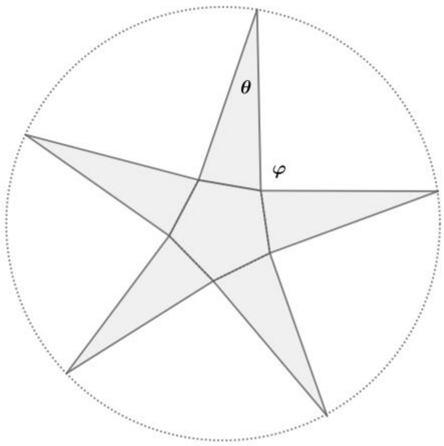
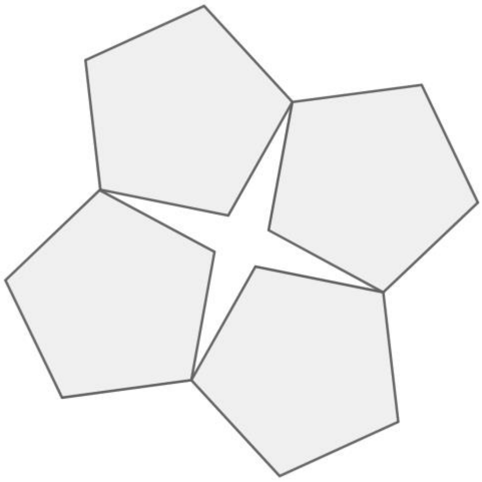
Polygon ring stars

This is a polygon ring star,
a regular p-gon at the centre
with congruent isosceles triangles
on its sides.



We choose φ so that it is the
interior angle of a regular n-gon.

In the example below, $n = 5, p = 4$.



We know $\varphi = \frac{(n-2)\pi}{n}$ but we'll leave it as φ
while we find out how θ depends on it.

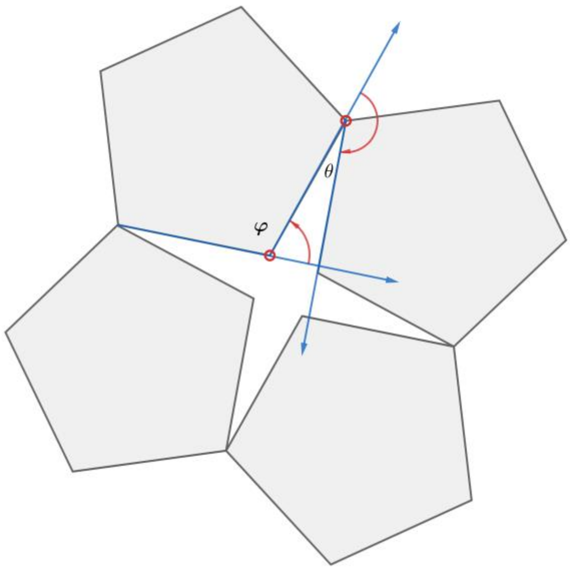
We'll work our way round a star using
Seymour Papert's Total Turtle Trip Theorem.
This states that, by the time the turtle is back
facing its original direction, it has rotated 2π.

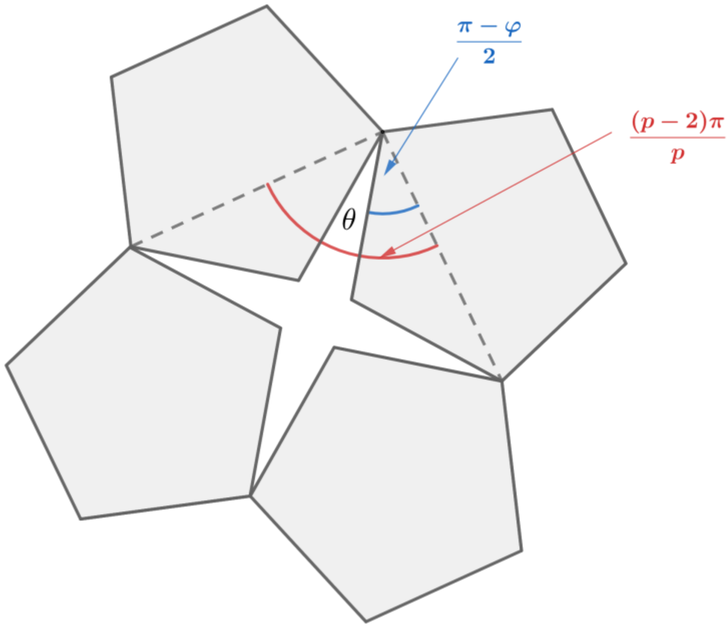
(Note that we can derive the φ formula itself
using the Total Turtle Trip Theorem.)

The turtle turns $-(\pi - \varphi)$ p times,
$(\pi - \theta)$ p times,
so we have:

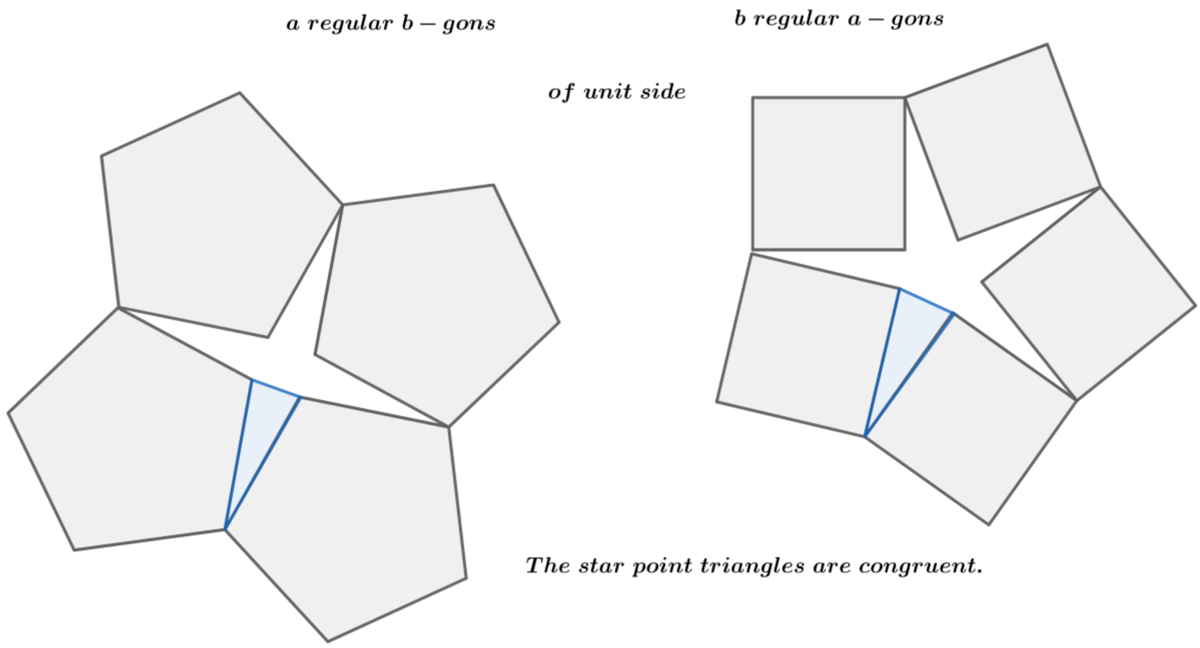
$p(-\pi + \varphi + \pi) - p\theta = 2\pi$,
$\theta = \frac{p\varphi - 2\pi}{p}$.
Substituting for φ and simplifying leads
to:
$\theta = \frac{[np - 2(n+p)]\pi}{np}$.

Alternatively we can find θ from the
figure below.





The interesting thing about our expression for θ is that it is symmetrical in n and p.
This means that θ for a ring of a b-gons is the same as for a ring of b a-gons. If the polygons have unit edge, the star point triangles are congruent:



What is the range of n values possible for a given p?

From our formula, $np \geq 2(n+p)$. Rearranging, $n \geq \frac{2p}{p-2}$.
If the n-gons are not to overlap,
$\theta + 2\varphi \leq 2\pi$. Substituting for θ and φ gives:
$np \leq 2(n+3p)$.

Rearranging, $n \leq \frac{6p}{p-2}$.
Putting those inequalities together, we have:
$\frac{2p}{p-2} \leq n \leq \frac{6p}{p-2}$.
What is interesting here is that, at either end of the range for a particular *p* value, if we have equality, that is $n = \frac{2p}{p-2}$ or $\frac{6p}{p-2}$, the figure is often part of a tiling or some other special case. Intermediate values may also give tilings as the table below makes clear.

p	n	θ	Special case
4	4	0	The tiling 4.4.4.4
	12	$\pi/3$	The tiling 3.4.3.12/3.12.12
5	4	$\pi/10$	
	10	$2\pi/5$	The 10-gons share edges
6	3	0	The tiling 3.3.3.3.3.3
	6	$\pi/3$	The tiling 3.6.3.6 This particular polygon ring star is A star polygon.*
	9	$4\pi/9$	The 9-gons share edges
7	3	$\pi/21$	
	8	$13\pi/28$	
8	3	$\pi/12$	
	8	$\pi/2$	Parts of the tiling 4.8.8 ‘chained’

* The difference in general between a polygon ring star and a star polygon is that the sides of the point triangles in the latter are continuations of the sides of the central polygon: