The Young Geometer’s Casebook

Training examples in elementary geometry for students entering maths contests

Contents

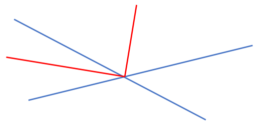
The focus is the circle and the way its symmetries permeate classical geometry.

A.1 What you need to know

(Most theorems without a name are to be found in Euclid's 'Elements'.)

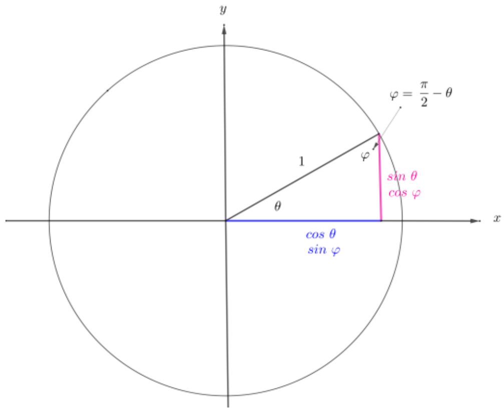
Angles:

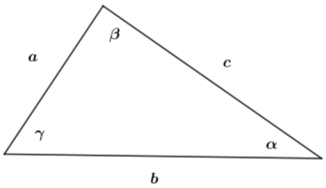
- Interior and exterior bisectors perpendicular



Trig. ratios, theorems and identities

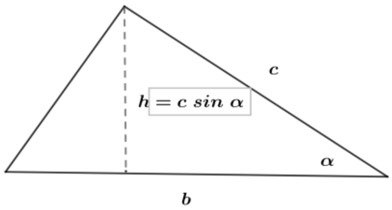
- The signs of the ratios in the four quadrants
- $(\sin\theta)^2 + (\cos\theta)^2 = 1$
- $\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$





- The sine rule: $\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$
- The cosine rule: $a^2 = b^2 + c^2 - 2bc\cos\alpha$

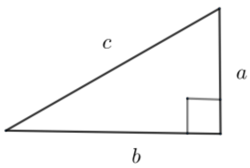
Triangles:

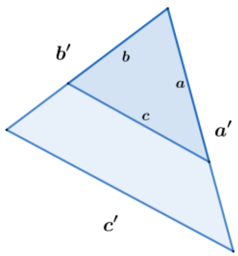


- Area of a triangle = $\frac{1}{2}bh = \frac{1}{2}bc\sin\alpha$

- Pythagoras' theorem and its converse
  $a^2 + b^2 = c^2$
- Ratios in similar triangles

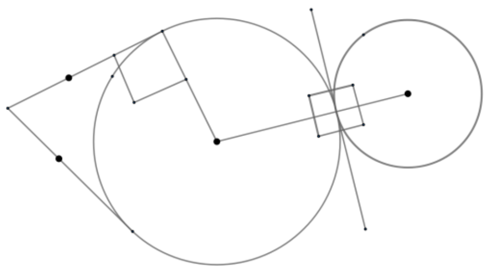
  $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$

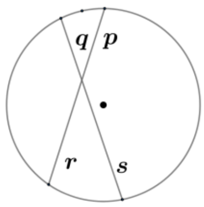




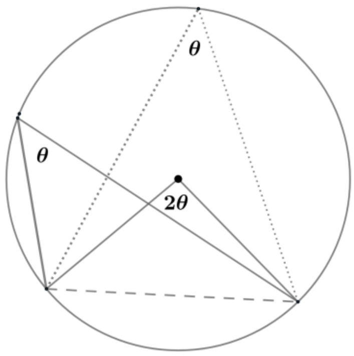
- Interior angle sum = π

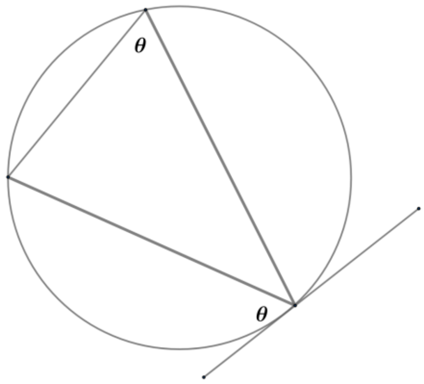
Circles: introduced in the text but summarised here:



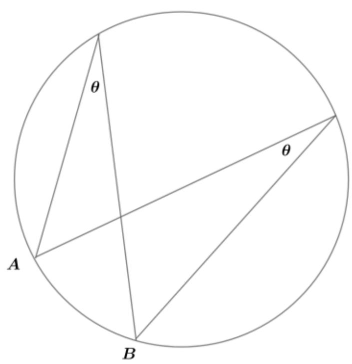


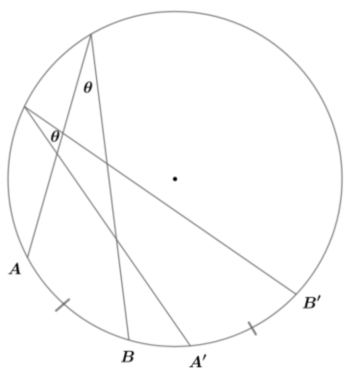
- Tangents from a point equal
- Tangent meets radius at right angles
- Point of contact of two lies in line of centres
- Common chord perpendicular to line of centres
- Intersecting chord theorem (converse also true): $pr = qs$





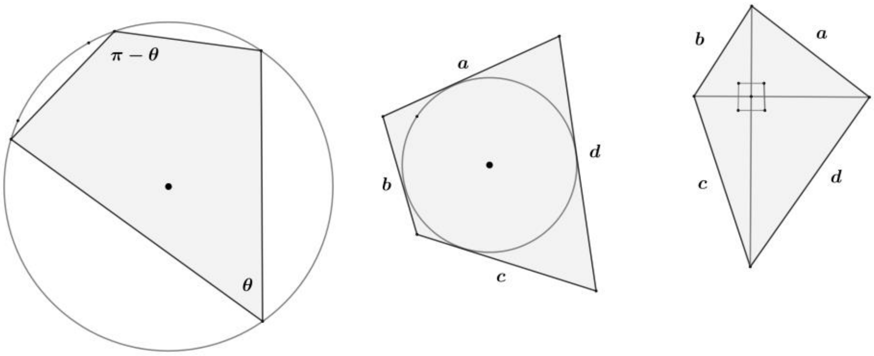
- Angles subtended by same chord/arc at all points on circumference in same segment equal (same segment theorem)
- Angle subtended at centre twice that subtended at circumference by same chord: special case: angle in a semicircle a right angle (Thales' theorem, converse also true)
- Alternate segment theorem
- Three non-collinear points define a circle





- Rotating 'same segment' figure about centre, we have *equal* segment theorem: angles subtended at points on circumference by equal chords/arcs equal

Convex quadrilaterals: introduced in the text but summarised here:



- Interior angle sum = 2π
- In cyclic quadrilateral: opposite angle pairs sum to same (converse also true)
- In tangential quadrilateral: opposite side pairs sum to same (Pitot's theorem, converse also true): $a + c = b + d$
- In orthodiagonal quadrilaterals: opposite side pair squares sum to same (converse also true): $a^2 + c^2 = b^2 + d^2$

Convex polygons in general

- Circumcentres of cyclic polygons found by bisecting sides perpendicularly
- Incentres of tangential polygons found by bisecting angles

Logic: introduced in the text but summarised here

- Necessary conditions
- Sufficient conditions
- 'If and only if ...'
- A theorem and its converse

The statement on the left is given symbolically on the right.

Logical implication:

A being true is a necessary condition for *B* being true. $B \Longrightarrow A$.
A is true if *B* is true. $B \Longrightarrow A$.
If *A* is false, *B* is false. $B \Longrightarrow A$.

A being true is a sufficient condition for *B* being true. $A \Longrightarrow B$.
B is true if *A* is true. $A \Longrightarrow B$
If *B* is false, *A* is false. $A \Longrightarrow B$.

Logical equivalence:

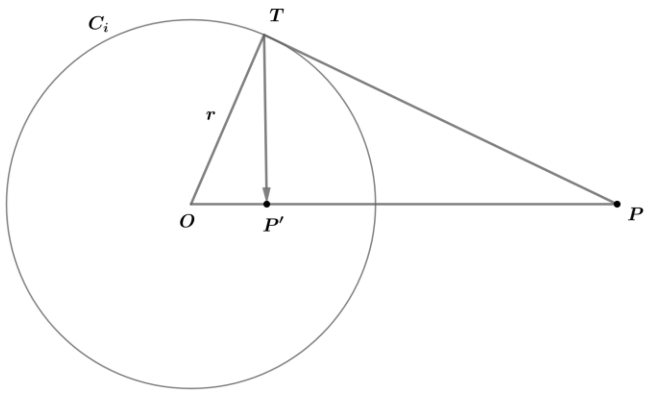
A being true is a necessary and sufficient condition for *B* being true.
$A \Longleftrightarrow B$.
B is true if and only if *A* is true. $A \Longleftrightarrow B$.

A theorem and its converse:

If a theorem states that $A \Longrightarrow B$, its converse states that $B \Longrightarrow A$.
A converse may or may not be true. If it is, we can again write $A \Leftrightarrow B$.

A 'characterisation' of a shape is definitive of it. So, again, if B is a characterisation of shape A, $A \Leftrightarrow B$.

We shall go outside Euclidean geometry only to use certain useful transformations. These comprise translations, reflections and rotations, and one other type: circle inversion.



The construction shows how we map one point, P, to another, P' by the 'circle of inversion', C_i.

The result is that $|OP| \times |OP'| = r^2$.

The transformation has useful properties. In particular, a circle goes to a circle, except one through the centre, O, which goes to a straight line through the intersection points of the circle in question and C_i.

A.2 Introduction

This book is aimed at those preparing for the geometry problems encountered in national or international contests for high school students. Because contestants may come from any educational system and have studied any curriculum, the contest must assume no specialised knowledge. The purpose of a problem is therefore to tax the student's ingenuity in using core high school geometry. The emphasis in a book like this must therefore be heuristics rather than algorithmics, strategy rather than tactics. The competitive nature of a contest should not spoil the student's delight in the mathematics. We enjoy watching a stage magician because we enjoy being fooled. An elementary knowledge of brain science only adds to our enjoyment as we witness how little control our conscious, rational minds exert over our unconscious, instinctive mental processes. For this reason, even when we read how the trick is done, our amusement is not lessened and may even be increased. In his Internet site Cut-the-knot, the late Alexander Bogomolny had animated geometric figures he called 'droodles'. The points were little eyes and when the figure hit an interesting configuration, the little eyes would look straight out at you in surprise. In mathematics a result is explained by a proof. But, as in stage magic, if the trick is a good one, our surprise should be replaced by the pleasure of watching the mathematical machinery work. I've picked results, most well known, which I hope are surprising on a first encounter, but to which elementary methods give immediate access.