A teacher's guide to Number City

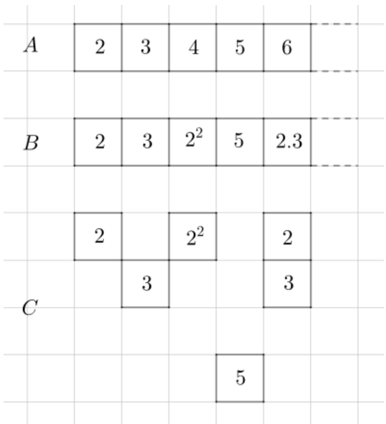
Introduction

This work forms a suitable project for a maths summer camp or term-time maths club. We ask the question: What do their prime factors look like when we take the natural numbers in order? Though the work is for Y6 and above, we won't use the number line but go back to a KS1 representation, the counting strip (**1*A***).

With the numbers factored into primes, the cells soon get crowded and the information they contain hard to read (**1*B***).

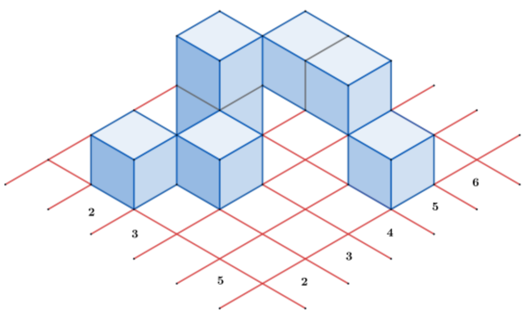
It takes two steps to sort things out. First, we give each prime its own row of squares (**1*C***).

1.



Second, we move into the third dimension so that we can show an exponent by a stack of cubes (**2**).

2.



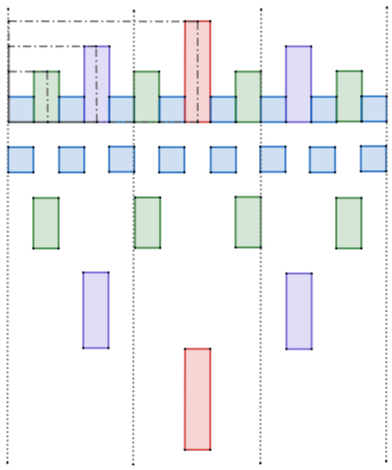
Our plan is for the children to work on squared paper, preparatory to building a model for a classroom windowsill. In the course of this work they will spot patterns, which they can then try to explain. We call the result Number City because it resembles an architect's model of a city laid out along a grid system, bordering a waterfront, which we shall assume runs west to east.

They all copy **1*C*** on squared paper, extending it as far as 10.

They repeat for 11 to 20, but merely noting primes > 10 (so here 11, 13, 17, 19) at the bottom of the sheet. They can do this from knowledge of their tables, breaking factors down as needed.

Before they proceed to the block 21 to 30, show **3**.

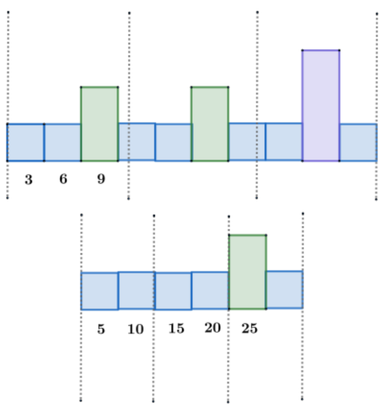
3.



Explain that it shows the even numbers up to 30. Ask what patterns the children spot and how those patterns might help check their diagram when they attempt the block from 21 to 30. [Looking left from 21, we know that everything left of the red tower will be reflected in it. Looking right, therefore, we shall meet first a blue tower, then a lilac tower, and so on.]

Then show **4**, which shows the multiples of 3 and 5 respectively.

4.



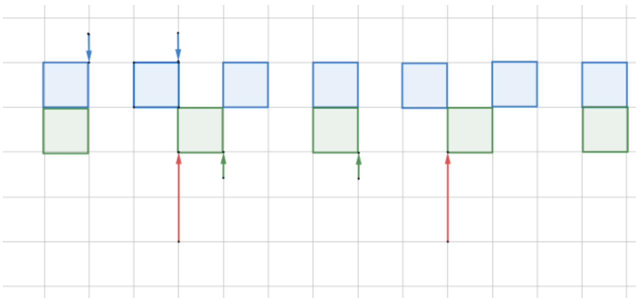
At the top, the new, higher, tower comes every 3; at the bottom, every 5. Make the general point that every 3rd number divides by 3, every 4th number by 4, every 5th number by 5, ... And, continuing in this way, every 5th multiple of 5 is a multiple of $5 \times 5 = 5^2$, and so on.

Now they attempt the block from 21 to 30. Again they can use their tables, using little tree diagrams to make sure they have all the factors. Do their results fit the symmetries just found?

Building the city

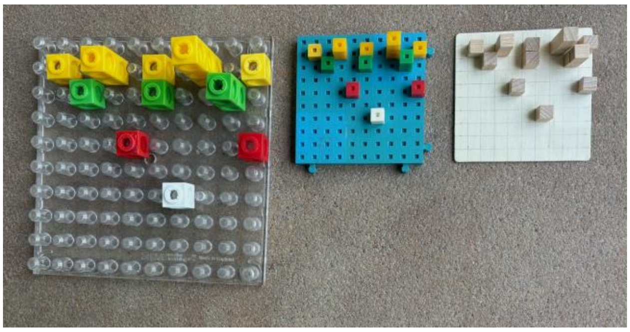
The next phase is the construction. You can allocate a decade to each pair of children, who, as before, will only record powers of the first four primes, 2, 3, 5 and 7. If working beyond 30, they should first prepare a plan on squared paper of the kind we have been calling a 'block'. To make use of the patterns identified, the children will need to consult blocks to the left of their own. If they ignore for the moment the exponents, the heights of the towers, they should note that a pattern in prime p and prime q repeats every pq, the lowest common multiple of p and q. In **5**, the blue pattern repeats every 2, the green pattern every 3. Therefore the blue-&-green pattern must repeat every $2 \times 3 = 6$.

5.



Now to the practical issues. See **6**.

6.



Is the display to be temporary or permanent?

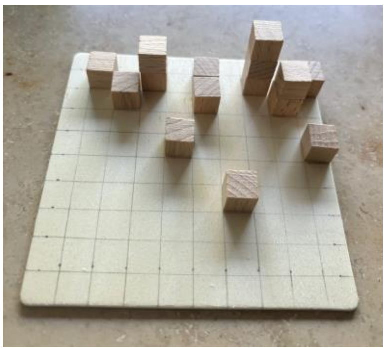
If temporary, the source must now be eBay or a wholesaler, not the initial supplier, but specify the supplier and product by name when ordering to ensure you get the right material. There is a choice of two scales for our model (more if it is to be a playground feature):

2 cm: Here are Multilink cubes by NES Arnold on a Multilink pegboard. With 15 decades, the model takes up 0.2 m by 3 m.
1 cm: Here are Osmiroid Centicubes on a Centicube baseboard. (Note that the baseboards interlock.) With 15 decades, the model takes up 0.1 m by 1.5 m. (At the time of writing these are only available from wholesalers in Australia, where the cubes are sold under the name 'Centifit cubes'.)

On either scale there is the immediate possibility here of colour-coding the primes. It is particularly desirable to distinguish the 2s and 3s in this way. We discuss this possibility further when considering a model using Centicubes for the first 5 decades but accommodating all the prime factors.

If permanent, one must make the same choice of scale. The obvious material is wood. Illustrated here is the first decade made using cm cubes (unpainted) on balsawood tiles from the supplier Etsy.

7.



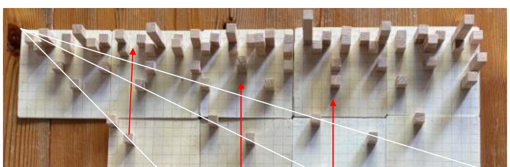
Whichever material is chosen, going out to 150, we shall need these numbers of cubes (rounded up):
140 for the 2s,
80 for the 3s,
40 for the 5s,
30 for the 7s,
i.e. around 300 in total.

Here is a view going out to 100:

8.

To understand more about the layout of Number City it is necessary to come out beyond the 7s. In **9** we have a complete plan of the part from 1 to 50. A classroom table might be devoted to it. We need duplicates of the first 5 units and a further 10, two of which are blank. Working in the same pairs, the children should organise the work among themselves. If using Centicubes, order 2 packs of 10 baseboards, and 500 cubes. These come as 50 in each of 10 colours. This is enough but 6 of the 15 primes displayed (29, 31, 37, 41, 43, 47) will have to receive the same colour (e.g. white or black).

9.



Navigating the city

With their eyes close to the table, they should sight along the aligned towers picked out in white. The numbers lying on the $45°$ line are the primes. The first red arrow picks out one and shows a clear view through to the top. The other white lines pick out the multiples of the primes. These sloping lines, radiating from 1, offer a complete description of Number City in the following sense. We can allot an address to every block in two ways:

1. 'Go $n = ab$ streets east, b squares south'.
2. 'Go to the b^{th} square centre on the a^{th} ray.'

Examples:

1. 'Go 45 streets east, 3 squares south.'
2. 'Go to the 3rd square centre on the 15^{th} ray.'

These locate the 3^2 tower belonging to $45 = 3^2.5$.

1. 'Go 45 streets east, 5 squares south.'
2. 'Go to the 5^{th} square centre on the 9^{th} ray.'

These locate the 5 block belonging to 45.

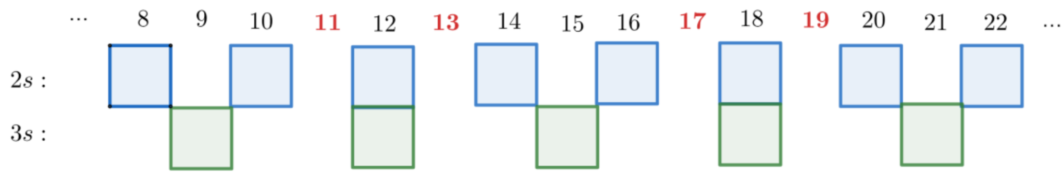
1. 'Go 39 streets east, 13 squares south.'
2. 'Go to the 13^{th} square centre on the the 3^{rd} ray.'

These locate the 13 block belonging to 39.

(Because of the precise alignment required, **2** is clearly the less practical system.)

The most conspicuous feature of the primes is that they're grouped in pairs around a multiple of 6 (*prime pairs*). Discuss **10** with the children, which shows why these are the only positions possible for a prime.

10.

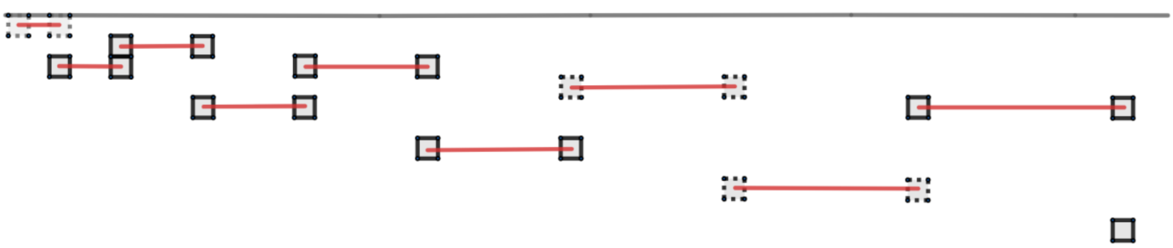


The second and third red arrows show numbers in these positions which might have been primes but are in fact composite: in the first case 5^2, in the second case 5×7.

The most conspicuous feature of the model is a negative one, something that doesn't happen: you never get two towers together parallel to the water. Ask the children why they think that is. [Every 3rd number divides by 3, every 2nd number by 2, every '1st' number only by 1. No pair of consecutive numbers shares a prime factor.]

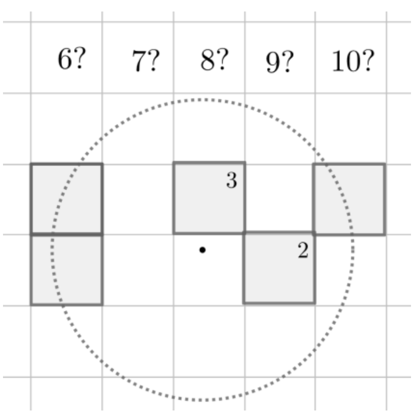
We can look at number types. Squares, cubes and higher powers occur when all blocks in a street perpendicular to the water are of equal height. Triangle numbers would make this nice pattern but for the fact that the dotted cells code composite numbers:

11.



We can focus the children's pattern-spotting if we imagine Number City at night. None of the buildings are identified: they are just towers of cubes, as in the model. A helicopter directs a searchlight beam downwards. How can we tell where we are?

12.

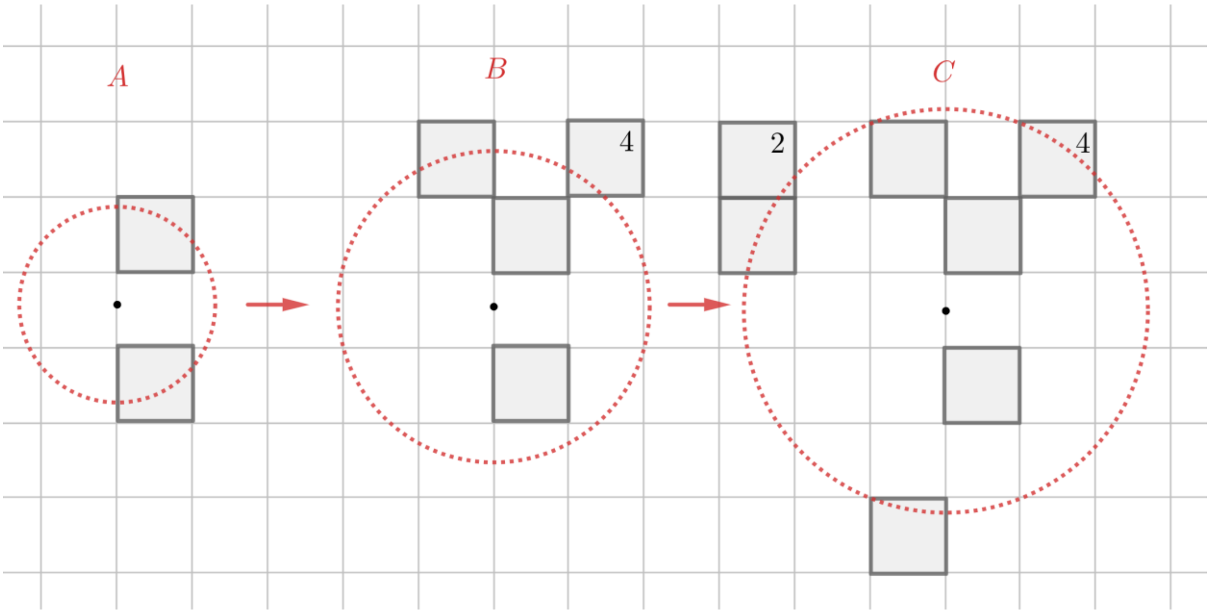


12 is an example. The small numbers show the heights of the towers. From the distances between repeats we see that the upper row is that of the 2s, the lower row, the 3s. (Also, only the 2s and 3s occupy consecutive rows.) We guess that we have a $2^3 = 8$ followed by a $3^2 = 9$.

These are the smallest possible numbers. If we move on in multiples of $2^3 3^2 = 72$, we shall soon find another match. $72 + 8 = 80$ and $72 + 9 = 81$ give us powers which are too high. But $2 \times 72 + 8 = 152$ and $2 \times 72 + 9 = 153$ fill the bill.

In **13** the helicopter rises, casting a broader and broader circle of light. In *A* we have no reason to favour one option over another. *B* tells us that the top two rows are the 2s and 3s respectively. This suggests a possibility, confirmed by *C*.

13.



Display and discuss one or two more examples. Then invite the children to devise their own examples to be shared with the class. For this purpose there are two possibilities: (a) They make physical drawings you put under a visualiser; (b) they make drawings on their tablets, using GeoGebra or other graphical software, for display via a data projector.

Overview

In fact every finite patch of the infinite pattern occurs an infinite number of times. But the pattern has no overall translational symmetry: it is aperiodic. As evidence of this, the number of primes is infinite. Thus, as we move along the counting strip, we keep meeting new primes.

Philosophically, the representation Number City is only possible because of The Fundamental Theorem of Arithmetic: every number has a unique representation in prime factors.

14. Here is a GeoGebra file prepared by Ben Sparks with which you can explore Number City virtually: www.geogebra.org/classic/jga6adzg .