

## A junior masterclass on the mystic rose with GeoGebra

Required:

For 30 children working in pairs:

- (15) laptops, loaded with **Geogebra Geometry** and the **.ggb** files for the session, labelled **Image 1, Image 2, Image 3, ...**, or, in some case, **Image 3.1, Image 3.2, ...** where additional files are needed.

For the instructor:

- one laptop loaded as above, but also with **.pdf** files for display only,
- a second laptop on which the instructor can monitor the detailed session plan.

All the computers networked and connected to a digital projector for display on a screen, the network being set up so that the work of a particular pair of children can be projected and if necessary corrected by the instructor.

Session structure:

There are a dozen ‘images’. Each part of the session begins with the instructor displaying for discussion the image on which the children are to work for the next 10 minutes or so, using their **.ggb** files. These are **.pdf** files labelled **Image 1, Image 2, Image 3, ...**, or, in some cases, **Image 3.2, Image 3.2, ...** where additional, explanatory graphics are needed.

Procedure for each **Image**:

Start:

- Open **Geogebra**
- Drag and drop **Image 1** into window

End:

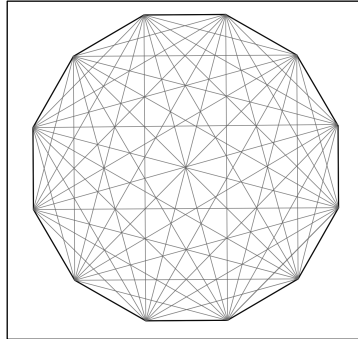
- Export completed file to desktop

Detailed session plan follows.

End of session: create folder, label **[Name]’s mystic rose**, store all the files.

N.B. The answers in blue are given on the teacher’s level, not how the children are likely to respond.

**Image 1**



**Image 1** .pdf projected.

Can you describe what you're seeing? You're describing it to someone over the phone. Your description must be so clear that the person at the other end could draw it.

Regular polygon. 12 sides. Every vertex joined to every other vertex so that polygon has all its diagonals. This means 11 lines come out of each vertex, 1 less than 12: two sides and 9 diagonals.

Bring up **Image 1.1** .ggb file, (which has order 12 rose on the left, the right side blank).

Choose a vertex. Starting at a side going out clockwise from that vertex, count round to a diagonal of your choice, the 3<sup>rd</sup> say.

Choose **Basic Tools: Move**. Click on your diagonal. Choose a colour.

Do the same for every 3<sup>rd</sup> diagonal round the rose.

If you've done that correctly, you should have a design which could be turned to any of 12 positions and you wouldn't know it had been turned. We say the design has *rotation symmetry, of order 12*.

The rose also has *mirror symmetry*. Colour a diagonal which crosses the centre. Choose **Construct: Midpoint**. Choose a pair of opposite sides for midpoints. Choose **Basic Tools: Segment** and join them. Colour it. You now have two symmetry axes. How many are there altogether?

12, 1 for each of 6 pairs of parallel edges, 1 for each pair of opposite vertices.

Choose **Polygons: Regular polygon**. Start it on the right. Choose 11 sides. Adjust it to fit page.

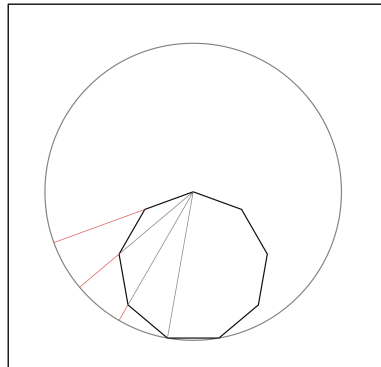
What main differences would there be if you clicked on **Basic Tools: Segment** and drew all the diagonals to make a rose of order 11?

No diagonals through centre.

How can you draw a symmetry axis?

Vertex to opposite midpoint.

**Image 2**



**Image 2** .pdf projected.

Describe what you see.

A circle. A (9-sided) regular polygon sits with 1 vertex at the centre, 2 on the circumference. 4 radii of the circle are drawn at equal angles. A side and 3 adjacent diagonals of the polygon lie in these.

Bring up **Image 2** .ggb file.

Choose **Basic Tools: Segment**. Make the figure symmetrical about a vertical mirror line. Use **Basic Tools: Segment** to join the ends of the radii to begin a bigger polygon. Estimate how many sides you think it'll have.

The circle has twice the diameter of the circumdiameter of the 9-gon, so  $9 \times 2 = 18$ .

Choose **Construct: Perpendicular bisector**. Bisect bottom edge. Choose **Construct: Perpendicular line**. Put line on perpendicular bisector just drawn and move to circle centre. Click on **Construct: Perpendicular bisector**, then **Edit: Hide**, then **Edit: Move**.

This defines 9 sectors in lower half, so answer:  $9 \times 2 = 18$ .

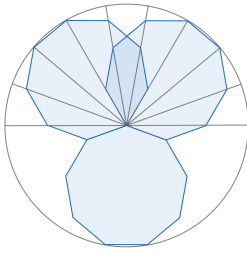
What is the angle of the sector?

$$360^\circ \div 18 = 20^\circ.$$

We could draw the rest of the radii by extending the existing radii across the centre as diameters. What if we're not allowed to do that?

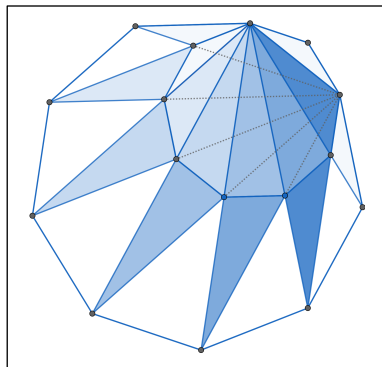
We know we can make the correct angles in the 9-gon so we can use that as a template.

Build 9-gons on the uppermost edges of the existing one, using **Polygons: Regular polygon**.



**Image 3:** seniors only

**Image 4**



**Image 4** .pdf projected.

What do you see?

2 9-gons, one within the other. A 1<sup>st</sup> diagonal of the small is a side of the large. Two sets of diagonals of the small lie over diagonals of the large. Are triangles of the same shade congruent?

Bring up blank page.

We're going to make a copy. How do you suggest we do it?

Draw the small one. Use a 1<sup>st</sup> diagonal of the small as a side to make the large.

Do so, using **Polygons: Regular polygon**.

Triangles of the same shade are in fact congruent. Can you see how to move one onto its copy?

Rotate about a vertex of the small 9-gon.

Choose **Polygons: Polygon**. Go round one of the triangles inside the small 9-gon. Choose **Transform: Rotate around point**. Click on your triangle. Click on vertex. Choose 140°. We'll see later why that is the correct angle.

Notice that the angle on the inside was carried round to the outside. Use **Basic Tools: Point** and put dots to show the two equal angles.

Choose **Polygons: Polygon**. Go round the triangle formed from your new triangle and the one inside it.

Because of the two equal angles what do you know about this new triangle?

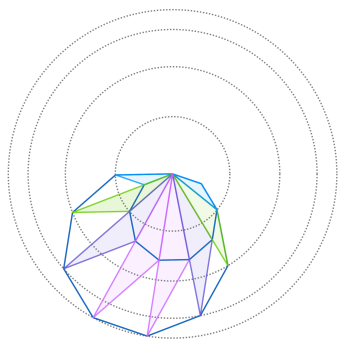
It's isosceles.

Put dots all the way round the top vertex of the little 9-gon, which are all equal. We saw that from the **Image 2** activities.

If you chose a different triangle to start with from your neighbour but the final isosceles triangle has the same base angle as your neighbour's, what can you say about all the isosceles triangles we've made?

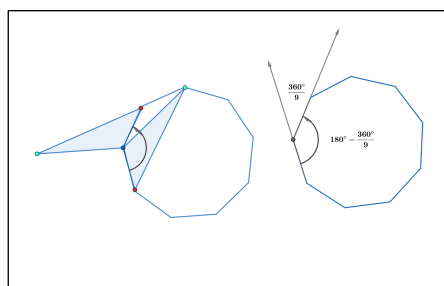
They're similar.

Choose **Transform: Translate by vector**. Click on the big 9-gon and move it away to the right. Choose **Transform: Translate by vector**. Move the small 9-gon to its original position on top of the big 9-gon. Click on **Basic Tools: Move**. Click on one of the 9-gons, then on the paint box symbol, and reduce the colour density to zero so that it goes white. Do the same with the other. Using the **Polygons: Polygon** tool again, draw in all the isosceles triangles (there are 8), giving each a different colour. But then use **Basic Tools: Circle with centre** and click on the top 9-gon vertex each time to find which triangles fit the circle and are therefore the same size, and give triangles the same size the same colour.

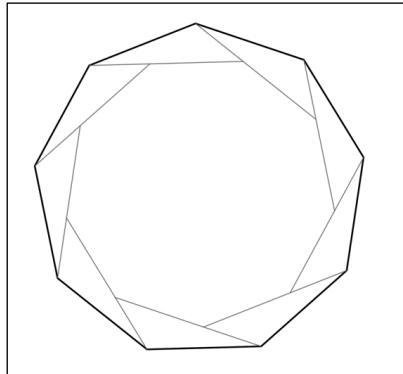


Now we must explain the  $140^\circ$ .

**Image 4.1** .png projected.



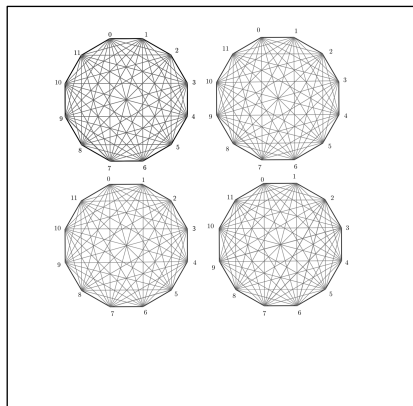
**Image 5**



**Image 5** .pdf projected. What do you see?

The same diagonal from each vertex is drawn but, proceeding clockwise, each new diagonal cuts off part of the previous one. Each diagonal is like the blade of an iris diaphragm on a camera. A mirror image would be an anticlockwise version. If we superimposed the two, we would produce a smaller 12-gon centred on the first, whose vertices are the intersections of those diagonals.

Bring up **Image 5.1** .ggb file, (which has order 12 roses in the four quadrants, each with vertices numbered anticlockwise 0 to 11).



*Take the top left 12-gon.*

Use **Polygons:Regular polygon**. Click on the intersection of 1<sup>st</sup> diagonals from adjacent vertices. Go anticlockwise till you reach the next such intersection. Choose '12'.

How could you make a smaller 12-gon?

Click on the intersections of 2<sup>nd</sup> diagonals from every 2<sup>nd</sup> vertex.

... and a smaller one still?

Click on the intersections of 3<sup>rd</sup> diagonals from every 3<sup>rd</sup> vertex.

Complete the sequence.

*Take the top right 12-gon.*

Instead of nested 12-gons which spiral round, we'll see if we can pick out regular hexagons. We begin by locating single vertices of the hexagons.

Choose **Basic tools: Point**.

Mark a vertex 0 of the 12-gon.

Mark an intersection of 2<sup>nd</sup> diagonals from vertices 0 and 1.

Mark an intersection of 3<sup>rd</sup> diagonals from vertices 0 and 2.

Mark an intersection of 4<sup>th</sup> diagonals from vertices 0 and 3.

From each of those points, go  $\frac{1}{6}$  of a whole turn,  $60^\circ$ , anticlockwise and mark a second point.

Using those point pairs as sides of your hexagons, use **Polygons: Regular polygon** to draw the sequence of hexagons.

Had we started everything at vertex 1, we would have got a second set of hexagons.

*Take the bottom left 12-gon.*

You might be able to pick out squares just by sight. Which vertices does the biggest join?

0-3-6-9 or 1-4-7-10 or 2-5-8-11.

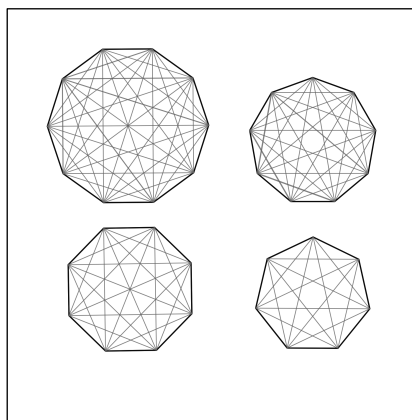
See if you can spot the right-angled corners.

In addition to the centred squares, see if you can trace a square with a vertex at the centre of the 12-gon and another, slightly smaller one, off-centre altogether.

*Take the bottom right 12-gon.*

Try equilateral triangles. There are only 2 sizes of centred equilateral triangle, but there are 5 off-centre.

Bring up **Image 5.1** .ggb file, (which has an order 10 rose top left, an order 9 rose top right, an order 8 rose bottom left, an order 7 rose bottom right).



Predict which regular polygons you can pick out in each case and try to draw them.

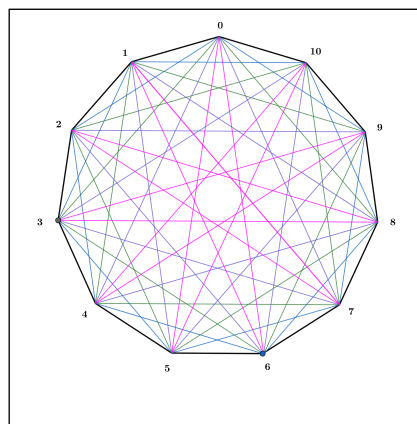
Number of sides:      10: 10, 5      9: 9, 3      8: 8, 4      7: 7

What is the rule?

The small number must be a factor of the large.

(You should find an off-centre equilateral triangle in the regular 9-gon.)

**Image 6**



**Image 6** .pdf projected.

This would be a rose of order 11, but the diagonals are coloured according to a scheme. Fix on one colour at a time. What rule do you think it follows? Discuss your ideas with a partner. What is the overall scheme into which your colour's particular rule fits?

The secret lies in the vertex numbers. For example, the lilac lines run in sequence 0 - 4 - 8 - 1 - 5 - 9 - 2 - 6 - 10 - 3 - 7 - 0. We add 4 each time but modulo 11. The figure it draws is the *star polygon* {11,4}. The scheme comprises all the star polygons which can be drawn in the order 11 rose: {11,2}, {11,3}, {11,4}, {11,5}.

Why is there not an {11,6}?

This would be {11,5} drawn in the opposite sense (anticlockwise).

What are the matches for {11,7}, {11,8}, {11,9}?

{11,4}, {11,3}, {11,2} respectively.

Starting at vertex 0, how many times does a star polygon go round the centre before it arrives back at its starting point? Fix on a particular example and see if your hypothesis stands up for the others.



$\{11,2\}$  goes round twice,  $\{11,3\}$  three times, etc.

How many vertices do all the star polygons based on the order 11 rose have? Check a few cases.

11, the same as the regular polygon itself.

Ordinary regular polygons have an interior angle: the equilateral triangle  $60^\circ$ , the square  $90^\circ$ , etc. But, if we imagine a vector drawing them as it moves along and turns at the vertices, we get the supplements of these angles:  $120^\circ, 90^\circ$ , etc. We call these *turning angles*.

Bring up **Image 6.1** .ggb file, (which has an order 12 rose with an inset key showing how to measure turning angle).

Pick out the star polygon  $\{12,5\}$  in colour in the order 12 rose. Choose **Measure: Angle**.

Measure the turning angle as shown.

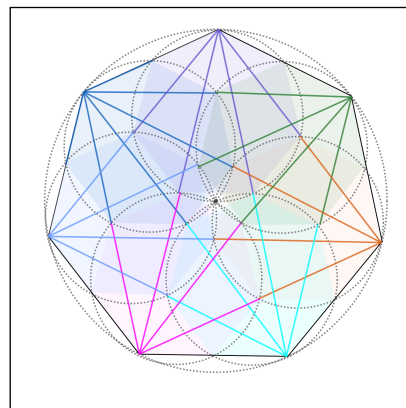
Multiply by 12 for the number of vertices.

Divide by 5.

Can you explain your answer?

Since  $\{12,5\}$  goes 5 times round the centre, the vector makes 5 complete turns of  $360^\circ$ . Each turning angle must therefore be  $\frac{360^\circ \times 5}{12}$ , which gets us back to the angle we measured. The inverse sequence of operations - multiply by 5 and divide by 12 - is what we've just done.

**Image 7**



**Image 7** .pdf projected.

What do you see?

*Upper figure*

7 circles. A diameter of each is a radius of a bigger circle. The outer ends of the diameters are the vertices of a regular 7-gon, of which the bigger circle is therefore the circumcircle. The diagonals of the big 7-gon are drawn. Each half is a diagonal of a smaller 7-gon, of which a

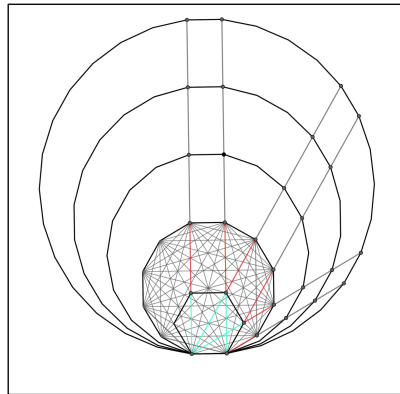
smaller circle is the circumradius. Alternatively we can think of the big figure as a small figure enlarged by scale factor 2 from one of the outer vertices. The small 7-gons are coloured. This reveals that they overlap in kites, of which a longer side is a side, a shorter side half a side, of a small 7-gon.

*Lower figures*

The two triangles are congruent. Likewise the two isosceles trapezia. How do we know?

If we were to number the vertices of each 7-gon, we would see that the same vertices belong to each shape.

**Image 8**



**Image 8 .pdf projected.**

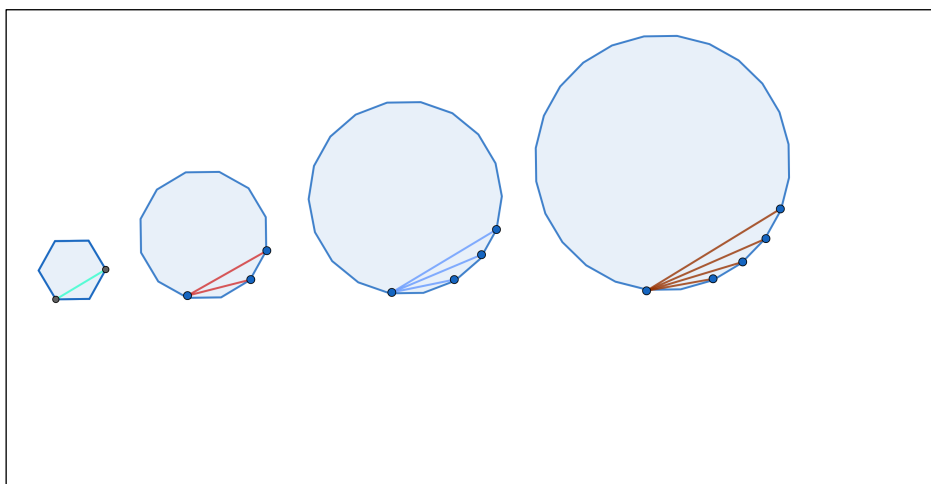
What do you see?

Regular 6-, 12-, 18-, 24-gons share an edge. Certain diagonal pairs can be traced right across the figure.

So we must try to explain the tramlines. There is a clue in the figure. Trace diagonals from the bottom left vertex. Any ideas?

There are two 12-gon diagonals for every one 6-gon diagonal.

Bring up **Image 8.1 .ggb** . We'll begin by writing above it the angle in degrees between adjacent diagonals for each shape.

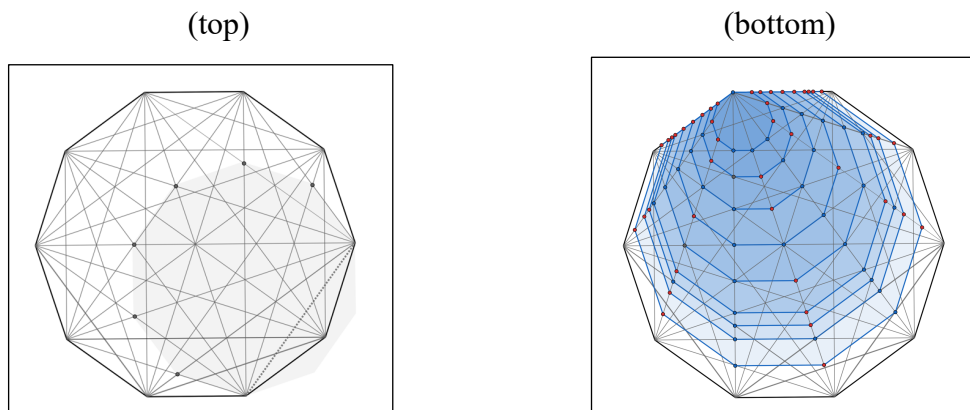


Choose **Media: Text**. Click above the hexagon and write ' $180 \div 6 = 30'$ '. Do the same for each shape.

Choose **Transform: Translate by vector**. Click on one of the coloured diagonals to the left and move it onto one to the right. Do this for several. Why is this always possible?

We can always make up the  $30^\circ$  with multiples of the other angles:  
 $4 \times 7\frac{1}{2}^\circ = 3 \times 10^\circ = 2 \times 15^\circ = 30^\circ$ .

**Image 9**



**Image 9** .pdf projected.

What do you see at the top?

2 10-gons, the vertices of the smaller one coincide with interior vertices of the larger.

What do you see at the bottom?

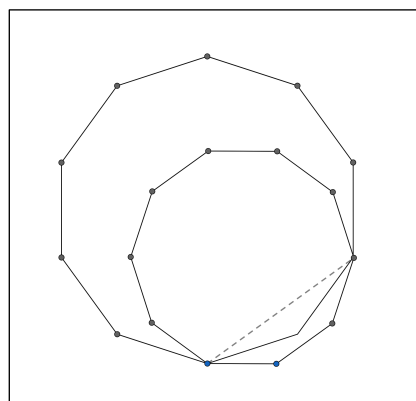
10 10-gons, a series of enlargements from one vertex. The blue dots show interior vertices; the red dots do not fall on an intersection. On two diagonals all dots mark interior vertices.

So all the black dots in the top figure are like blue dots in the bottom figure. What is the difference between the two figures?

In the top figure the small 10-gon shares 2 vertices with the large one; in the bottom figure it shares just 1.

We'll try to find out what's special about the top figure.

Bring up **Image 9.1** .ggb .



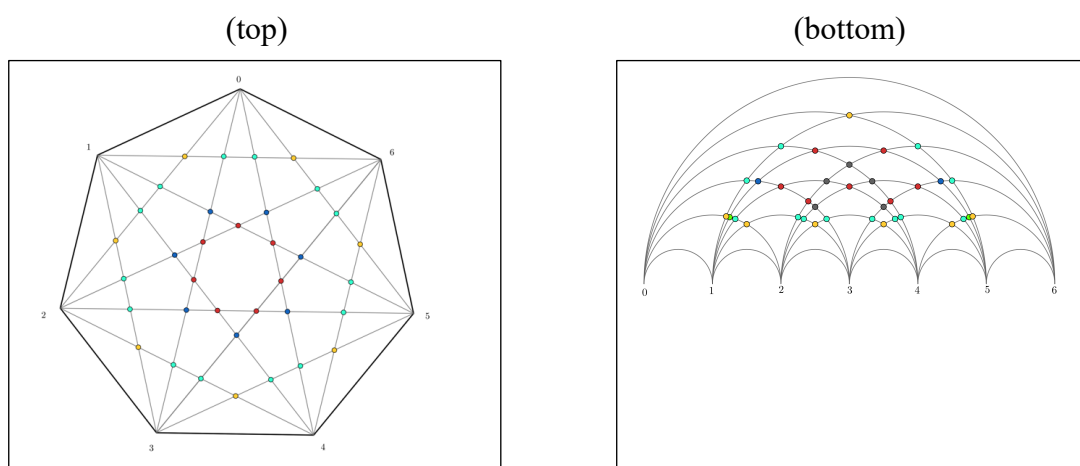
With **Basic tools: Move** colour the bottom one of the two vertices shared by the two shapes red; colour the top one blue. Using **Basic tools: Segment**, join the red point to each vertex of the big shape. With the **Basic Tools: Move** tool colour the lines red. In the same way, make blue lines from the blue vertex to each vertex of the big shape. What do you find?

Every vertex of the small 10-gon inside the big 10-gon lies on both a red line and a blue line. So it is an interior vertex of the big 10-gon.

Why do diagonals of the small 10-gon lie on top of diagonals of the big 10-gon?

Because the angles between them are all equal and all the same and the two shapes start with a shared diagonal.

**Image 10**



**Image 10** .pdf projected.

*The upper figure*

What do the colours code?

We could draw circles, concentric with the centre, on which each set of coloured dots fall: they mark points at equal distances from the centre.

Count the dots of each colour. What do you find?

The numbers are multiples of 7.

How do you explain this?

Every dot is multiplied by 7 due to the order 7 rotation symmetry. If a dot lies off a symmetry axis, its number is further multiplied by 2 because of reflection symmetry.

*The lower figure*

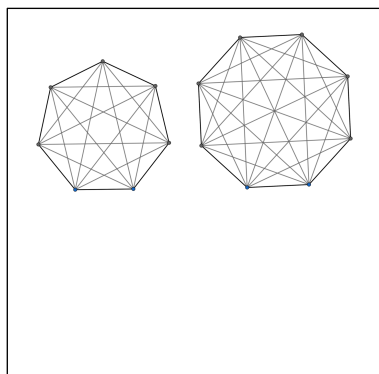
What are the differences between the upper figure and the lower one?

In the lower one the straight lines of the diagonals have been replaced by semicircles. But the polygon itself, which lies on a circle in the upper figure, has been broken at vertex 0 and straightened out. The edge between vertices 0 and 6 has become the biggest semicircle.

The points where the diagonals cross in the upper figure are the points where the semicircles cross in the lower one.

**Images 11, 12: seniors only**

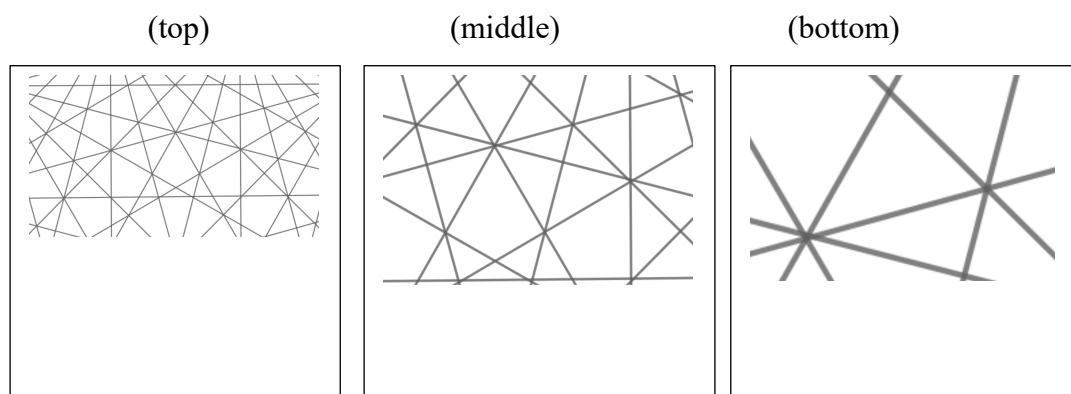
**Image 13**



**Image 13** .pdf projected.

Bring up **Image 13** .ggb . Using **Polygons: Polygon** colour the regions in such a way that no two regions sharing an edge have the same colour. Use as few colours as possible - just two if possible!

**Image 14**



**Image 14** .pdf projected.

What d'you think the three pictures show?

Zooming into a rose in three stages.

To find out which rose it is, the easiest picture to use is the top one. Why?

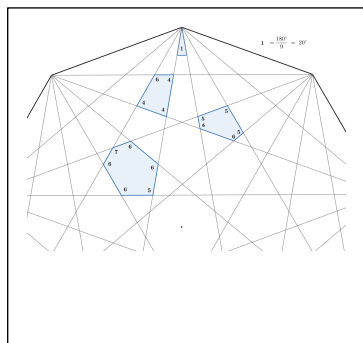
There are lines which look as if they converge on the centre or on a vertex. We can measure the angle between lines which form the smallest angle and divide that angle into  $180^\circ$ .

There are such lines in the middle picture. How could you check that they are the ones you need?

Look at the shapes either side. Identify two lines which would carry the same shapes round with them under a rotation.

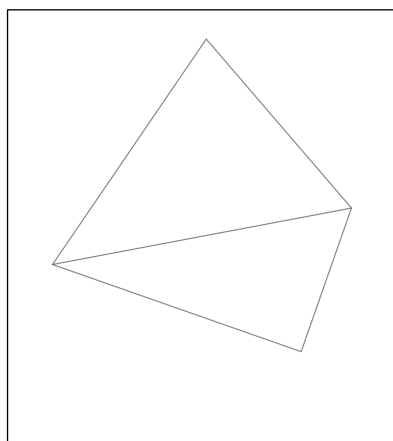
The bottom picture is the hardest, but there's a principle we can apply.

**Image 14.1** .pdf projected.



The principle is that every interior angle is a multiple of the smallest one. And the smallest one is the angle between two adjacent diagonals at a vertex. So here we have a 9-gon. Our unit, our smallest angle, is  $180^\circ \div 9 = 20^\circ$ . We just call this **1**. So, in the figure, **4** means  $4 \times 20^\circ = 80^\circ$ . The problem is of course that, in the example we are studying, we're working backwards. We can measure the angles, but then we have to work out what the unit is. We do that by finding the numbers' highest common factor. Look at the quadrilateral. We would have measured  $80^\circ, 120^\circ, 80^\circ, 80^\circ$ . Their highest common factor is  $120^\circ - 80^\circ = 40^\circ$ . But now look at the hexagon. Here we would have measured  $100^\circ, 120^\circ, 120^\circ, 140^\circ, 120^\circ, 120^\circ$ .  $140^\circ - 120^\circ = 120^\circ - 100^\circ = 20^\circ$ . So it looks as if  $20^\circ$ , not  $40^\circ$ , is the angle we're after.

Bring up **Image 14.2** .ggb. So try.



Use **Measure: Angle**. Click on the two arms of your chosen angle. (You may get the *supplement* of the angle you want and have to subtract from  $180^\circ$ .)

The right triangle yields an h.c.f. of  $30^\circ$ , the quadrilateral,  $15^\circ$ . So we guess that our polygon has  $\frac{180^\circ}{15^\circ} = 12$  sides.

Paul Stephenson

1<sup>st</sup> draft

23.5.21