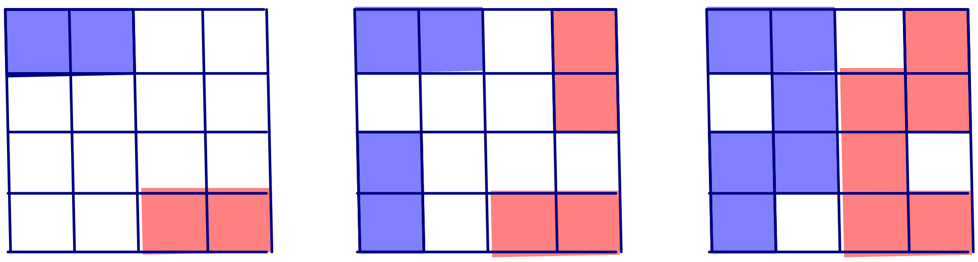
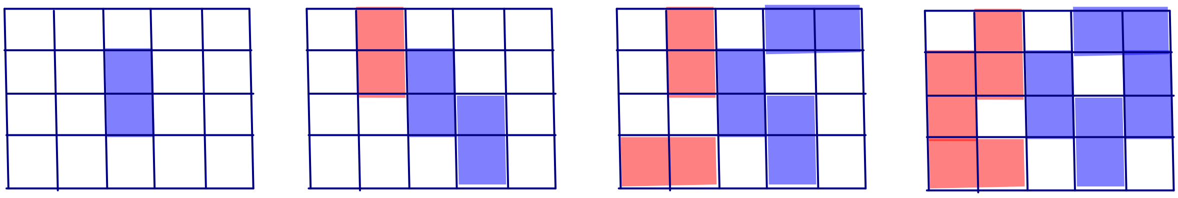
More on ‘Domino Block’

Some years ago I described the following game for 2 players (generally known as ‘Cram’). There is a 4 x 4 grid and each player has 4 dominoes. They lay their dominoes alternately to cover pairs of adjacent squares. The winner is the last to be able to do so. It turns out that the second player wins by complementing the position of the first to produce a design with half-turn (point) symmetry. Blue starts:



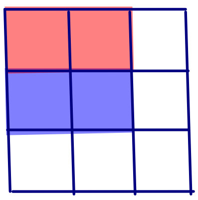
Recently a year 7 pupil asked what happens if you add a column to the grid and each player has 5 dominoes. We found that the first player (blue) can steal the strategy of the second (red) by laying his/her domino bang in the middle:



A game with 6 dominoes each on a 6 x 4 board looks like the first; a game with 7 dominoes each on a 7 x 4 board, like the second. What about an odd number each way?

We’ll start with a 3 x 3 board. (As you can work out, each player needs just 2 dominoes.)

This is a win for Red, just by placing a domino alongside Blue’s first:

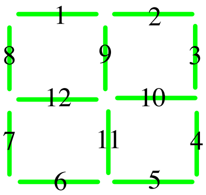


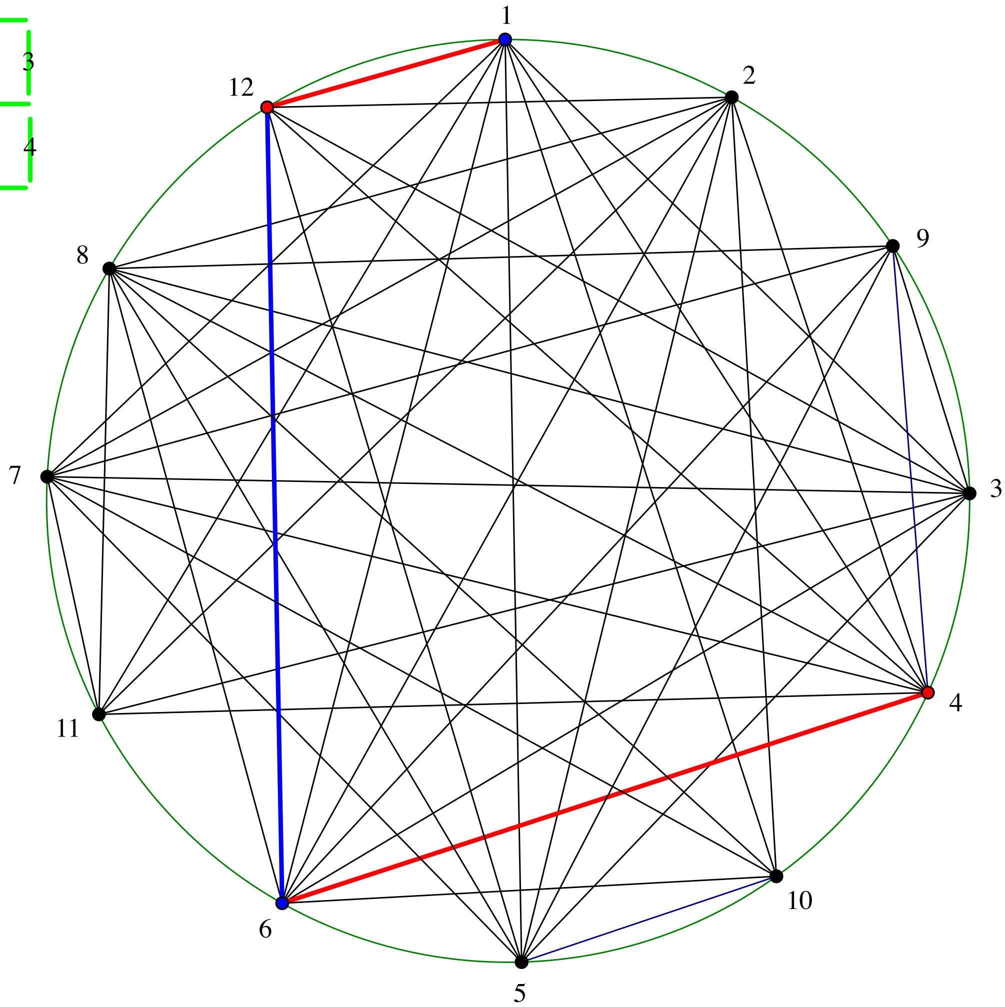
There are 4 possible moves for Blue, but each leaves room for Red’s domino.

What happens on a 5 x 5 board?

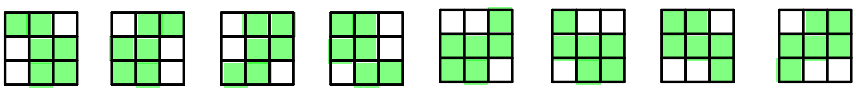
Make your own game with coloured card and squared paper. After an hour or so’s play, you’ll realise how complicated a game this is. When we can’t deal with a difficult problem, we try a simpler one. Can we learn more from the 3 x 3 game? Yes, if we transform it in some way.

Here is a graph in the form of an incomplete ‘mystic rose’. The inset diagram shows the domino positions. The state of the board following Blue’s first move corresponds to a node at the edge of the circle. In a game you trace a path on which each arc is a move. We’ve picked out a game in which Red uses our simple ‘parallel’ strategy:

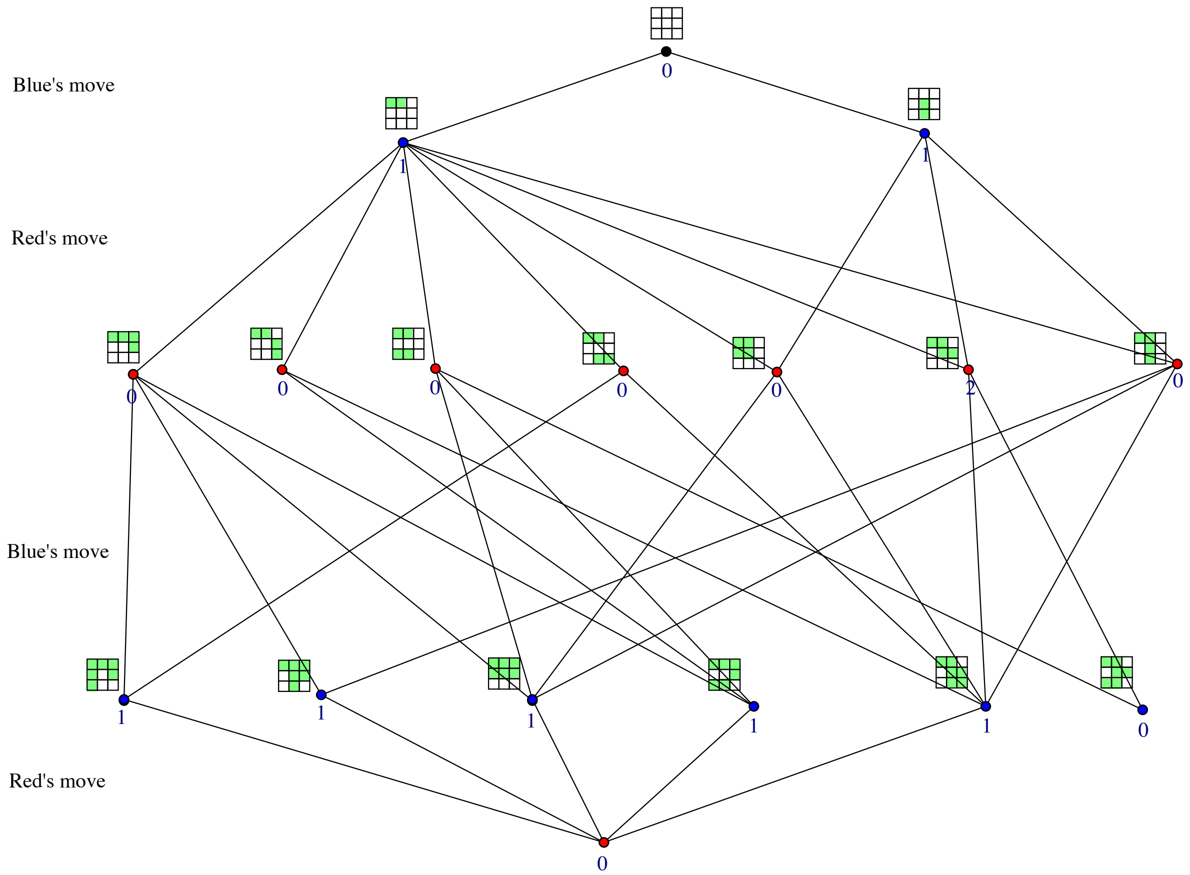




The graph is complicated but notice the symmetries. These symmetries are inherent in the game. So we must be able to simplify the graph representing it. So far we’ve used colour to show whose domino has been placed. But, for the next player, all that matters is the shape the dominoes make. We’ll use green for this. Also, it doesn’t matter which way round the board is. Think of the green shape drawn on an acetate sheet, which you can turn or flip over, using all the symmetries of the square. Thus, for example, these 8 states of the board are one and the same:

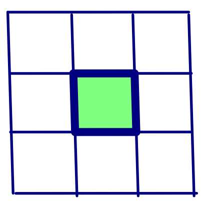


We now make a graph, to be read from top to bottom, showing the progress of all possible games. As before, a node is a state of the board; an arc, a move:



We now label the vertices with SG numbers (after the mathematicians Sprague and Grundy, who devised the system). A ‘0’ means that player has either already won, or can move in such a way that he can win by moving to another vertex labelled ‘0’ on the next go. By contrast, the unfortunate player who finds him- or herself on a different number cannot reach such a vertex. The scheme works like this. You use the set of numbers {0, 1, 2, 3, …}. The ‘victory’ nodes receive a ‘0’. Work back along an arc. Label the next node with the smallest number which doesn’t label neighbours immediately downstream. Thus, if the only downstream node is labelled ‘0’, the new node receives a ‘1’. Note the particular node on our graph labelled ‘2’. It receives that number because ‘0’ and ‘1’ are immediately downstream. Upstream of that is a node labelled ‘1’ It receives that number because ‘0’ and ‘2’ are used, so the smallest number available is ‘1’. Right at the top we have a ‘0’. This means that Red can win a game on a 3 x 3 board. All the player need do is follow a route which alternates ‘0’ and other numbers (just ‘1’s in our case). Notice that, of the 10 possible first moves for Red, only one is disastrous. (Here the board ends up with 3 empty squares, as opposed to 1 in the other cases.) *Trace our original solution on the graph.*

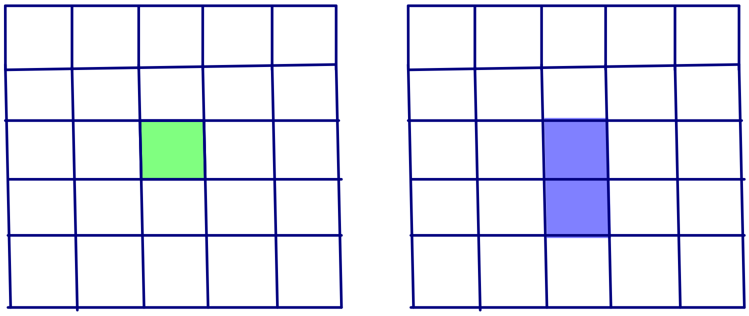
Try your hand at graphing a simpler game: a 3 x 3 square with the centre square out of bounds:



We know immediately that Red can win by half-turn symmetry matching, *but see how this route appears on your graph.*

Simple though the 3 x 3 games are, by taking a cross-section of our last graph at the level of Blue's second move, you find that it cuts 18 arcs, each representing a different game. Imagine then how complicated the 5 x 5 graph must be. In 2009 the mathematician Martin Schneider ran a computer program which showed that the top SG number in a 5 x 5 graph is again '0', meaning that, if Red is lucky enough to have the graph to hand, he or she can trace at least 1 winning route (I would guess many) down the graph. The question is, how can you navigate without a map? As the game nears the end, you can look ahead and anticipate the other player's moves, but how should you begin? I know how Blue should begin:

Blue thinks: "If the centre square were out of play, we would have a game like the first, and Red would win. Therefore I must cover that square on my first move."



But what should Red do? Remember: the SG number tells us Red can win *whatever* Blue does. I've no idea, but perhaps you can find a new way of thinking about the problem.

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