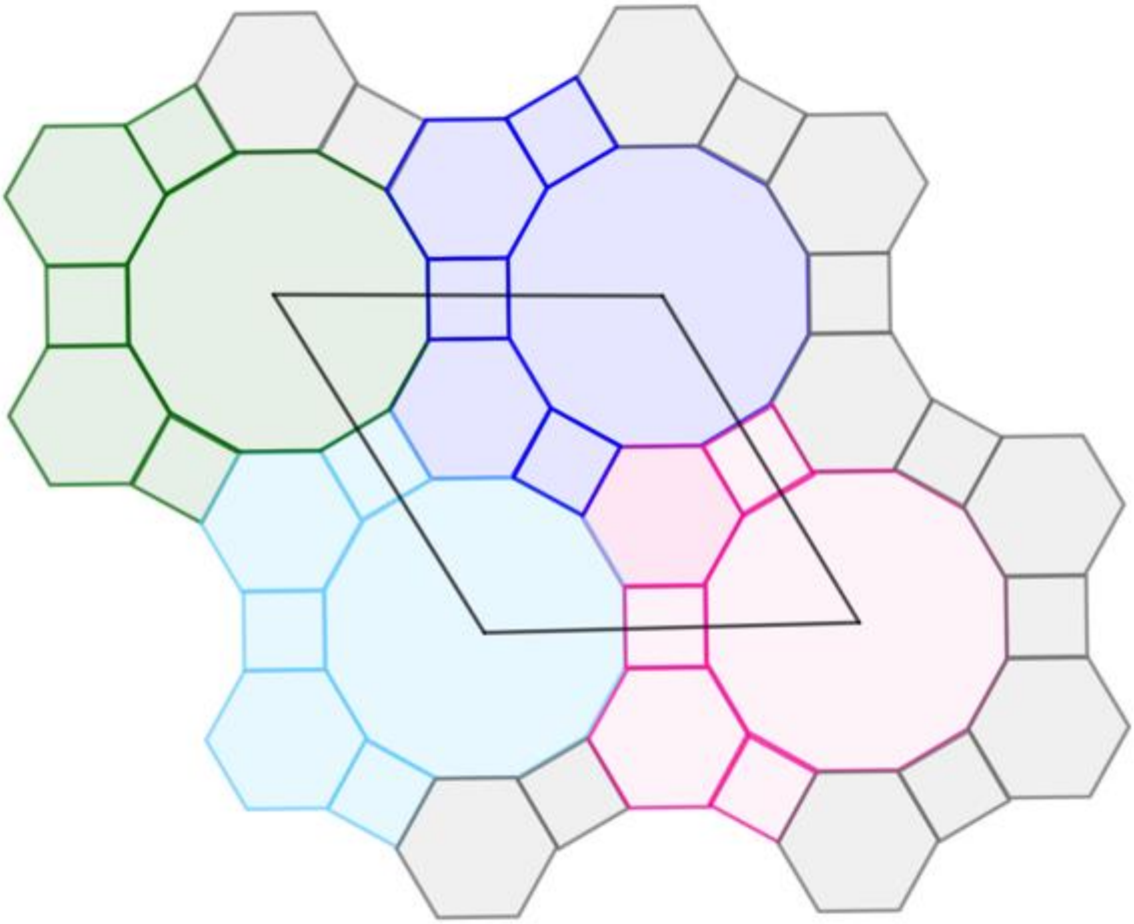
**Shape counts in a tiling**

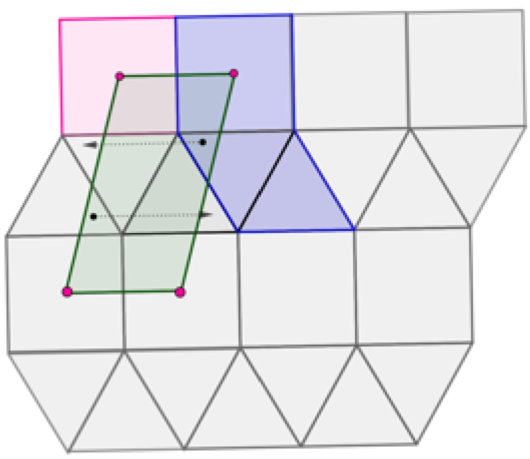
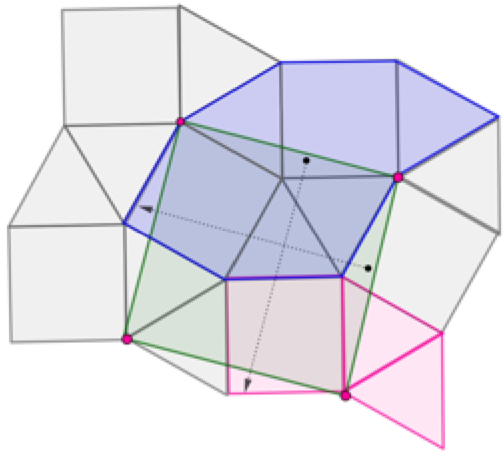
A *periodic* tiling is one you can make by translating a repeat unit. Particularly interesting are the *uniform* tilings. There each tile is a regular polygon and each vertex is the same. For that reason, having labelled a polygon by the number of sides, you can name such a tiling by recording each shape you meet in a tour round a vertex: 4.6.12, 4.8.8, 3.3.4.3.4, and so on. Take the first case. How do I know how many squares, hexagons and dodecagons I need to make the tiling, or rather, since a tiling is infinite, what *ratio* of squares to hexagons to dodecagons do I need? All I need do is sample a repeat translation unit:



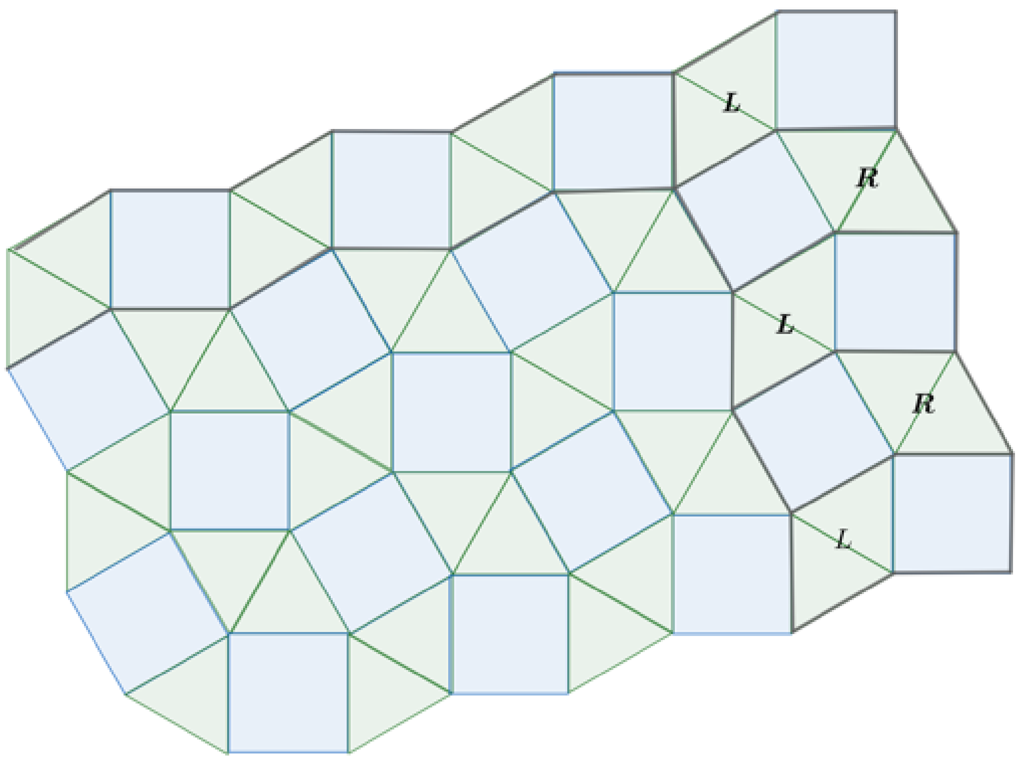
As you can see, within the black rhombus, we have one whole and four half squares, two hexagons and parts of one whole dodecagon.

We don't have to work out each of the four parts of the dodecagon. We have a part for each interior angle of the quadrilateral, whose sum 360°, so we catch the whole shape. This is a special case of what we'll call the circuit area theorem. We'll meet it again.

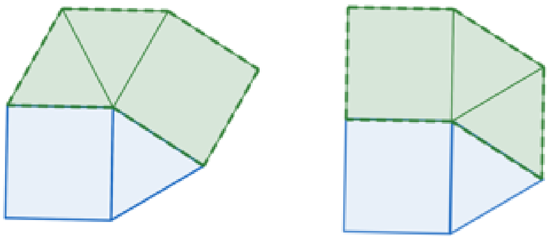
The ratio we require is therefore 3:2:1. But the repeat unit need not be a parallelogram. In pink, cyan, green and mauve I've coloured the same shape. This motif too tiles the plane and we can count shapes in it without summing fractions. This ploy is particularly useful in the respective cases of 3.3.3.4.4 and 3.3.4.3.4, shown on the left and right respectively below. It's much easier to use the mauve figures as unit cells than make counts in the green cells ...

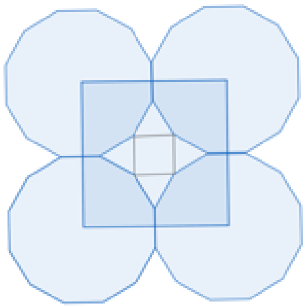
... but let's try. On the left we have one square by the circuit area theorem and we can make up parts of two triangles by translation. On the right, applying the circuit area theorem to the pink shapes, we have one square and two triangles. There are two further triangles within the cell and we can make up one further square by translation. The ratio in both cases is two triangles to one square. Though the combined pink shape on the right consists of one square and two triangles, unlike the same shape in mauve on the left, it is not a repeat translation unit. But we can generate 3.3.4.3.4 by flipping alternate shapes over (the orientations denoted 'L' and 'R' below) to make the 'zonogon' down the side and repeating the shapes across in the same orientation (or making our first zonogon across and repeating the shapes downwards):



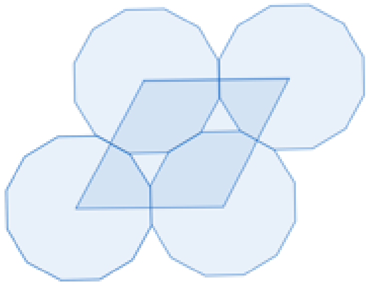
Notice that turning our shape over converts a 3.3.4.3.4 vertex into a 3.3.3.4.4 vertex, and vice versa:



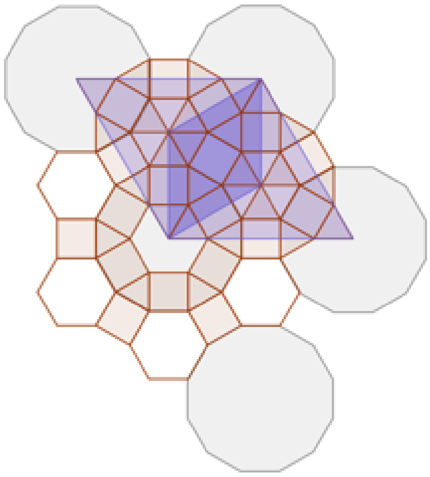
Call 3.3.3.4.4 tiling **A**, 3.3.4.3.4 tiling **B**. If we are to create a periodic tiling of squares and equilateral triangles where the two shapes are present in a ratio other than 2 triangles to 1 square, we must create more than one vertex type (so that the tiling is no longer uniform). The following are obtained from uniform tilings by dissection:



Tiling **C**: arrange 4 dodecagons in a square: this leaves a space in between consisting of 1 square and 4 triangles. Split each dodecagon into 6 squares and 12 triangles.



Tiling **D**: take 3.12.12 and split each dodecagon in the same way.



Tiling **E**: take 3.4.6.4 and split each hexagon into 6 triangles; or, what leads to the same ratio, take 4.6.12 and split each hexagon into 6 triangles and each dodecagon as above.

The light mauve cell belongs to 4.6.12, the dark mauve cell to 3.4.6.4.

You may confirm that the triangle : square ratios which result, in order of the tilings poorest in triangles to the richest, are as follows:

| | | |
|---|---|---|
| **A**, **B**: | 2:1 | 112:56 |
| **C**: | 16:7 | 112:49 |
| **D**: | 7:3 | 112:48 |
| **E**: | 8:3 | 112:42 |

We can always check our tile counts by a further method. We'll return to our first example, 4.6.12:

A 12-gon shares edges with 6 squares but a square only shares edges with 2 12-gons. Therefore the ratio of squares to 12-gons is 6:2, 3:1.

A 12-gon shares edges with 6 hexagons but a hexagon shares edges with 3 12-gons. Therefore the ratio of hexagons to 12-gons is 6:3, 2:1.

Thus, for every 12-gon, there are 3 squares and 2 hexagons, as we found above.