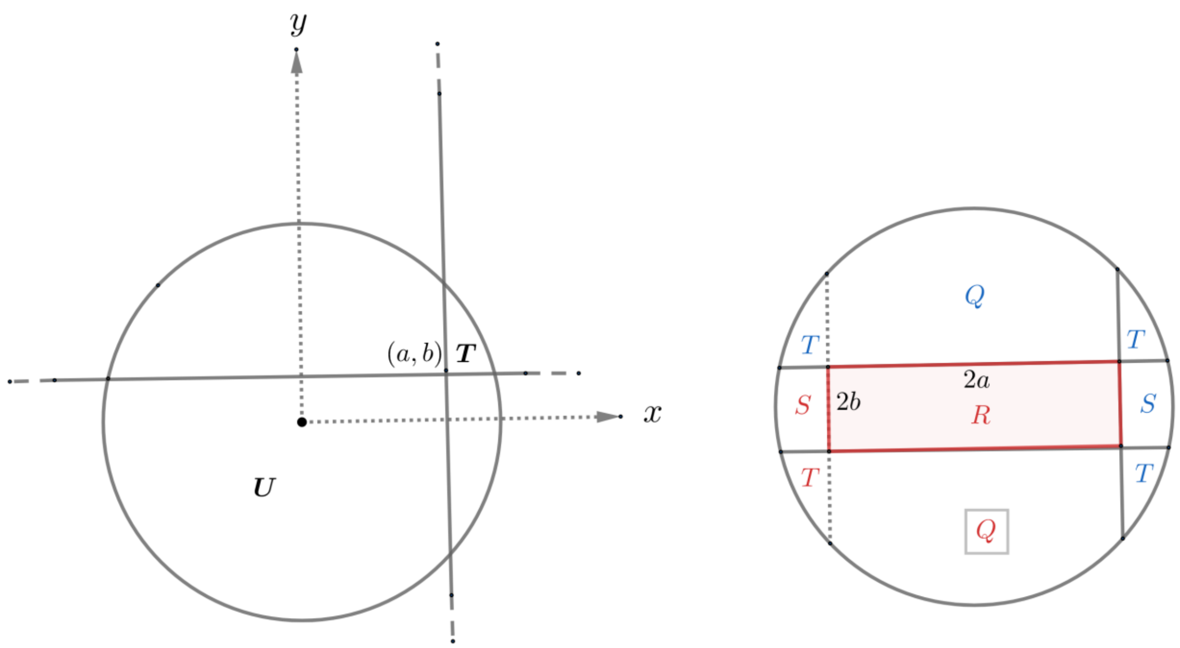
1.1 Circle symmetry problems

The following thirteen problems depend on particular aspects of circle symmetry.

Problem 1.1.1

The symmetry of the circle is nowhere more apparent than in the following little problem.

A ring is thrown on a floor of square tiles in such a way that the part covering one tile (U) is equal in area to half the ring. *How does the area t of the vertically opposite region (T), depend on the magnitudes of the coordinates of the tile junction, (a, b)?*

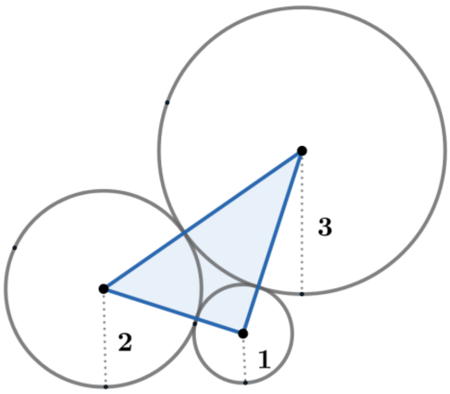


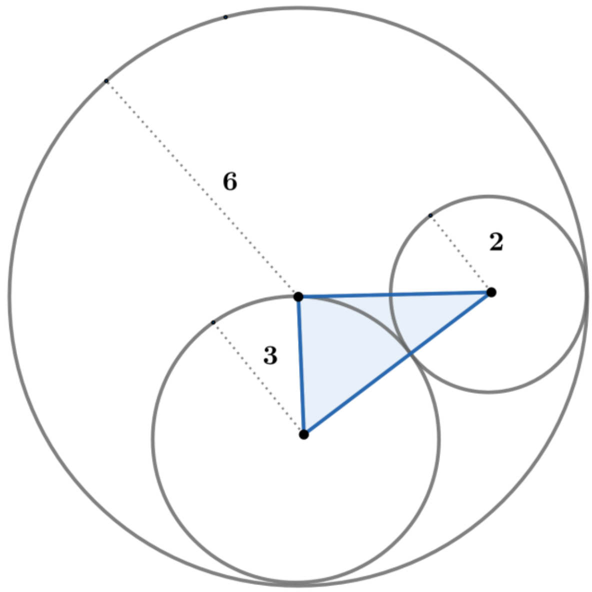
The mirror symmetry of the circle allows us to delimit the congruent regions on the right. Let the region Q have the area q, etc. We equate the red areas within U to the blue areas outside:

$q + r + s + t = q + s + 3t$,
$r = 4ab = 2t$,
$t = 2ab$.

Problem 1.1.2

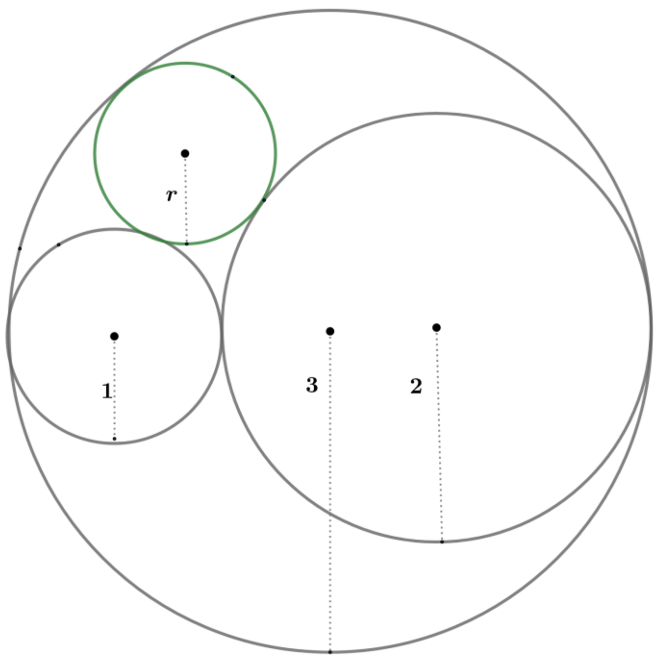
Recalling that the point of contact of two circles lies on their line of centres, confirm that both these triangles contain a right angle.

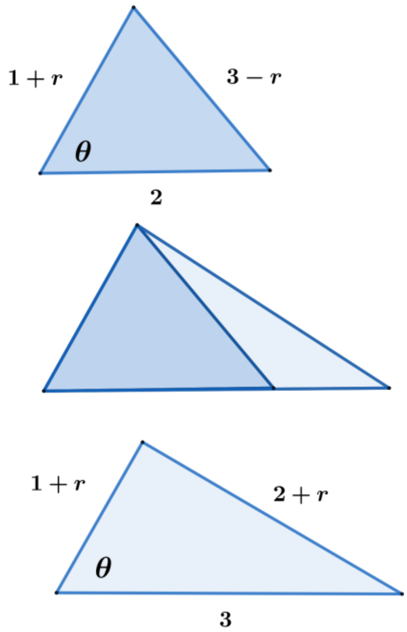




Problem 1.1.3

In the next problem we must find the radius of the green circle. *Confirm that we can pick out of the figure the triangle middle right.* We separate this into the upper and the lower one. *Apply the cosine rule in each. Eliminate* $\cos\theta$ *from the two equations you obtain to leave* r. (You should get 6/7.)



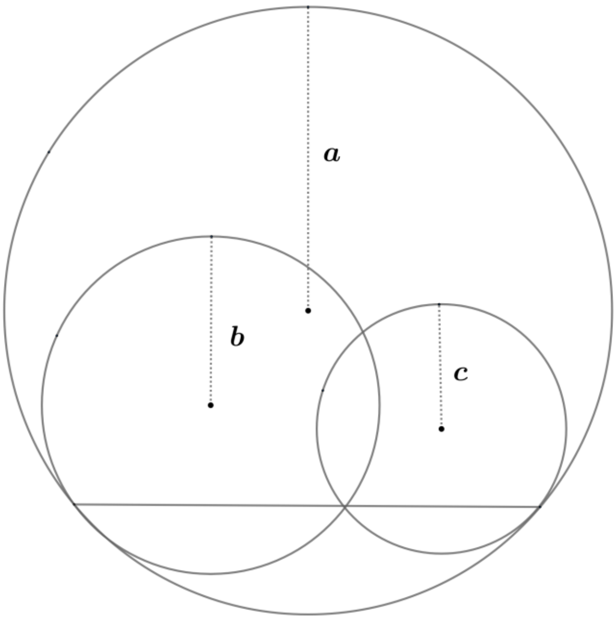


You may also use a formula due to Descartes. *Go to the Wikipedia entry 'Descartes' circle theorem'.*

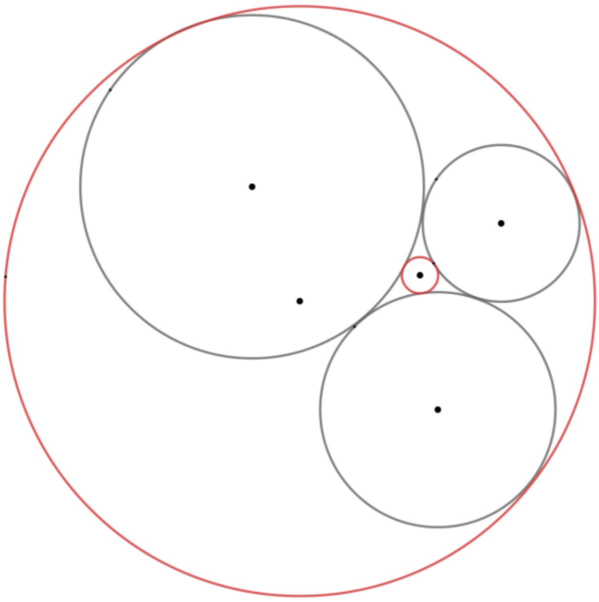
A symmetrical half of the black figure has been studied from ancient times and yields a lot of interesting geometry. *Go to the Wikipedia entry 'Arbelos'.*

Problem 1.1.4

In the this next example, extract three isosceles triangles, two contained in the third, to show that $a = b + c$.



As another consequence of the symmetry, any three circles, each touching the other two, touch a fourth externally and internally:



Joining the centres of the three black circles, we have a triangle. This reminds us that the vertices of any triangle can be the centres of circles touching in pairs. To express this fact algebraically, if the triangle side lengths are a, b, c and the circle radii e, f, g , we have:

$$\begin{aligned} a &= e + f \\ b &= \quad\;\; f + g \\ c &= e \quad\;\; + g \end{aligned}$$

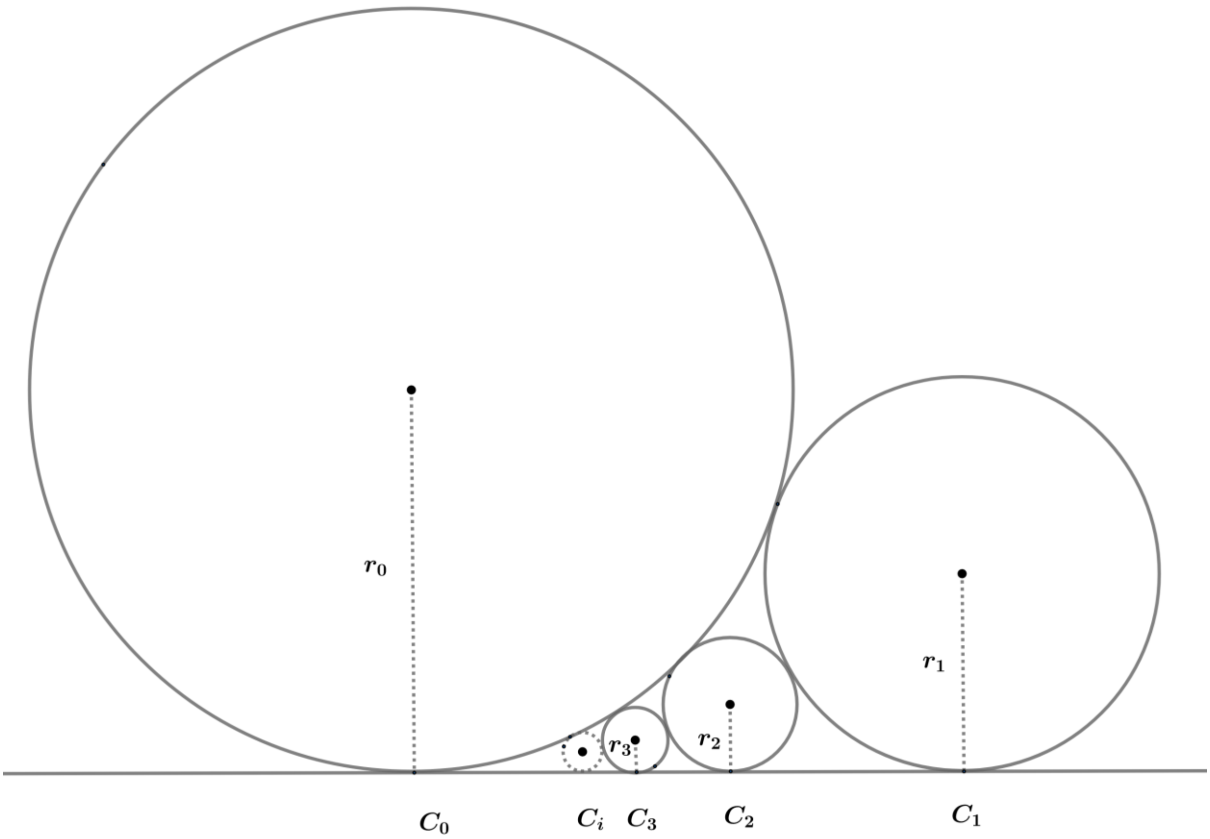
And, solving these equations:

$$e = \frac{a-b+c}{2}$$
$$f = \frac{a+b-c}{2}$$
$$g = \frac{-a+b+c}{2}$$

Problem 1.1.5

Here is another problem exploiting the same symmetry.

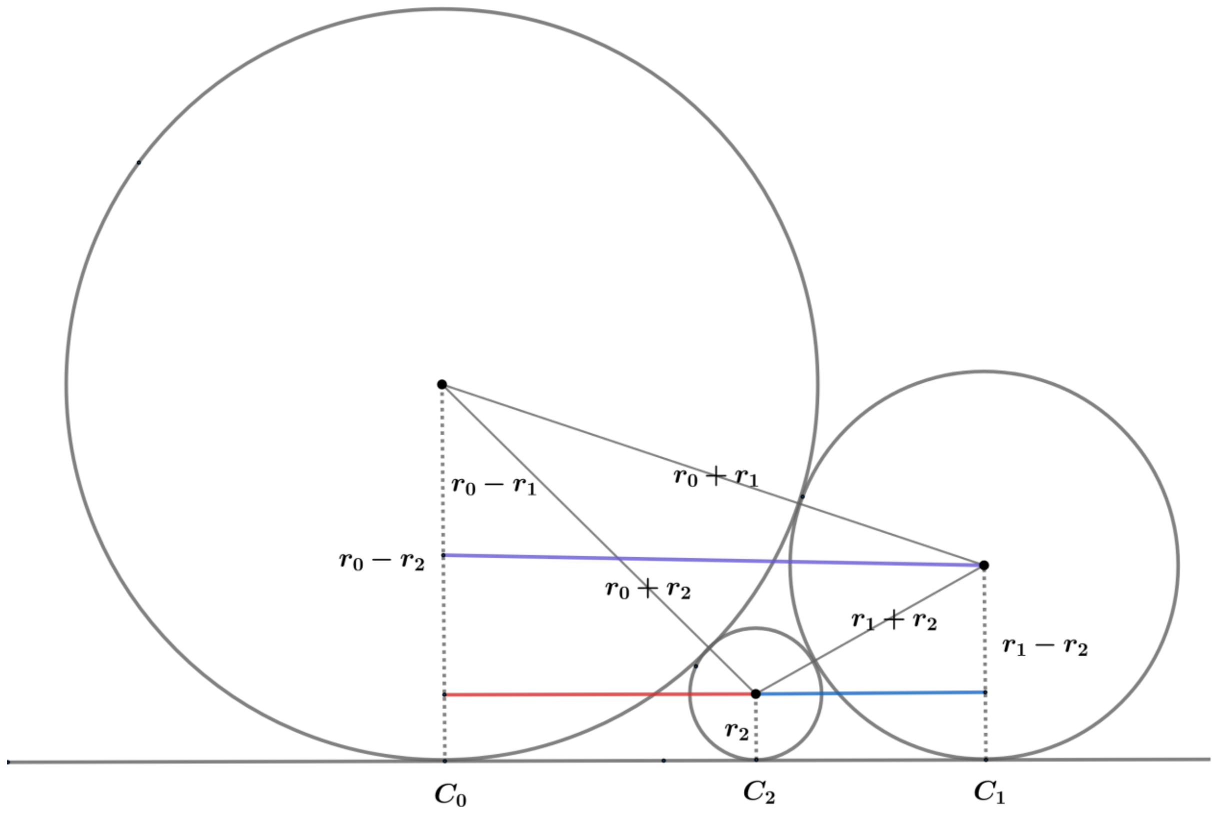
The figure shows an infinite series of circles, $C_1, C_2, C_3, \ldots, C_i, \ldots$. C_i touches C_0, C_{i-1} and the line. $r_0 = p^2, r_1 = q^2, p > q$. *Express r_i in terms of r_{i-1} and r_{i-2}. Also find r_2, r_3, r_4 in terms of p and q and comment on the general sequence of denominators.*



We have the right triangles picked out below and can use Pythagoras' theorem to find lengths. We then equate the lilac length to the sum of the blue and the red.

$\sqrt{r_0 r_1} = \sqrt{r_0 r_2} + \sqrt{r_1 r_2}$, or $\frac{1}{\sqrt{r_2}} = \frac{1}{\sqrt{r_1}} + \frac{1}{\sqrt{r_0}}$.
This generalises to: $\frac{1}{\sqrt{r_i}} = \frac{1}{\sqrt{r_{i-1}}} + \frac{1}{\sqrt{r_{i-2}}}$, or $r_i = \frac{r_{i-1} r_{i-2}}{\left(\sqrt{r_{i-1}} + \sqrt{r_{i-2}}\right)^2}$.

We find $r_2 = \left(\frac{pq}{p+q}\right)^2, r_3 = \left(\frac{pq}{2p+q}\right)^2, r_4 = \left(\frac{pq}{3p+2q}\right)^2$. The denominators are squares of consecutive numbers in sequences of Fibonacci type, i.e. those where each term is the sum of the preceding two.



Go to the Wikipedia entry 'Ford circles' for more properties of such circle sequences.

Problem 1.1.6

This example illustrates most aspects of the circle's symmetry.

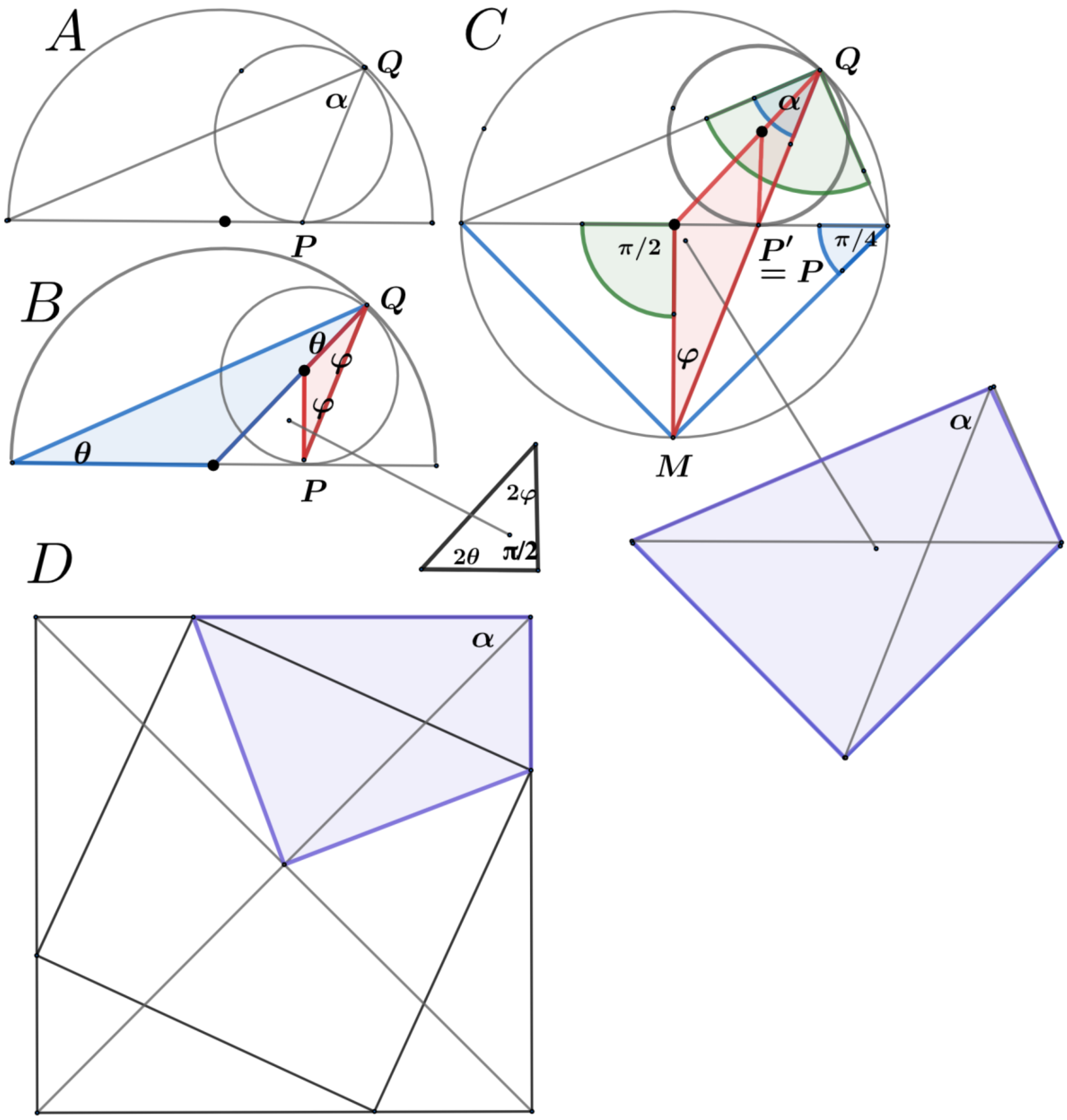
A Prove that $\alpha = \frac{\pi}{4}$.

B Adding the essential lines to the figure gives us the two important isosceles triangles and the perpendicular from the centre of the small circle. Chasing angles into the small right triangle, we have $2(\theta + \varphi) = \pi - \frac{\pi}{2}, \theta + \varphi = \frac{\pi}{4} = \alpha$.

C Completing the big circle, we can construct the large red isosceles triangle, similar to the small one in ***B***. We identify *P'*, the point where *MQ* cuts the horizontal diameter, with *P*. The blue lines mark sides of big and small right isosceles triangles. We can use either the property that the angle at the centre is twice the angle at the circumference, or the 'same segment' theorem, to determine α.

Marking the right angle at the top draws our attention to the fact that *PQ* bisects this angle.

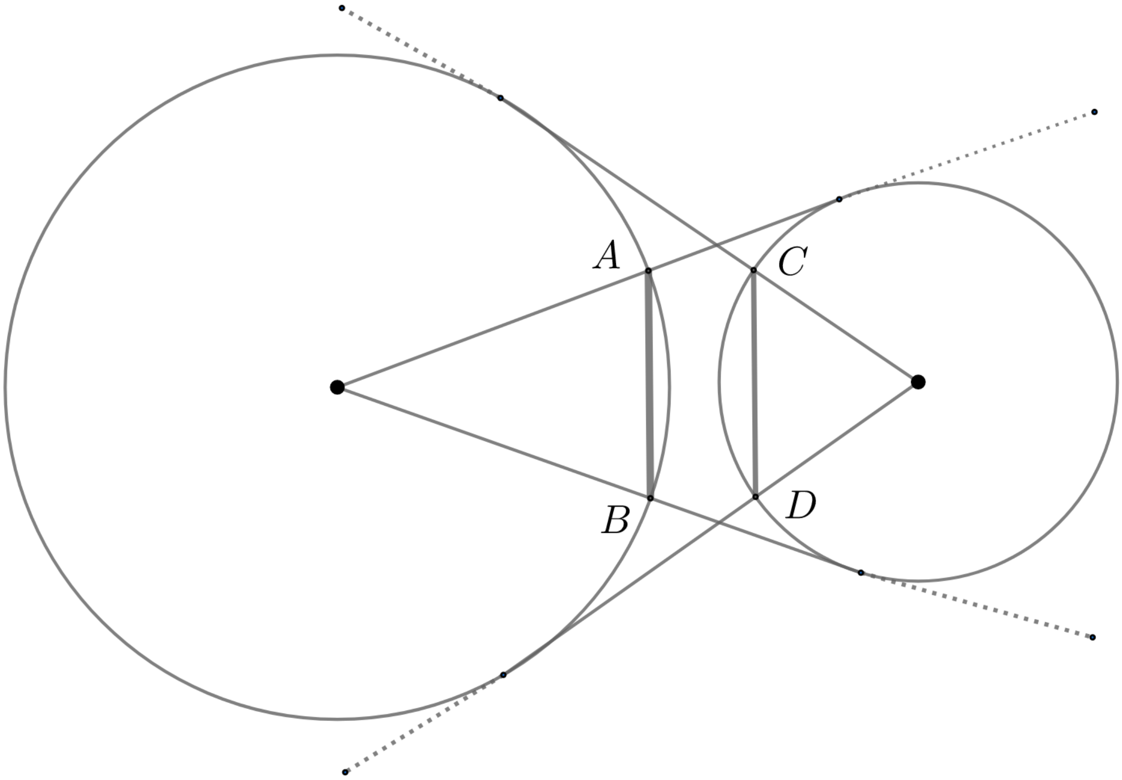
D Incorporating the lilac quadrilateral into a classic 'Pythagoras' figure, this bisector becomes the diagonal of a square.



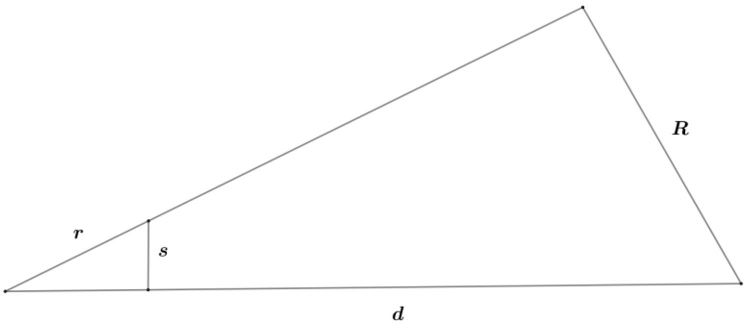
Problem 1.1.7

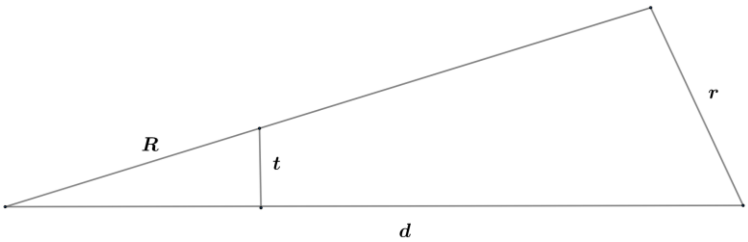
In this problem we can take advantage of the symmetric relation between the two circles.

Show that the line segments AB and CD are of equal length (the ‘eyeball’ theorem).



The line of centres is a symmetry axis for the whole figure so we only need consider half. We can make the symmetric relation between the two circles clear by giving them the same orientation:





d is the distance between the circle centres; *R, r,* are the circle radii, *s,* t are half the lengths to be compared.

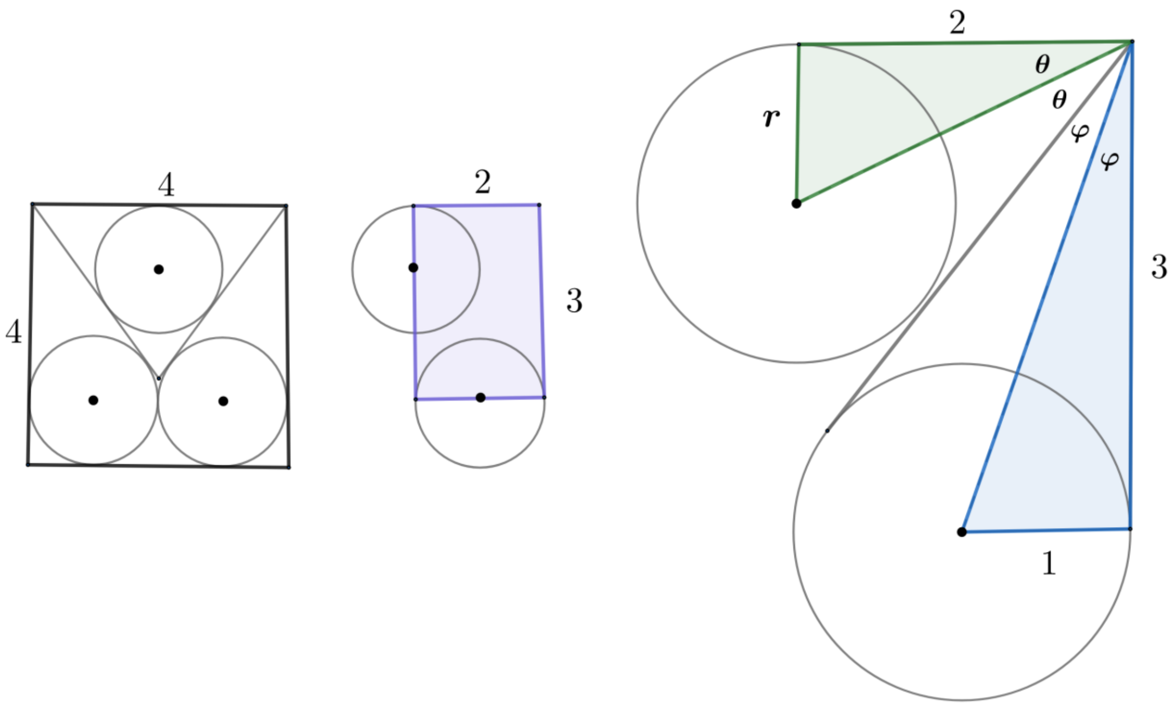
Because radii meet tangents at right angles, we have two right triangles with equal hypotenuses.

Within each, the smaller and the larger are similar.

Write down equal ratios of corresponding sides and thus prove the claim.

Problem 1.1.8

The figure shows two unit circles in a square. Tangents to these extend from the upper corners. The tangents and the upper edge circumscribe a third circle. What is its radius?



The diagram for this classic problem contains two equal pairs of tangents from a point and therefore provides right triangles in which to use trigonometry. The symmetry of the figure means we need only use the part shaded.

Notice in the enlargement:
1) The blue triangle gives us φ,
2) $\theta = \left(\frac{\pi}{4} - \varphi\right)$.

Solve the green triangle for r.

Generalise your result for an initial rectangle of the same width and height h.

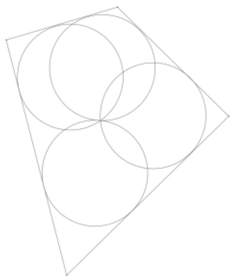
Problem 1.1.9

For this see **4.3.1**: Turning a *C* into a *CO*.

Problem 1.1.10

A problem of Ross Honsberger:

We have 4 like circles passing through the same point. *Show that the quadrilateral circumscribing them is cyclic.*



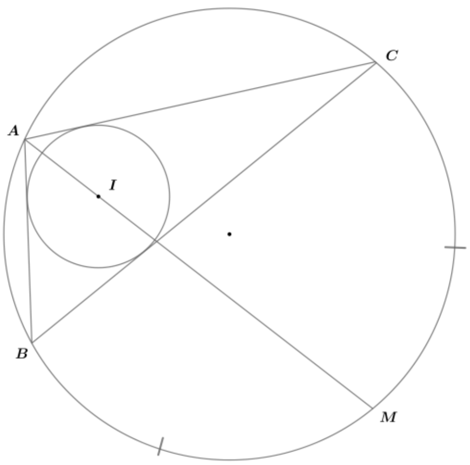
You will immediately want to insert the circle centres. Being equidistant from the intersection point, they are concyclic. And therefore the quadrilateral joining them is likewise cyclic.

When you join the centres in pairs, what can you say about the sides of this quadrilateral and the sides of the original one?

Proceed from there.

Problem 1.1.11

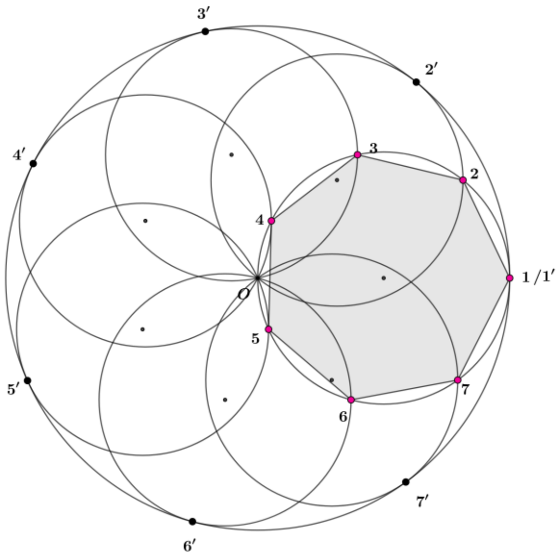
This requires you to bring toge
lengths.



Show that A, I, M are collinear.

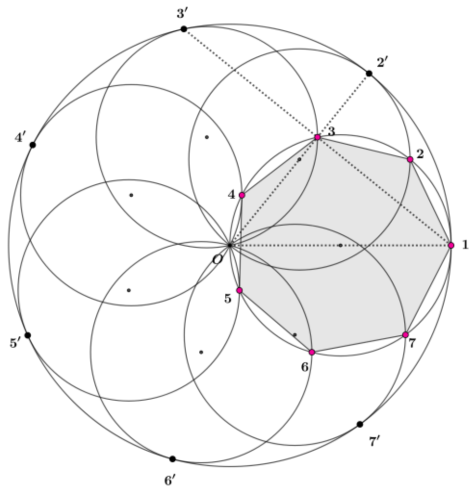
Problem 1.1.12

1’2’3’4’5’6’7’ is a regular 7-gon.
We label the circles with the points where they touch a circumscribing circle.
Vertex *2* is the intersection of circles *1’, 2’*,
Vertex *3* is the intersection of circles *1’, 3’,*
Vertex *4* is the intersection of circles *1’, 4’,*
etc.
Show that *1234567* is a regular 7-gon.

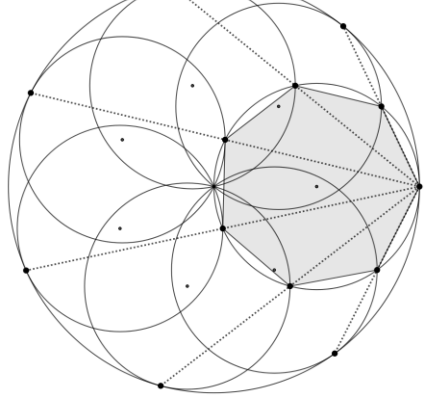


By symmetry *1* is the reflection of *2’* in the line *02*.
By Thales’ theorem angle *021* is a right angle.
Therefore *122’* is a straight line with *2* the midpoint.

We can argue similarly for each vertex of *1234567*:



We thus end up with the last figure,
from which we see that *1234567*
is an enlargement of *1’2’3’4’5’6’7’*,
scale factor ½, centre *1/1’,* and therefore also
a regular 7-gon.

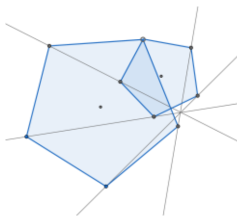


Though we have chosen the value
n = 7, the argument can be
extended to any regular *n*-gon.

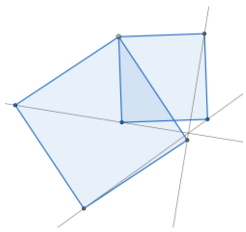
Problem 1.1.13

This problem, 8.13 from Arseniy Akopyan's collection 'Geometry in figures', is special in that it is not clear whether we are being presented with a special or the general case. **1.** is the figure. Two regular pentagons share a vertex. We must show that the apparent coincidence of the four lines joining corresponding vertices is real. **2.** We ask ourselves if the regular pentagon can be replaced by another regular polygon. We try one and find it can. **3.** With dynamic geometry software we can move the point of coincidence. It seems to follow the circumcircle of the fixed pentagon and indeed to fall on the the second point of intersection of the circumcircles of the two pentagons. **4.** We now guess that the problem is not about regular polygons at all, but about points in corresponding positions on two circles.

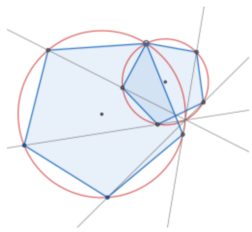
1.



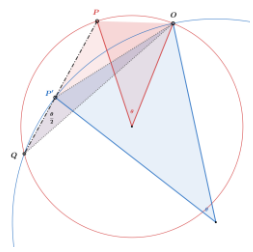
2.



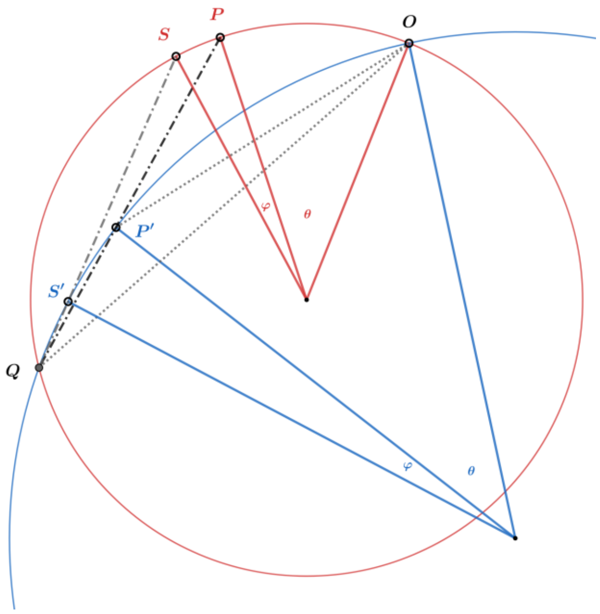
3.



4.



The two circles cut at *O* and again at *Q*. We take as reference line the radius joining *O* to the respective circle centres and consider point *P* on the red circle such that *OP* subtends an angle θ at the centre of the red circle, and point *P'* on the blue circle such that *OP'* subtends the same angle at the centre of the blue circle. Since the angle at the centre of a circle is twice that at the circumference, angles *PQO* and *P'QO* are both $\frac{\theta}{2}$. This requires that *P'* lies on *PQ*, i.e. *PP'Q* is a straight line. Taking two other points with equal corresponding angles, *S, S'*, we therefore know that *SS'Q* is also a straight line, and similarly for all such point pairs.


S
P
O
P′
S′
Q
φ
θ
φ
θ