Soap Films

The workshop is designed for Y8 students but can be tailored to older or younger groups. (The material presented in these notes spans the age range Y5 to Y13.) The session falls in two parts of an hour-and-a-quarter. The idea is to perform physical experiments (**E**) but to do small pieces of maths (**M**) in parallel to convince ourselves that what the experiments tell us makes sense. Sufficient apparatus is provided for 30 children, working in pairs. The **M** units to include depend on the children involved. None is essential. What *is* necessary is to ensure that the children grasp on an appropriate level the meaning of the following mathematical terms: *length, area, angle, simple proportion,* and the following terms from physics: *potential energy, work, force*, because the experiments concern these quantities. For younger children the workshop can be used as an opportunity to explain them. The piece of physics **P**, presented without experimental justification, relates the 3 physical quantities and enables us to explain the observation that 3 ribbons (and walls in general) meet at 120°. We can omit this and just state as a fact that the force (per unit width) is the same for each of the soap ribbons. In the same way we can omit all experiments which are there only to justify a mathematical result.

Abbreviations: KB – 'Kubik Bubbles' kit for making skeleton polygons in **Part 1** and polyhedra in **Part 2**
MP – 'motorway plates' kit for **Part 1** (perspex sheets which can be separated by means of movable spacers)
OHP – overhead projector for display of 'motorway' figures. Be sure to put thin spacers (counters or coins) under the perspex sandwich. Otherwise the wet model will stick to the platen. Projection on to whiteboard.

Part 1: structured work in 2 dimensions, **Part 2**: structured and free work in 3 dimensions. **Part 3**: extension work. This may be developed in a school maths club. There is overlap here with the masterclass 'Surfaces', of which 'Minimal surfaces' forms a section.

Part 1

We want to use soap films to solve a problem:
What is the smallest total length of motorway needed to join n *cities?*
We use a ribbon of soap film of constant width to model a motorway.

We prepare the children for the 'motorway' experiments by showing them MP and describing how we are going to use the apparatus.

We claim that the ribbon will try to attain the shortest possible length.
What is the evidence for that?

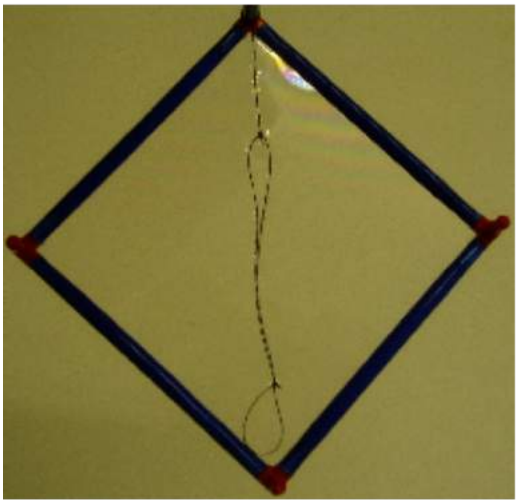
E1:

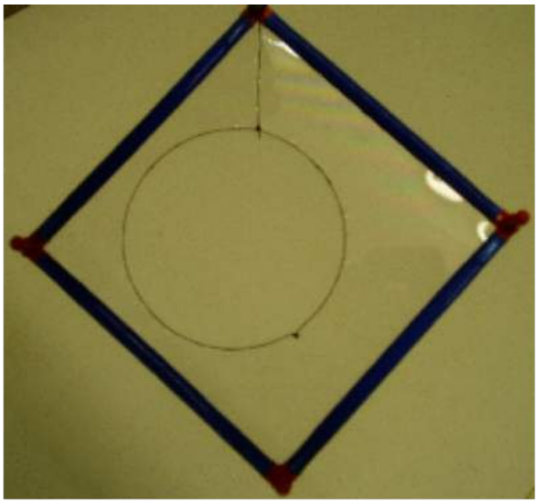
Apparatus: Square frame with handle, made from KB; cotton loop tethered to corner in which handle inserted; single KB straw.

Test: Dip frame in soap solution and withdraw slowly so that film forms across frame.

Use single straw to pop film within loop.

Observation: Loop adopts circular shape.





Inference: The area within the square is least when the area within the loop is greatest. This is the case when the loop adopts a circular form. The soap film has therefore adopted the smallest possible area for its boundary.

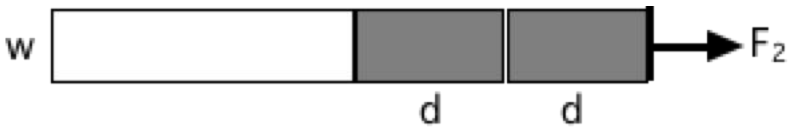
Our ribbon has a constant width so its length is proportional to its area.

P: When you stretch a rubber band, the force you need increases as the band gets longer. This is not so with a band of soap.

Here is a band of soap of width w cm.
In the first diagram we pull with a force F_1 through a distance d cm.

We have added an area dw cm ^{2}. Each cm ^{2} added increases the potential energy of the band. Call this increase E. We have achieved this by doing work: moving the force F_1 through the distance d. Work done = F_1d = potential energy gained = E.

In the second diagram we pull with a force F_2 through a further distance d.
The increase in energy is just the same as before so we can write F_2d .



But $F_1 d$ also equals E, so $F_1 = F_2$. In other words the force stays the same.

E2:

We use lengths of string to model the soap ribbons, in which, we now know, the tension forces are equal.

This experiment involves just the one group of children who volunteer for it.

Apparatus: A ring. Into this, 3 strings are hooked. The strings run over pulleys. Their axles have a handle at each end. The strings support 100 g masses so that the tension force in each is equal. An improvised protractor, consisting of a folded sheet of stiff card.

Test: A child holds each pulley. You hold the ring still. The children stand in a circle. When you the ring, tell the children to adjust their positions till the ring stops moving. When this happens, check the angles between the strings with the protractor.

Observation: The angles are equal (and therefore each $= \frac{360°}{3} = 120°$.).

Inference: Symmetry alone tells us that this is what we would have predicted. But, since soap ribbons adjust their lengths to give the minimum total length, we now also expect that, where 3 ribbons meet in our ‘motorway’ experiment, they will do so at angles of 120°.

E3: ‘Motorway’ experiment no.1.

Apparatus: MP, spacers defining equilateral triangle, 3 dots placed similarly on central whiteboard.

Prediction: Child volunteer invited to draw figure expected. Rival figures drawn in different colours. Vote taken.

Test: Teacher’s MP dipped in soap solution.

Observation: 120° trigon obtained, displayed on OHP.

E4: ‘Motorway’ experiment no.2.

As above, triangle scalene but having no interior angle as great as 120, and experiment performed by all pairs.

Observation: 120° trigon persists.

M1:

It is hard to show from scratch that the figure is a trigon but, given that it is, we can show that the angles must be equal.

E5: Teacher demonstration.

Apparatus: single sucker on whiteboard. Single string looped round this with dry-wipe pen in end loop.

Draw the result, a circle (Figure 1).

Point out that:
1) The black, blue, red and green lengths are all the same (the *radius* of the circle).
2) The *tangent* makes the same angle – a right angle - either side of the radius.

E6: Teacher demonstration.

Apparatus: Single sucker replaced by two. New string looped around both suckers.

Prediction: Ask what shape will appear this time.

Draw the result, an *ellipse* (Figure 2).

Call the two 'centres' into which the original one has been split *foci* and point out that, again,
1) The black, blue, red and green lengths are all the same.
2) The tangent makes the same angle either side of the radius.

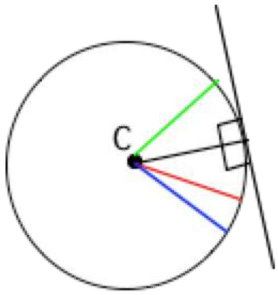


Figure 1

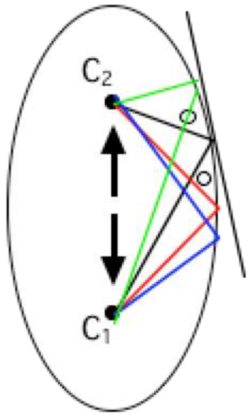


Figure 2

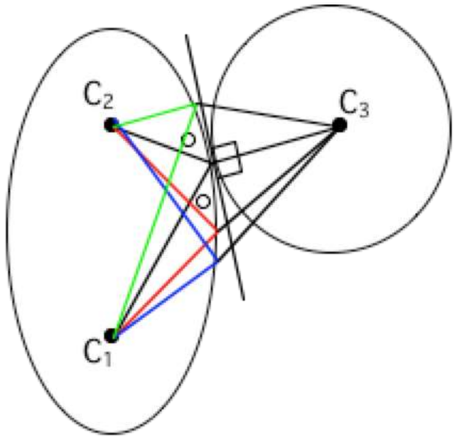


Figure 3

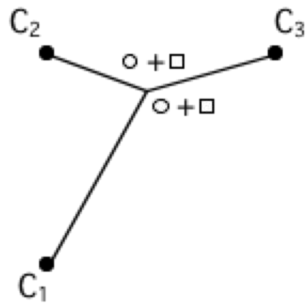


Figure 4

The argument runs as follows.

In Figure 3, , C_1, C_2, C_3 are our cities. Centred on C_3 we've drawn a circle which shares a tangent with the ellipse.
The combined length of the two black or two blue or two red or two green roads is the same. However the road which comes from the junction to C_3 is shortest at the point of tangency. At that point therefore the total length of the three roads is as small as possible.
Notice the pair of equal angles, both equal to 'circle' + 'square' (Figure 4).

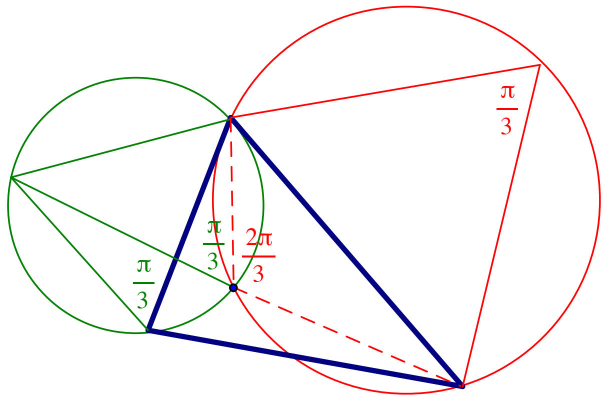
Ask the students to suggest how the argument might continue.

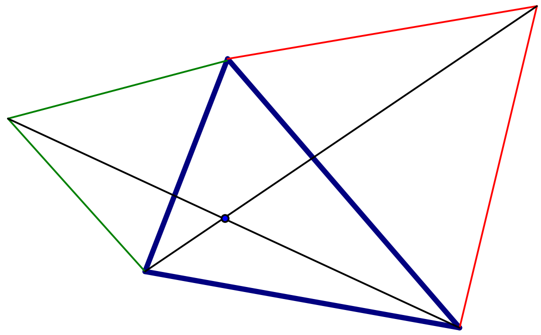
If we'd drawn our ellipse round C_2 and C_3 and our circle roundC_1, or our ellipse round C_3 and C_1 and our circle roundC_2, we'd have found a different pair of angles equal in each case.
Therefore all three must be equal.

M1a

Given a triangle, how do we locate the Fermat point? On the left we have drawn equilateral triangles on two sides of our triangle and also their circumcircles. Because of the cyclic quadrilaterals like that shown in red, we know that the angle enclosed by the dashed red lines is the supplement of $\pi/3$, $2\pi/3$. The same would be true for all 3 such circles, therefore the point of intersection of any two, in particular the red and the green, is our Fermat point.

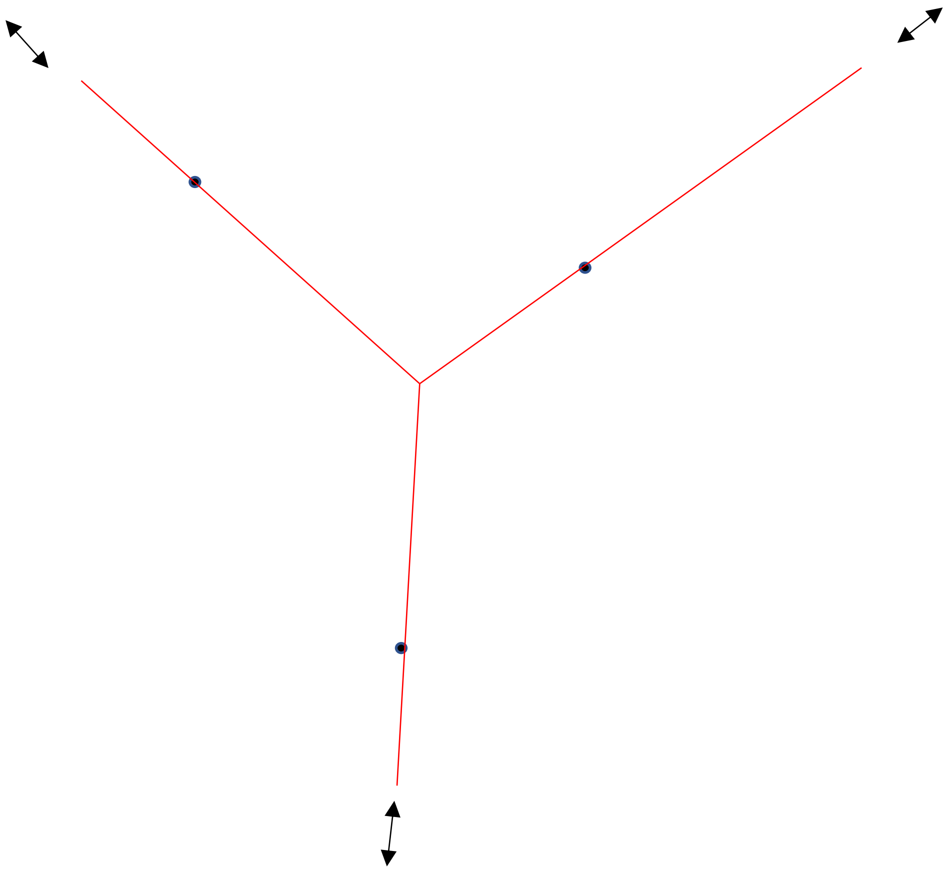
But we also note two angles in the same circle segments of the red and green circles respectively. These total π. So at the vertex marked by a dot the green line is extended by the red dashed line. And we could draw a similar line comprising a solid red segment on the right and a dashed green segment on the left. All we need draw therefore are the two lines shown in the figure on the right.





E6a

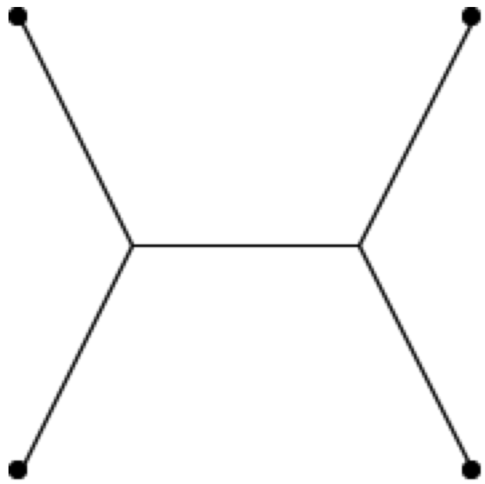
We can use a 120 ° trigon to find the Fermat point.



E7: 'Motorway' experiment no. 3.

A re-run of **E3** but for 4 cities in a square.

Observation:



Repeat **E3** and put result alongside **E7**. Children invited to compare. (We now have *two* 120° trigons.)

E8: Lengths measured on the whiteboard and compared with those which result by simply drawing diagonals to make an 'X'.

E9: 'Motorway' experiment no.4.

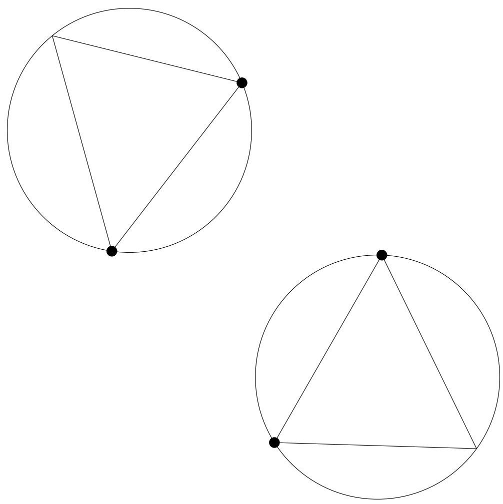
A re-run of **E4**, i.e. irregular quadrilaterals tried.

Challenge: what will happen if an angle of your quadrilateral is 120° or greater?

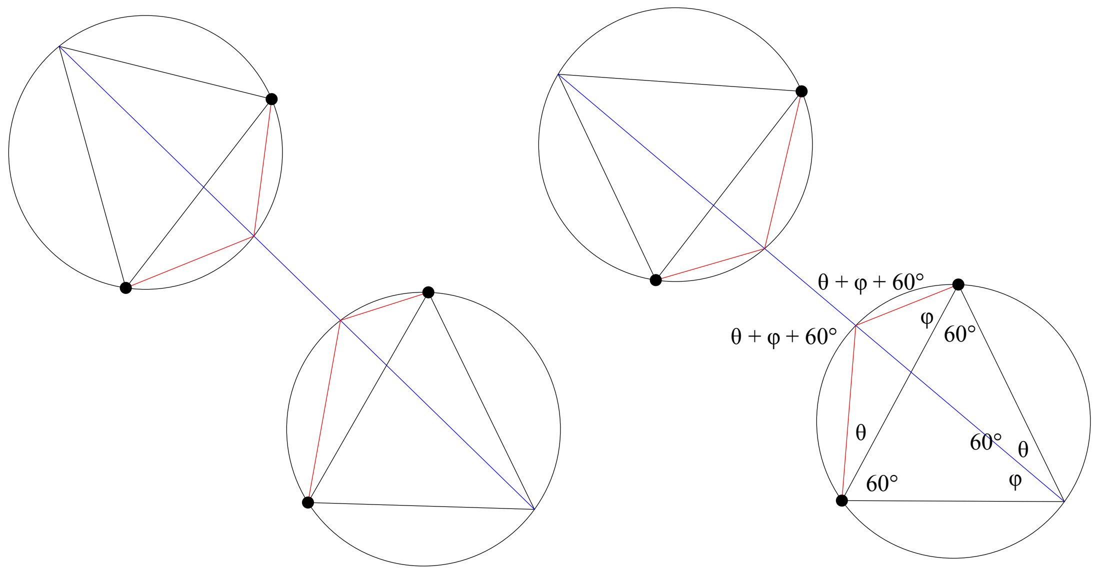
Observations: 120° trigons persist. Where angle is 120° or more, junction 'retreats' to that city.

M2 Demonstrate the following construction.

Draw equilateral triangles on two sides of the quadrilateral and draw circles through them:



Draw the blue line.
The red lines are your final figure.



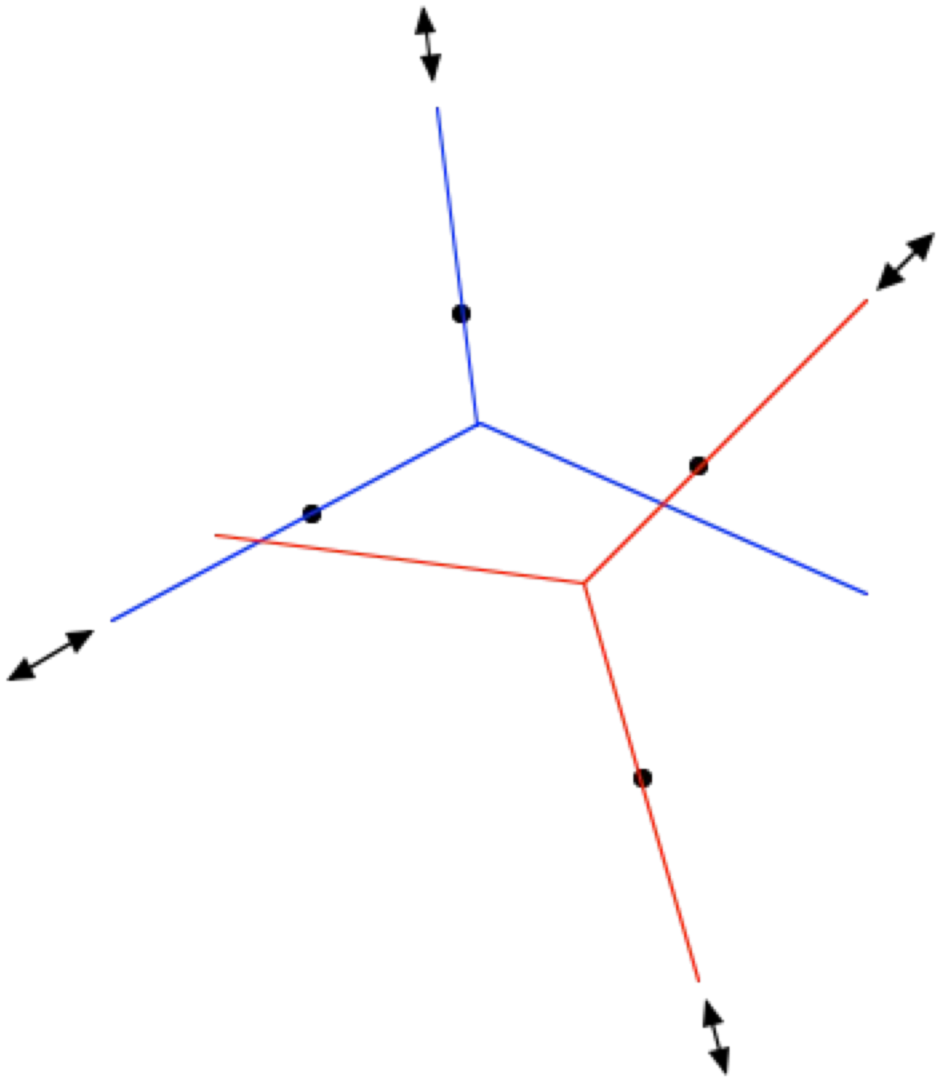
To show why the construction works we've labelled angles in the right-hand diagram.
We've split a $'60°'$ into a θ and a φ.
The 'angles in the same segment' theorem repeats these angles where shown.
We can now use the 'exterior angle' theorem in the two triangles separated by the blue line to work out the sums shown.
But, since we know that $\theta + \varphi = 60°$, we know that this total is $120°$ in each case, and so, by subtraction, must our third angle be.

E10: Teacher demonstration on OHP.

A practical alternative to **M2**.

Apparatus: One acetate with dots for the 4 cities. Two acetates with 120° trigons.

Adjust the two acetates so that two limbs of each trigon always pass through a pair of points and the third limbs are in line:

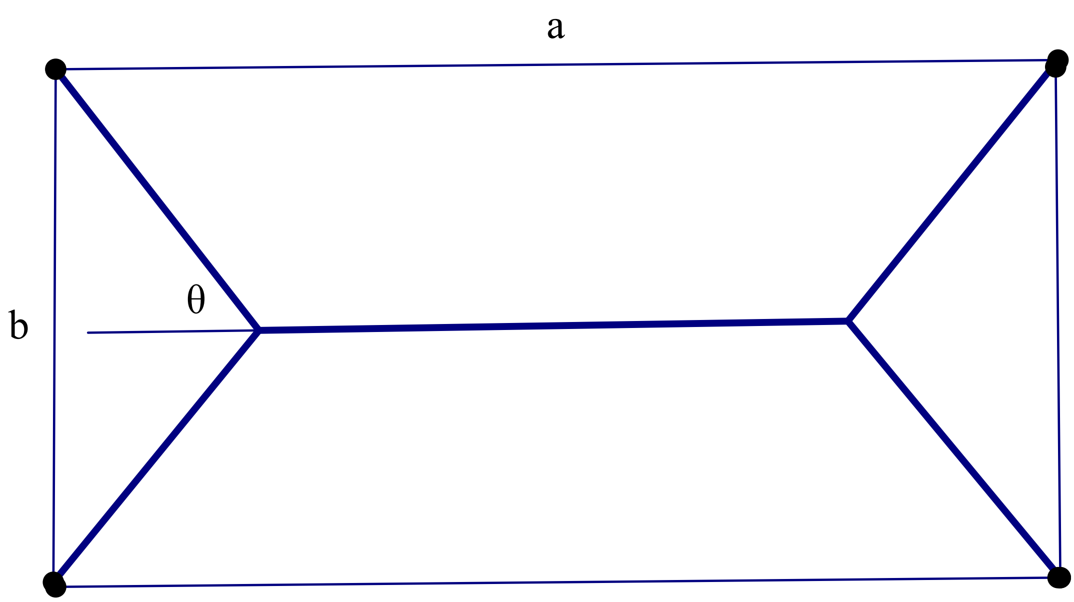


Unfortunately, there are generally two positions in which the correct angles are obtained. But, as long as the total distances are very different, the soap should choose the better solution.

M3 A suitable exercise for older students:

This motorway network, joining cities at the vertices of a rectangle ($a > b$), and with the symmetry of the rectangle, is to be built.

Find the value of θ ($\leq 90°$) if it is to have the smallest total length.



The total length, L, $= 4\frac{b}{2sin\theta} + (a - 2\frac{bcos\theta}{2sin\theta}) = a + b(\frac{2-cos\theta}{sin\theta})$.

We wish to minimise the quantity in the bracket, $l, = \frac{2-cos\theta}{sin\theta}$.

$\frac{dl}{d\theta} = \frac{sin\theta sin\theta - (2-cos\theta)cos\theta}{sin^2\theta} = 0,$

i.e. $\frac{1-2cos\theta}{sin^2\theta} = 0.$

This equation is satisfied by $\theta = 60°$.

We need to differentiate again to confirm that this does indeed give a minimum.

$$\frac{d^2l}{d\theta^2} = \frac{sin^2\theta\,(2sin\theta) - (1-2cos\theta)2sin\theta cos\theta}{sin^4\theta} = \frac{2(sin^2\theta - cos\theta + cos^2\theta)}{sin^3\theta} = \frac{8\sqrt{3}}{9} > 0.$$

Remark: When we have 4 cities at the vertices of a square, we might be surprised that the resulting figure has lower symmetry than the square itself. It is an example of *symmetry-breaking*. Nature has chosen one of the two possible ways to orient the bent 'H' but can easily be persuaded to set it the other way.

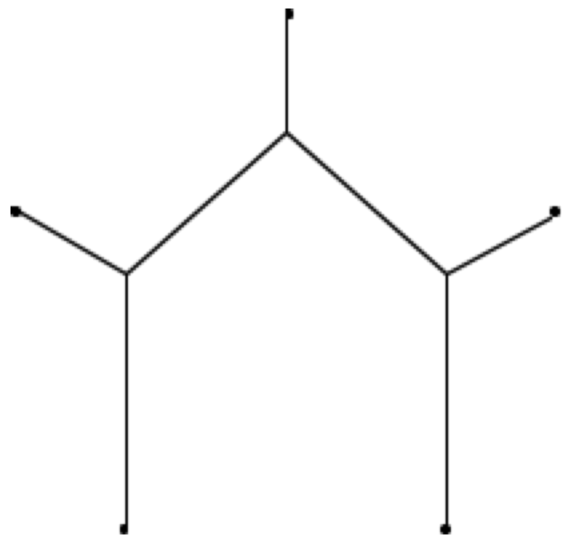
E 11: 'Motorway' experiment no.5.

A re-run of **E3** but for 5 cities forming a regular pentagon.

Challenge:
For the square we had 2 trigons. Will we have 3 this time? If so, how will they be arranged?

Mark the vertices on the whiteboard and invite volunteers to draw the solution.

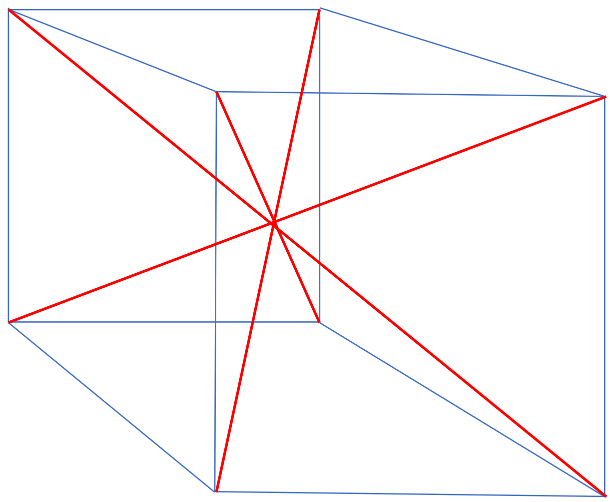
Observation:



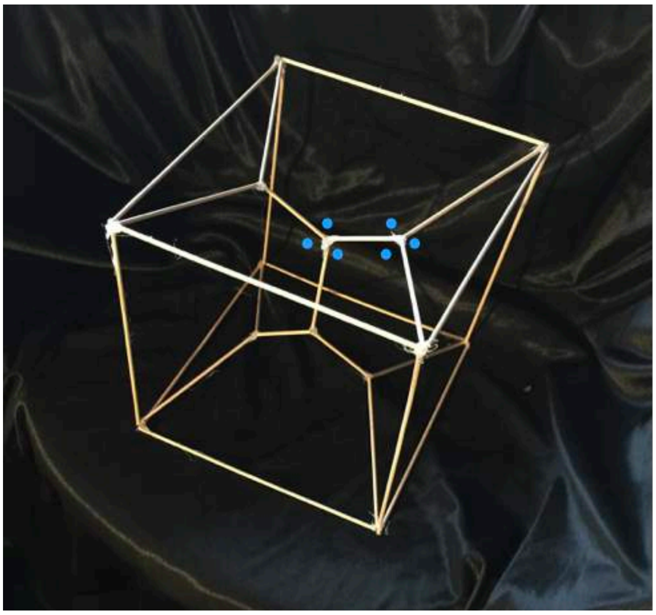
E12: Teacher demonstration

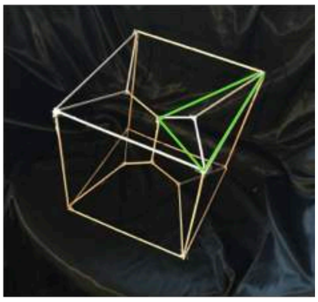
Here is a 3-dimensional problem of the same kind. A spider lives in a cube and needs to reach its 8 corners. What configuration should his web take if he is to use the smallest amount of silk?

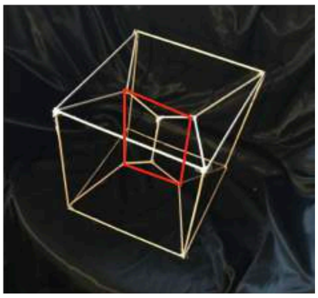
Because of what happened with a square, you won't now be expecting this to be the solution:



This is what I think the solution is, but I can't prove it: it might be something completely different. But I'd like you to look at the three pictures and tell me why my guess is at least plausible.

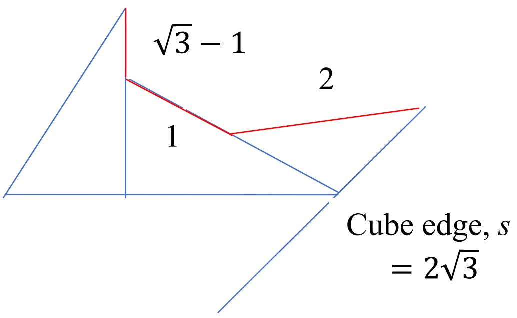






M4: Calculating the length of silk

Build the figure from 'set square' (1-2-$\sqrt{3}$) triangles:



We need:
2 lots of $(\sqrt{3}-1)$,
4 lots of (1),
8 lots of (2).
For the total, l, in terms of s we then need to divide by $2\sqrt{3}$.
$l = 3\sqrt{3}+1$.

This is a saving of about 10% on the 4 crossed space diagonals, total length $4\sqrt{3}$.

Part 2

E13: ‘Polyhedra’ experiments no.1.

Apparatus: KB. The straws and connectors allow the children to build a selection of regular and semiregular polyhedra.

Test: The pairs of children are free to choose which particular polyhedron they build, and dip it in the soap solution.

Observation: Exhibit the results and asks the children if they notice any feature which is common to all the models *and which they were expecting from work with MP*.

With MP the angles between the walls were displayed automatically because the wall edge was perpendicular to the perspex plates. In three dimensions the children have to look for what are in fact the *dihedral* angles and spot that these are equal in all cases.

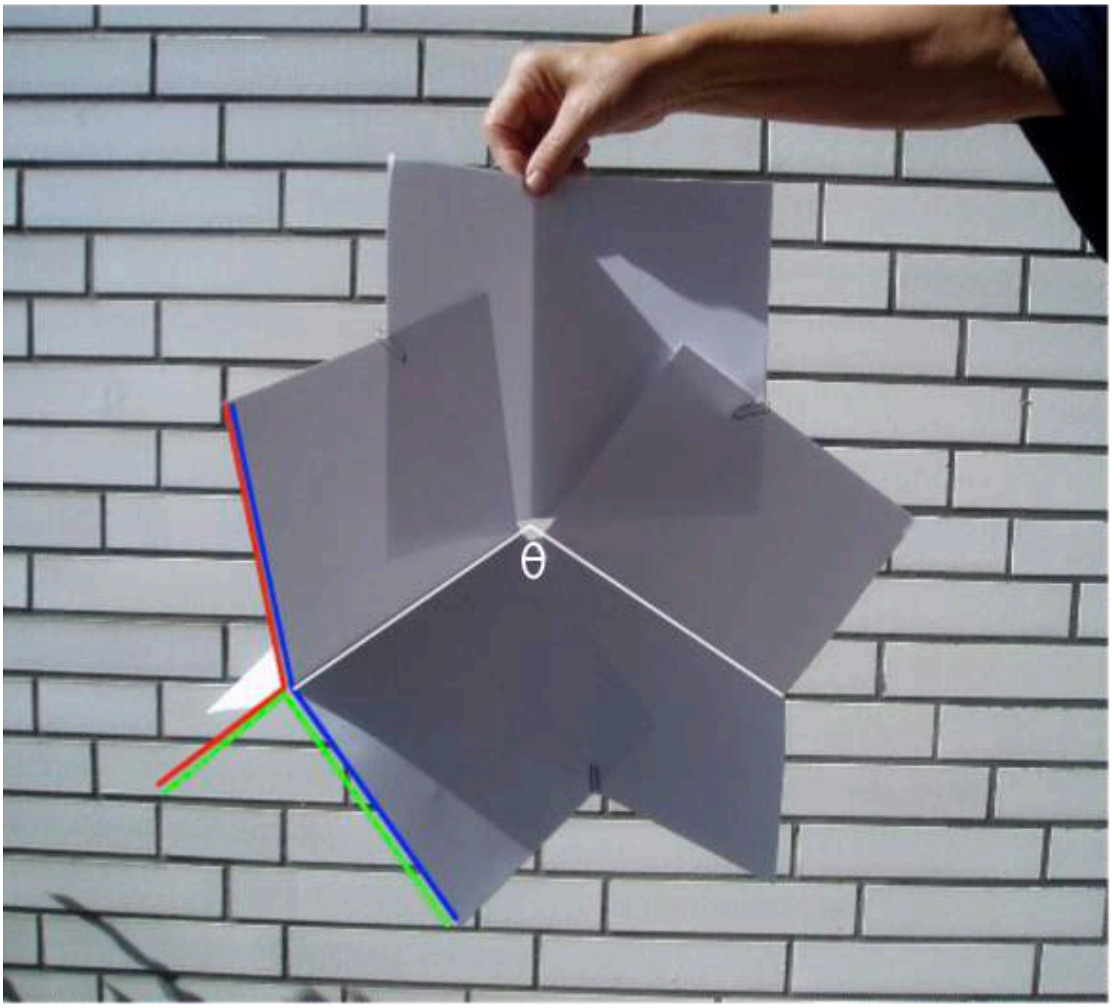
Then ask the children if they notice any other features which are common to all models. You may have to draw attention to all of the following:
1.) 4 edges meet in a point.
2.) Every angle between edges is the same.
3.) 6 walls meet in a point.

Finally ask for features *not* common to all models. Notable here is the observation that not all edges are straight.

E14: Experiment. This picks up on observations **1**, **2** and **3**.

Apparatus:

3 A5 sheets of paper for each of 4 students working as a group; 6 paper clips, 1 Pritt stick for the group.



4 120° trigonal prisms made by folding 3 sheets of A5 paper width-ways and sticking them together along the red, blue and green edges as shown. Each prism represents 3 walls of soap meeting in an edge.

Test: Fit 2 prisms together so that a pair of walls merge – the corresponding sheets slide over each other. You can vary the angle θ and fix the prisms together with a paper clip. But in order to fit a 3rd prism to these two you find you have to adjust θ to a special value. When you've found this, fix the 3rd prism in place with paperclips, one for each pair of overlapping sheets. You now find that the 3 prisms in place allow a 4th to be added without any further adjustment. Secure each of the 3 overlapping pairs of sheets with paper clips.

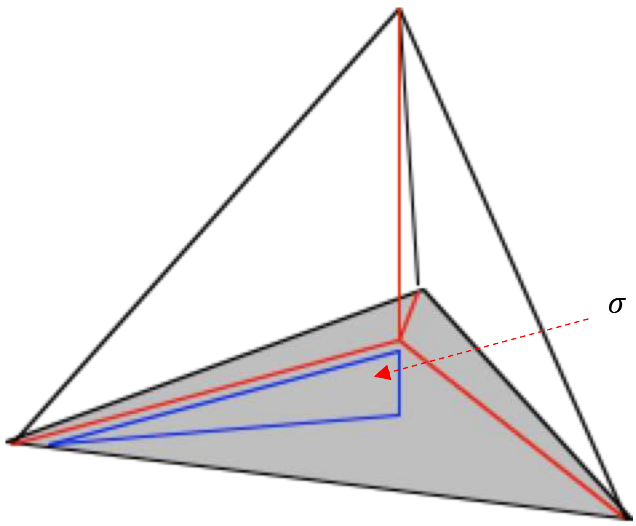
Observation: The final model has the greatest possible symmetry.
There are 6 walls. Each defines a plane of symmetry.
There are 4 edges. Each defines an axis of rotation symmetry of order 3. Each lies at the intersection of 3 planes of symmetry.

Every pair of edges makes an equal angle – about $109\frac{1}{2}$ ° - which we find accurately in **M5**.

We find this angle at the centre of a methane molecule, which has a C atom at the centre and H atoms at the vertices of a regular tetrahedron, and in all structures with a C atom at the centre.

Show a model to the children (e.g. one made from the Orbit molecular modelling kit).

M5: Finding θ.



The red lines divide the tetrahedron into 4 congruent pieces.
Each must therefore have $\frac{1}{4}$ the volume of the whole.
Since this is proportional to height, the blue height must be $\frac{1}{4}$ of the total height.
The blue height therefore stands to the red length as $1:3$.
This gives us the cosine of angle σ
and θ is the supplement of this = $arccos\left(\frac{-1}{3}\right)$.

E15: θ by paper-folding. Teacher demonstration.

(See figures for **M5**, following.)

Take a sheet of A4. Fold P on to P'.
Show that the red angle is θ by inserting into the **E14** apparatus.

M6a: Proof of **E15**.

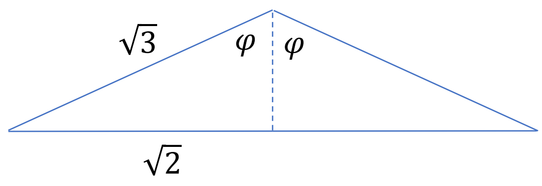
The last figure on the next-to-last line below shows the folded sheet and the ghost of the original.
Look at the sequence of 4 figures on the last line.

First figure:
If the left-hand angle is φ, so is the right-hand angle by the symmetry of the fold.

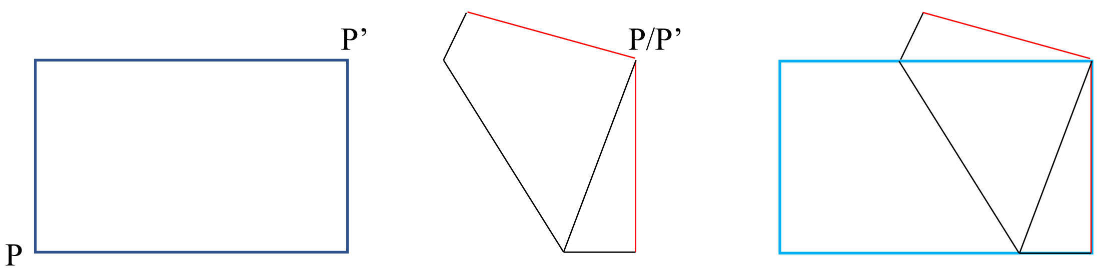
Second figure:
If the left-hand angle is φ, so is the right-hand angle (alternate angle between parallel lines).

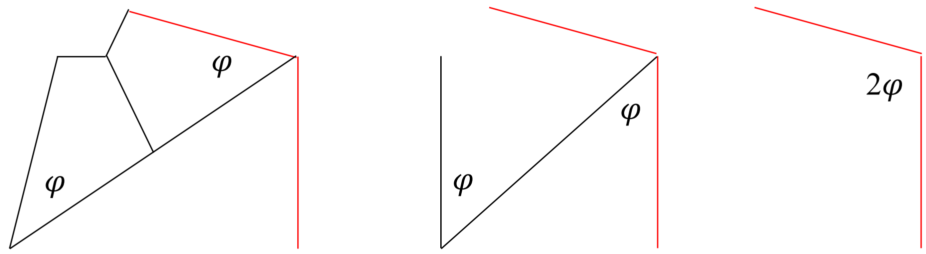
Third figure:
Therefore $\theta = 2\varphi$.

But from the proportions of a sheet of metric paper we have this isosceles triangle:



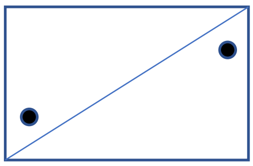
From the cosine rule we find that $\theta = 2\varphi = arccos\left(\frac{-1}{3}\right)$.





M6b: Alternative proof of **E15**, using similar triangles instead of trigonometry.

The children rule a diagonal across a sheet of A4, marking a big black dot for each of the angles $\theta/2$ in these positions:

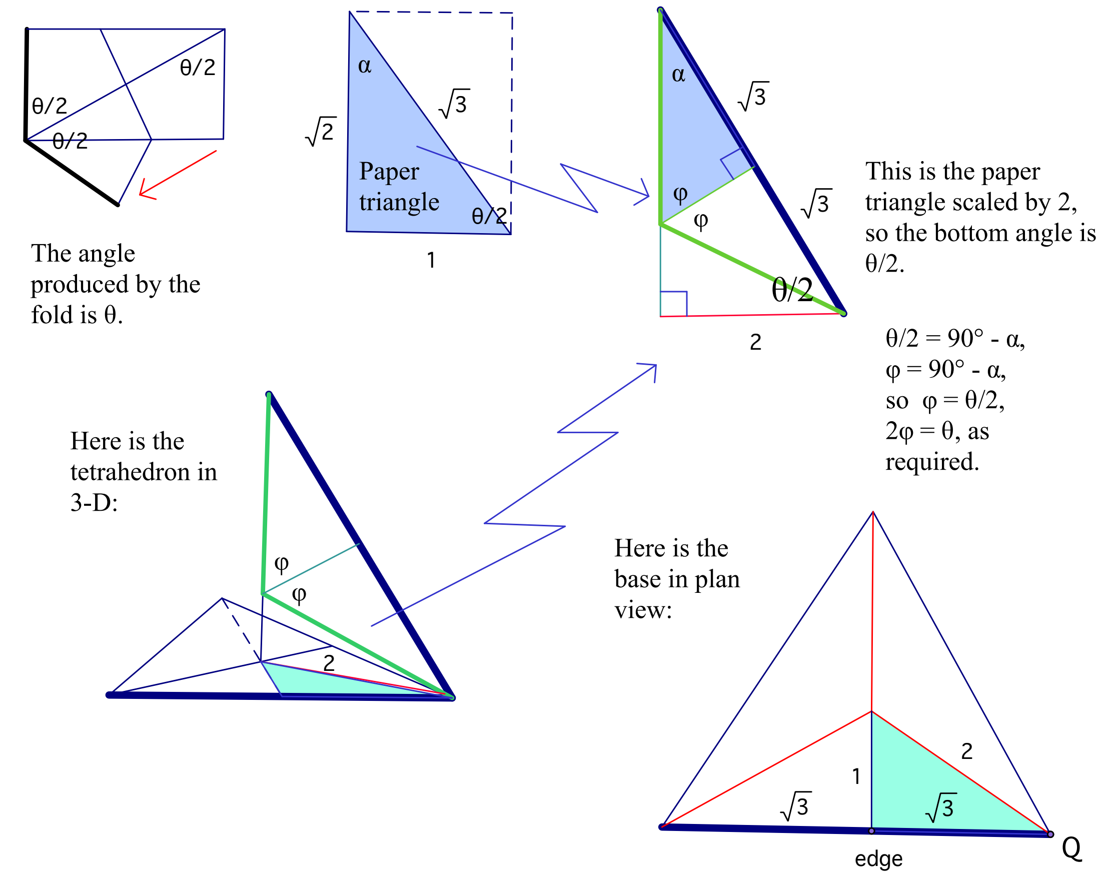


They make the above fold and see that the angles come together to make the required angle.

Project and explain this diagram:

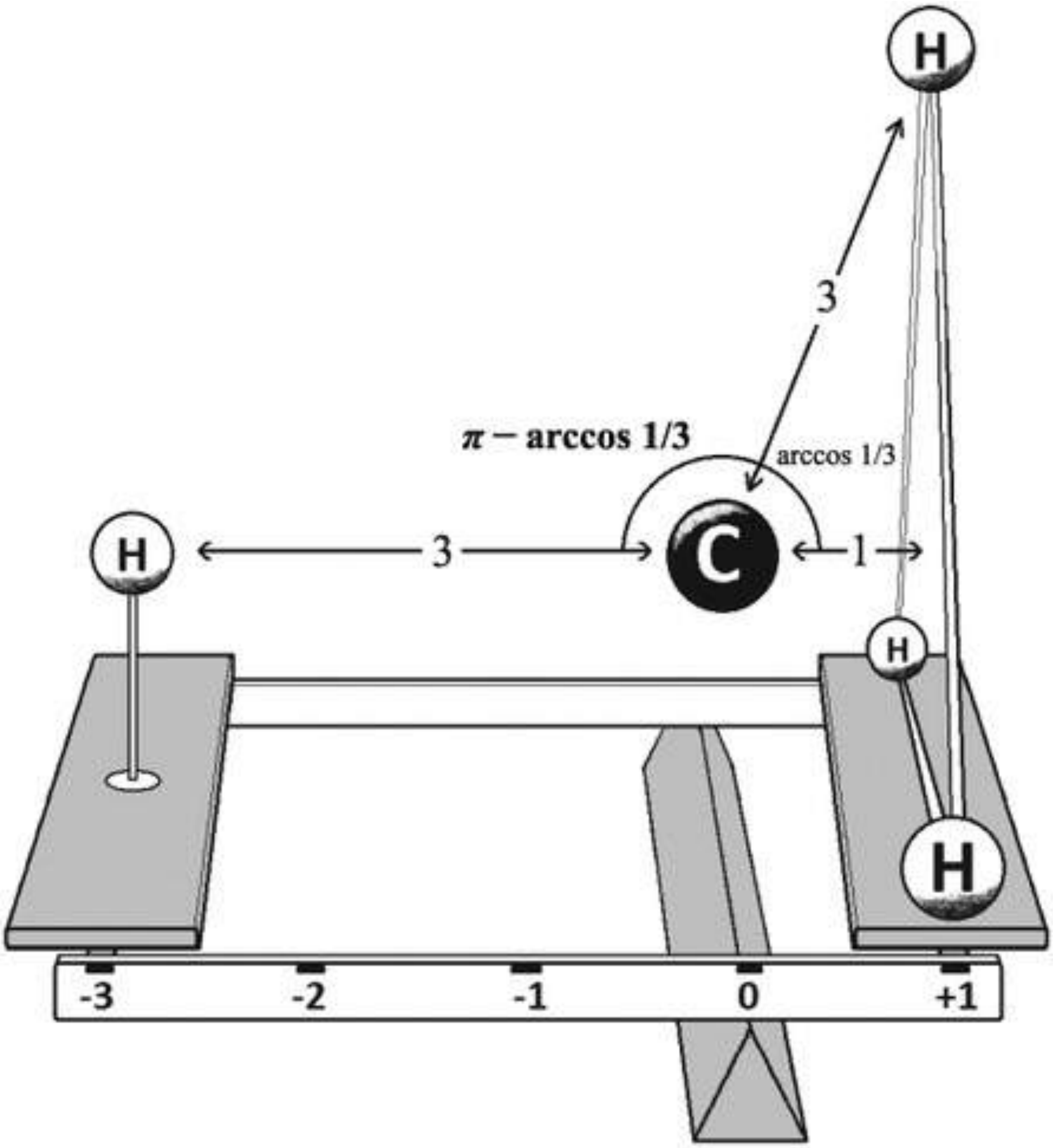
The 'tetrahedral angle' by folding a sheet of A4

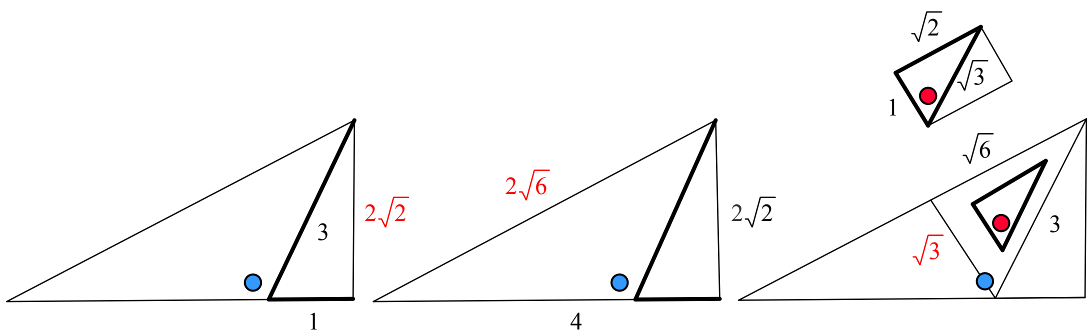
We need to show that the black angle (θ) is the same as the green angle (2φ).



M6c: a proof-without-words by a Hong Kong student:

Project and explain ff. diagrams, which confirm the blue angle as the one we want.





E16: ‘Polyhedra’ experiments no.2.

Invite the children to investigate what happens when they repeat **E12** but pop selected walls.

Observation: Now, not only do we obtain curved edges but also curved faces.

E17: 'Polyhedra' experiments no.3.

Finally the children can join the KB pieces in any way they please and see what results.

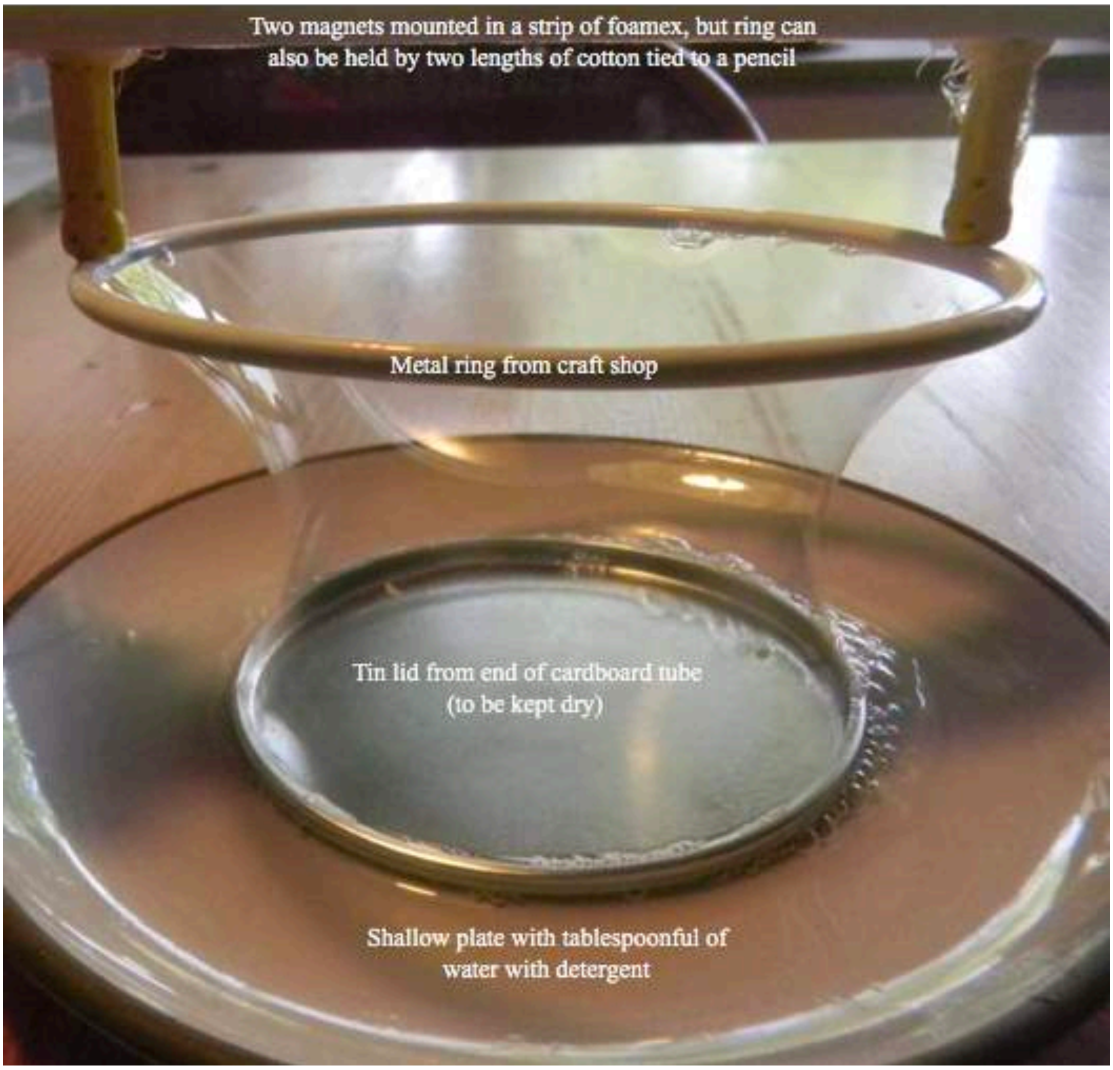
Part 3

In the course of their **Part 2** experiments, particularly **E17**, the children will have produced curved surfaces. What feature do they all share? [They are saddle shapes. If answer not forthcoming, leave question open pro tem].

E18 *Class experiment*

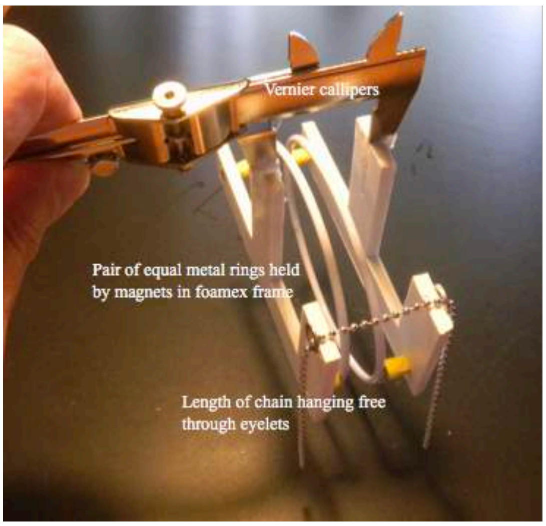
Shrink-wrapped pack of water bottles, x 1 Piece of mirror vinyl with acetate attached, x 2 Red, blue felt pen, x 1 Craft knife, x 1	Exhibit the pack and show how the polythene is sucked on to the bottles. Tell the children that this negative pressure results in a surface of minimum area. Point out the saddle. Explain it is a property of minimal surfaces that the tightest curvatures are equal and opposite and lie in perpendicular planes. Demonstrate with your cupped hands. Make the red cut. Invite one child to hold a piece of mirror in the cut, a second to trace the section in red. Make the blue cut but, because of the effect of the first cut, in the symmetrical position on the other side of the pack. Invite the children to repeat their drawing, but in blue. Remove the acetate pieces and, before overlaying them on the OHP, ask the class what they hope to observe. [A black curve because the blue curve should match the red]

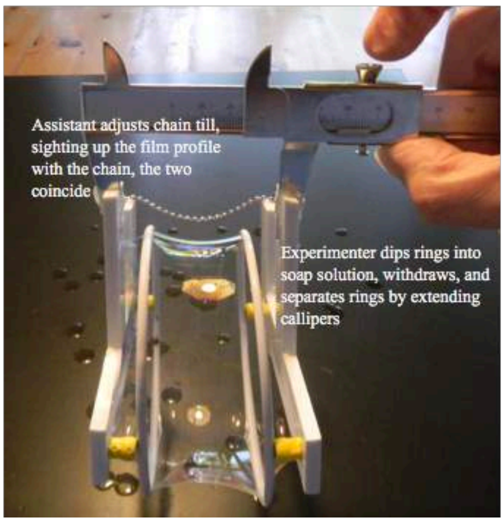
An interesting curved surface is that spanning two rings. Here are two ways to produce it. In the first, suitable for experimentation at home, the axis is vertical. The second, where the axis is horizontal, is designed as a teacher demonstration for a workshop and is the subject of **E19**.



Soap solution tub, x 1, apparatus as shown, x 1

E19 *Teacher demonstration*





(a) With a volunteer set up the apparatus as described in the pictures.

Compact light source, x 1, screen x 1

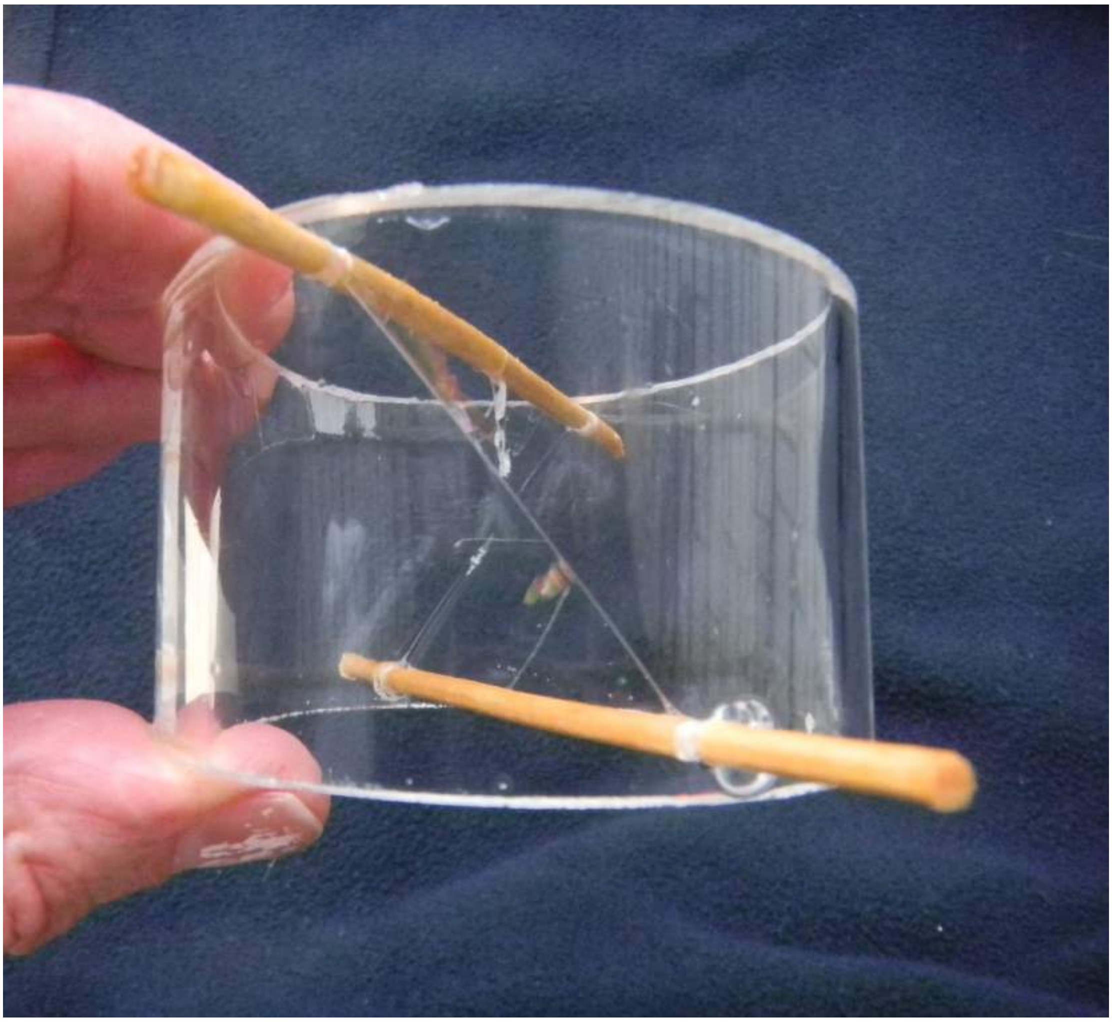
Hang the apparatus in the path of the beam to project the film profile and chain as shadows. Explain that the chain (Latin *catena*) hangs in a *catenary* curve and that the soap film forms the solid of revolution of this, a caten*oid*.

Chain lengths, x 15

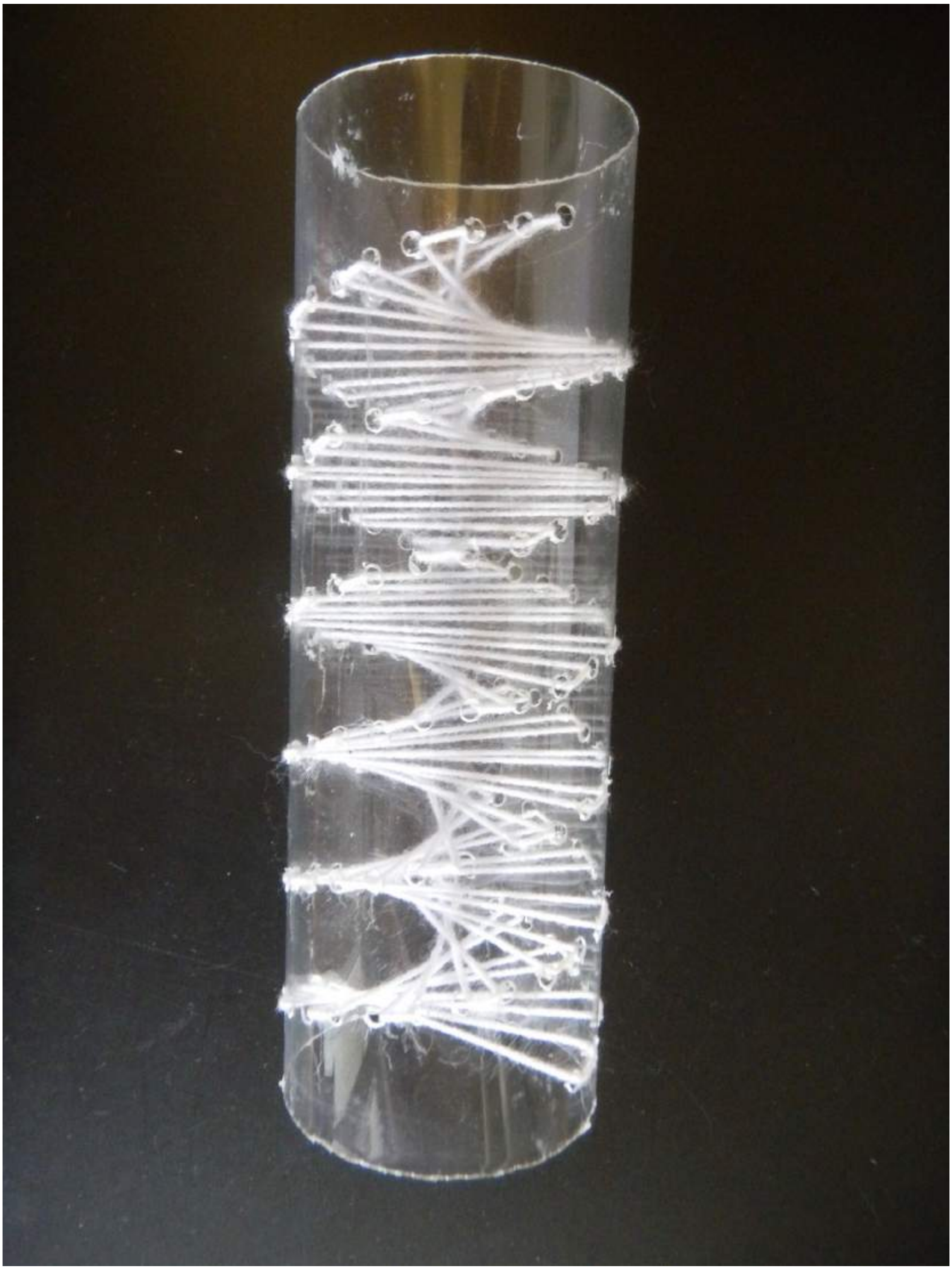
(b) Project the acetate showing a cocktail table and the same with fabric stretched over it. Point out the parallel with experiment **(a)**. Set the acetate at such an angle that the symmetry axis of the curve is vertical and invite the children to hang their lengths of chain so that they coincide with the profile of the table cover.

E20 *Teacher demonstration*

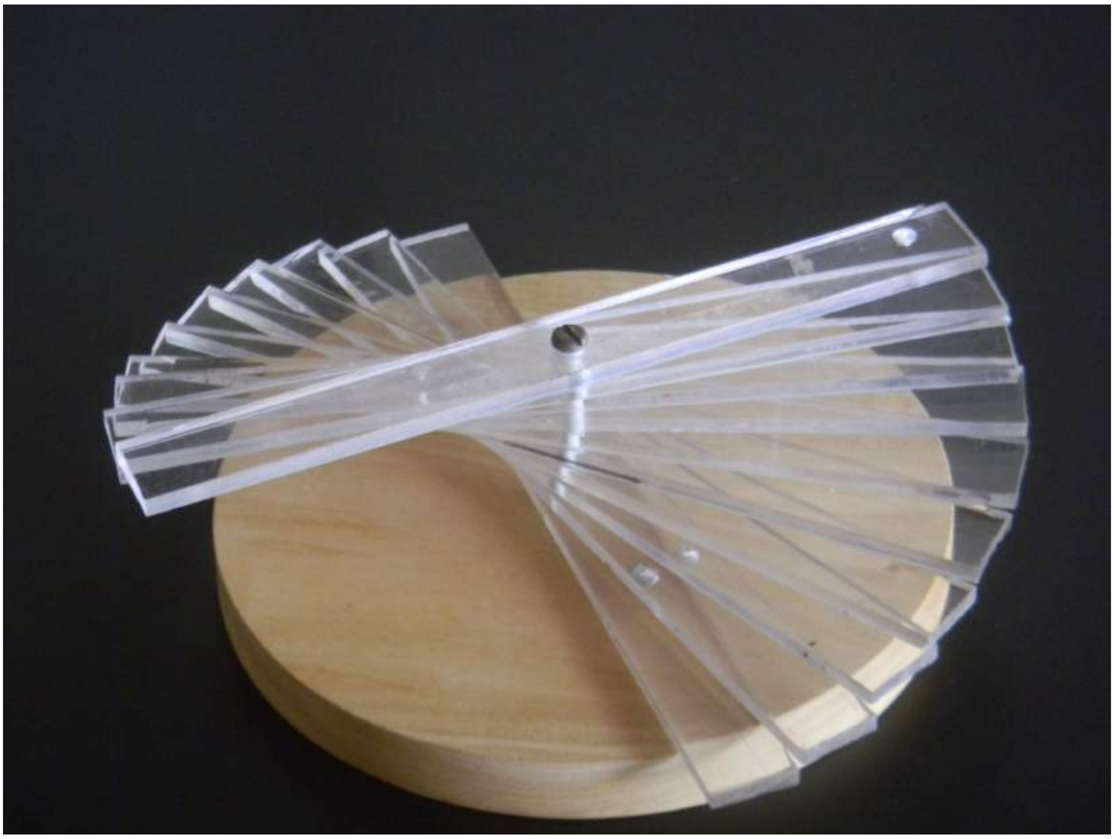
Tub as above, x 1 Apparatus as described, x 1	Take a model consisting of a clear cylinder open at both ends with rods at 2 different heights, and in perpendicular planes, passing diametrically through it. Submerge and withdraw. Display the resulting *helicoid*, tracing a *helix* on the cylinder.



Helicoid model, x 1	Point out this is one of the ruled surfaces, and one of the special ones which glides on itself. Point out that it is in fact the only ruled surface which is minimal.



Model corkscrew staircase, x 1   Exhibit.

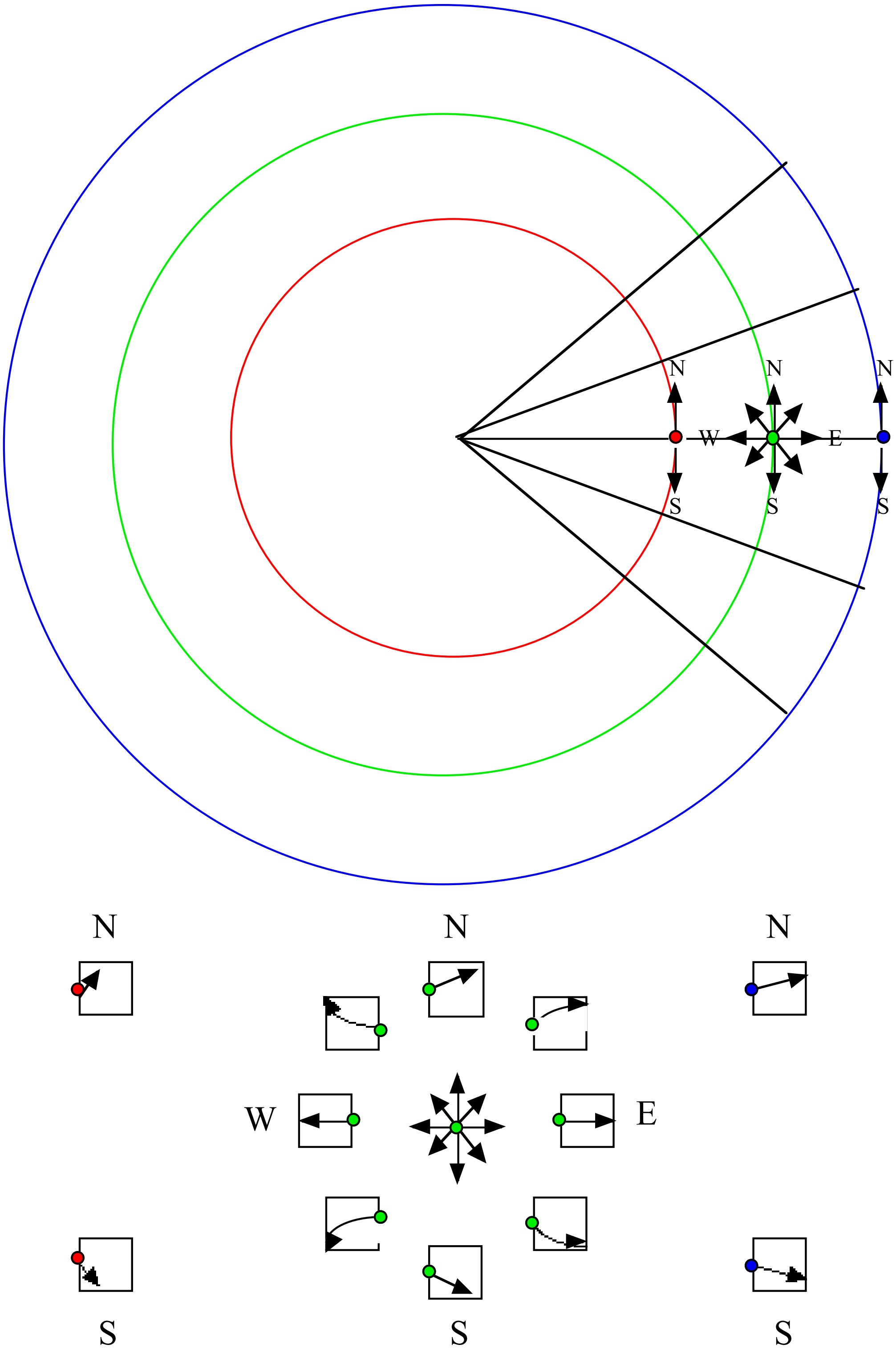


If you're lucky enough to have access to a ***real*** *corkscrew staircase, use it.*
Ask the children to recall running up a corkscrew staircase to the top of a tower. Keeping to the centres of the steps, the gradient was constant. What was it best to do when they got tired: run in towards the centre or out towards the wall?

Acetate as below

Display the acetate showing an aerial view of a child and two friends climbing a spiral staircase. 'Blue' keeps to the outer edge, walking a long way but following an easy gradient; 'Red' keeps to the inner edge, a shorter but steeper way. 'Green' generally keeps to the middle. The slope eases when s/he veers to the right, increases when s/he veers to the left. Ask the children to imagine that the steps are infinitesimally small so that the three are walking on a smooth surface. NE-SW the surface has a con*vex* bend; NW-SE, an equal con*cave* bend. Point out this is characteristic of all minimal surfaces:
At every point there are two, perpendicular directions in which the curvature is greatest; in these two directions the surface bends equally but opposite ways. (The plane is a limiting case in which there are no bends in any direction.)

Your green self and your blue and red friends are climbing a spiral staircase which ascends anticlockwise. The arrows show directions of climb. The thumbnail transects show the gradients experienced.



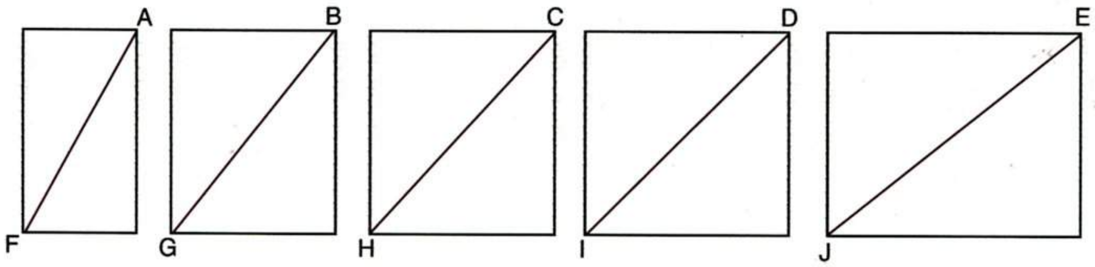
E21 *Teacher demonstration*

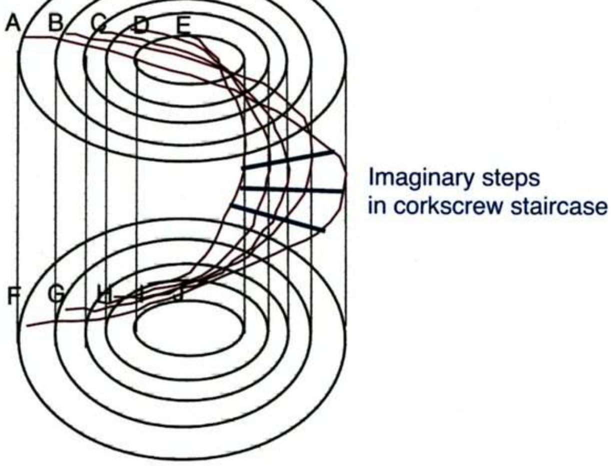
Acetate
Corresponding model, x 1

Offer the children the following intuitive way of seeing why the helicoid must be a minimal surface.

A helix is a straight line wrapped around a cylinder. It is a 3-dimensional curve, a *space curve*.

Because a straight line is the shortest distance between two points in the plane, a helix is the shortest distance between two points on the cylinder.





Here AF becomes a helix of one turn on a thin cylinder; BG a helix of one turn on a fatter, concentric cylinder; CH a helix of one turn on a fatter cylinder still; and so on.

Imagine that each helix is not a space curve but a thin ribbon and that each ribbon is stitched along its edges to its neighbours on either side.

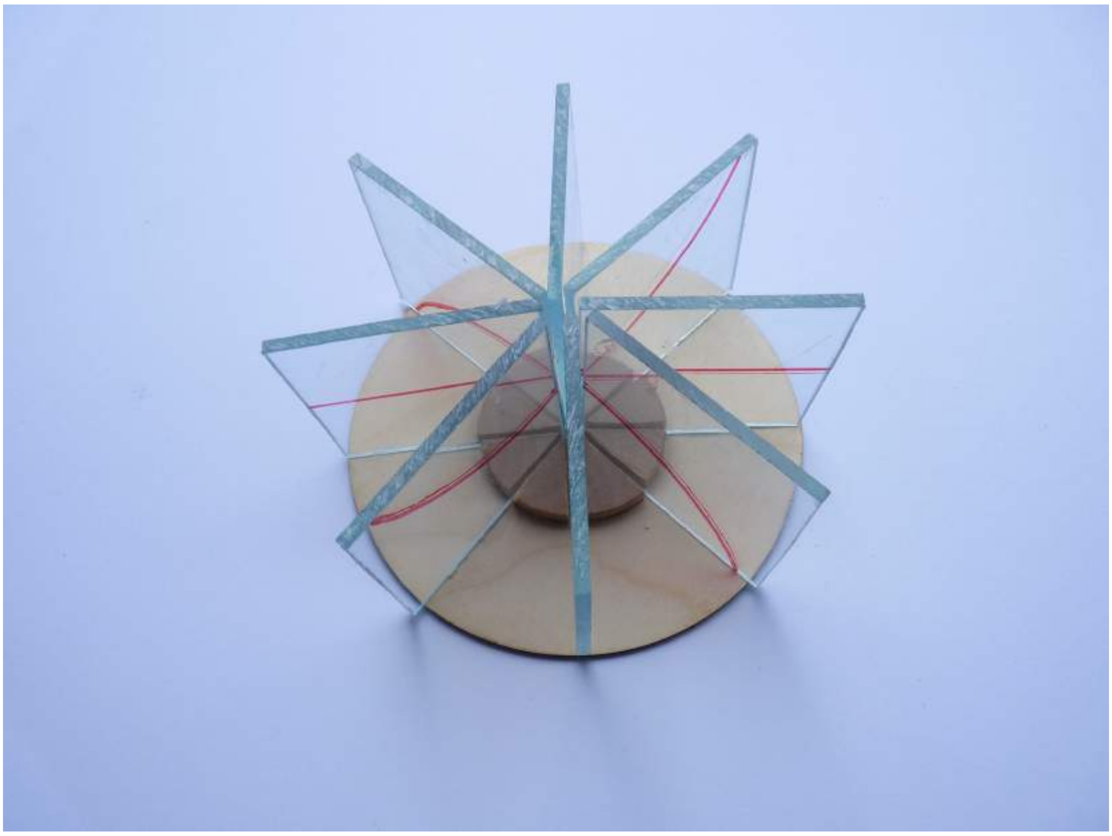
Now let the ribbons become infinitesimally thin, and their number correspondingly large, and you have a smooth surface, a helicoid.

Since each helix follows the shortest route between its ends, each ribbon shows the surface of least area between its edges, and all the ribbons together the surface of least area between the innermost edge and the outermost edge.

E22 *Teacher demonstration*

Base + 8 clear plastic plates, 4 of one type, 4 of another (B25), x 1

Ask the children to cast their minds back over all the surfaces t they've seen (**E15** – **E19**) and visualise an infinitesimal part. Ask for 2 volunteers to assemble the corresponding model.



Ask the children how we know that the alternate segments
Are straight, even though their length is infinitesimal.
[If the surface is continuous, there must be a point between an 'up' curve and a 'down' curve where the surface is level.]

E23 *Pupil experiment*

Repeat **E20** by performing the 'saddle dance': with your head as the point and your arms as the cross-section of the surface, rotate through a whole turn. The children repeat.

E24 *Teacher demonstration*

Soap solution,
Straw, x 1,
Lenart sphere, x 1

Blow a bubble.

Point out on the sphere that the principal curvatures go the *same* way, not opposite ways. Point out that, unlike in the other soap film experiments, there's no boundary and there's a positive pressure difference across the surface. The soap film represents the smallest possible surface area for that particular pressure difference.