How Descartes might have arrived at the result that the total angle deficit for a polyhedron is 4π

Around 1630 Descartes wrote the following. (In this translation of *Progymnasmata de solidorum elementis* P. J. Federico puts contemporary interpretations in brackets.)

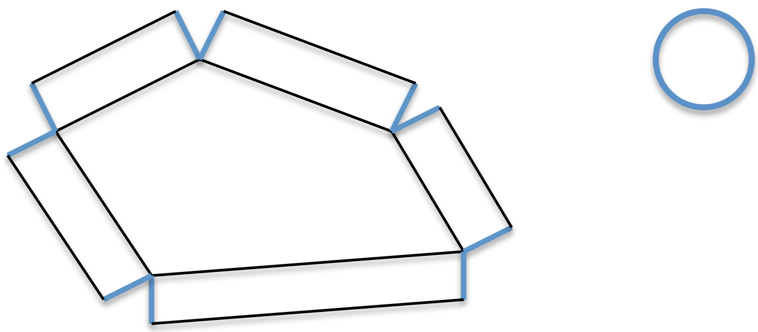
> As in a plane figure [polygon] all the exterior angles, taken together, equal four right angles [2π], so in a solid body [polyhedron] all the exterior solid angles [angle deficits], taken together, equal eight solid right angles [4π].

As you see, his statement is an analogy. Breaking it down into 3 subsidiary statements, we have:
1. There is a relation between the exterior angles of a polygon.
2. There is a relation between the angle deficits of a polyhedron.
3. This relation is the same, namely that they have a constant sum, (which Descartes specifies in both cases).

Descartes' realisation that the constancy holds across dimensions adds to the geometric significance of either sum.

Descartes does not provide a proof for his theorem. He makes a series of statements, leaving the reader to guess the thought processes connecting them. In *Induction and Analogy in Mathematics* (ch. 3, note 37) George Pólya conjectures the precise analogy Descartes might have made.

We take the 2-dimensional case. We represent the exterior angles of a polygon by sectors swung round so that, proceeding clockwise (say), we realise that the right edge of one sector is parallel to the left edge of the next so that they all fit together to make a complete circle:



We move up a dimension. Instead of the rectangles attached to the polygon sides, which separate circle sectors, we attach prisms to the polyhedron faces, separating sphere sectors. The illustration shows one such sphere sector centred on a vertex in which 3 faces meet. (The argument can be extended to include vertices where more than 3 meet.) The sphere centre is marked by a red dot. Red arcs mark the faces of the sphere sector. These faces are sectors of great circles. In our two-dimensional case we noted the parallel sector edges at adjacent vertices. Here correspondingly we have parallel sector faces. As the circle sectors complete a circle in the former, the sphere sectors complete a sphere in the latter. In units of solid angle, therefore, the total angle deficit of the polyhedron is 4π steradians.

The interior angles of the faces meeting in the vertex are α, β, γ.
The angles of the spherical triangle PQR, the dihedral angles between the faces of the sphere sector, are the supplements of these, viz. $(\pi-\alpha),(\pi-\beta),(\pi-\gamma)$.

Now the angle defect at our vertex in steradians = the area of the spherical triangle PQR on the unit sphere = $(\pi-\alpha)+(\pi-\beta)+(\pi-\gamma)-\pi=2\pi-(\alpha+\beta+\gamma)$.
But $2\pi-(\alpha+\beta+\gamma)$ is the angle defect at the vertex expressed in plane angles.
The total angle deficit of the polyhedron, expressed in plane angles, is therefore 4π radians.