Mobility

In this workshop we explore *machines* whose *mechanism* is a *linkage*. Our interest centres on the *mobility* of the linkage, which we quantify via Kutzbach-Grübler's equation.

For the children this provides a simple example of mathematical modelling and algebra as generalised arithmetic.

For work in 2 dimensions we use Meccano; in 3, Geomag. In both cases we anchor the piece to a baseboard, which provides our reference grid. The hydraulics which communicate forces via the sliders in a JCB jib are realised pneumatically in a Fischertechik model. The children can also use the baseboard to perform calculations.

Pictures of a dozen examples surround the whiteboard we sketch on. We refer to these throughout the session and invite the children to comment on them.

Th headings are as follows. In a morning of 2 $1\frac{1}{2}$ - hour sessions, the break might be taken between the 2-D and 3-D examples. **Introduction**, **Key concept: degrees of freedom**, **Combining degrees of freedom**, **2-D examples**, **3-D examples**.

The pictures. Key features are noted here:

A	Hurdler	achieves big vertical rotation about hips and knee joints
B	Shot putter	achieves big horizontal rotation about shoulders and hips
C	Gymnast	achieves big rotations about many joints
D	Dancer	likewise
E	Pianist	highly complex series of rotations at the service of finger pads, the interface with a second linkage, the hammer mechanism
F	Lazy tongs	employs slider; allows operation at distance from user
G	Umbrella	allows compacted sheet to be opened out to large area, employing parallelogram loop where lazy tongs chain rhombuses (cf. space vehicle antenna, hood on convertible car)
H	Anglepoise lamp	slider attached to spring to allow position to be held (cf. up-&-over door)
I	Wheelbarrow	simplest linkage: single lever; achieves mechanical advantage
J	JCB jib	makes much use of sliders, operated by oil pressure
K	Robot arm	up to 6 chained links, allowing great mobility, needs to pick the equivalent of a flower out of a nettle patch
L	Dentist's chair	lamp and drill head must have their own arms

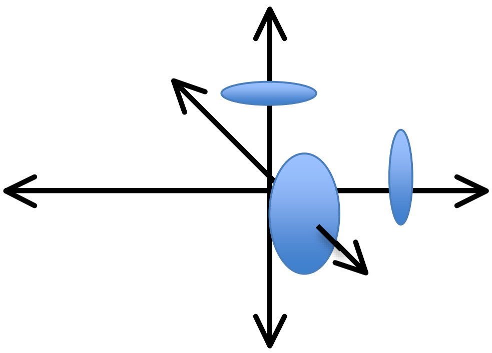
Introduction

All the pictures, even those of people, show *machines*. A machine is a device for transforming effort, either reducing it at the expense of having to move the point where the effort is applied through a big distance, i.e. gaining a *mechanical advantage* (case **I)** or moving the point applied to the load to a convenient point (case **F**).

A machine contains a *mechanism*, the thing which enables the machine to do its job. In all 12 examples this mechanism is a *linkage*, a system of levers. We shall refer to the levers as *links*, the connections as *joints*. **I** is the simplest possible linkage, a single lever. There's one joint, the wheel axle, and one link, the barrow. What we want to know is how *mobile* a linkage is, how great the rangle of positions it can adopt. **C** and **D** are highly mobile; **F** and **G** not so. But how can we put a number to this?

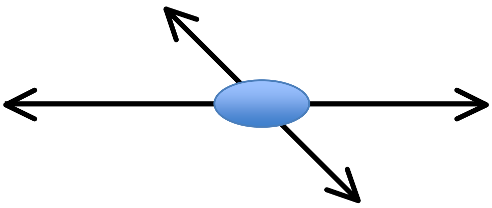
The key concept: degrees of freedom (DOF)

Ball with ends of one diameter marked	**E1a**	*Teacher demonstration* Here is a ball in space. It can move in any of these 6 ways - the disks represent planes of rotation:



It has 6 *degrees of freedom* (DOF). This is its mobility, the maximum possible for 3 dimensions.

Disk with one diameter marked	**E1b**	Here is a disk, which we shall confine to the horizontal plane. It can move in these 3 ways:



It has 3 degrees of freedom, the greatest number possible for 2 dimensions.

What is the maximum for 1 dimension? [1: movement backwards or forwards along a line]

Combining degrees of freedom

We relate a complete linkage to a fixed reference link. For the mobility of a complete linkage we imagine all *l* free links to have the greatest possible freedom and then subtract the factors which restrict the movement at each of the *j* joints.

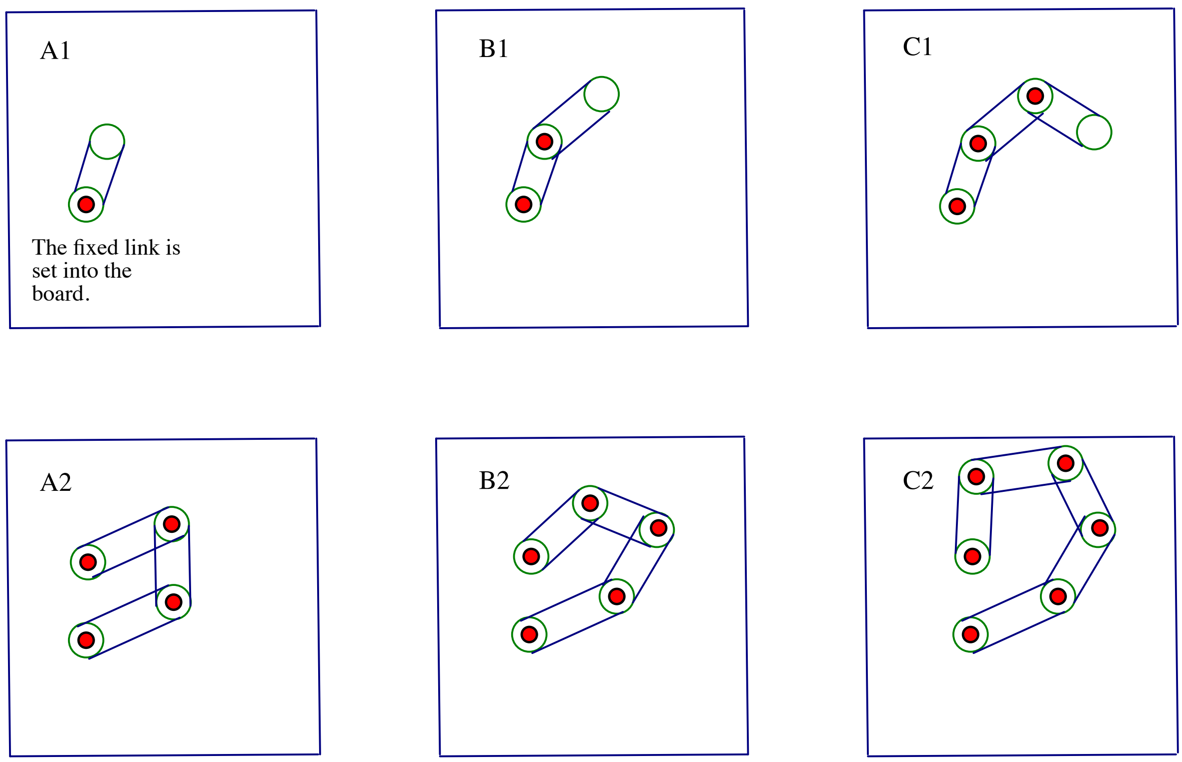
In 2-D therefore, we start with 3*l*. Because each joint only allows a rotation, therefore has DOF 1, we must subtract the constraint = 3 - 1 = 2 at each, giving a final figure *M* of ***3l* - 2*j*.**

In 3-D, we start with 6*l*. This time the rotation has DOF 3, so we subtract 6 - 3 = 3 at each joint, giving a final *M* of **6*l* - 3*j***.

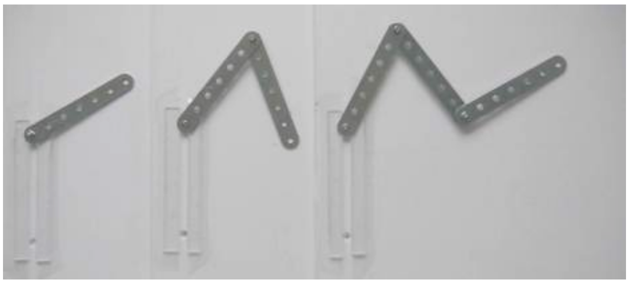
(If your parents ask what you've been doing this morning, you can tell them that you've been studying Kutzbach-Grübler's equation.)

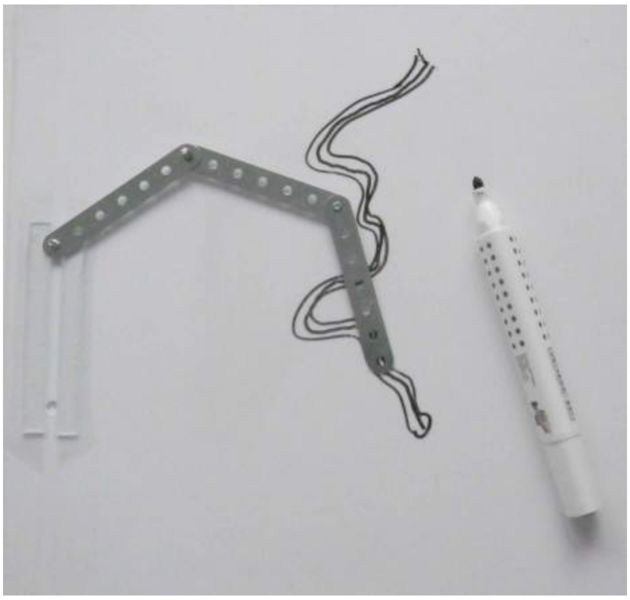
2-dimensional examples

Meccano on baseboard x 15	**E3**	*Pupil experiment*
	Build these in sequence. [Demonstrate with your own model.]	



The children insert a dry-wipe pen in a chosen hole and investigate the range of positions into which it can be brought and the route it can follow to get there. They should draw a twisty corridor they need to steer it down. They can imagine a long rivet gun descending vertically from the end which has to reach a point in a panel surrounded by a tangle of electrical wiring in a car assembly line.





What is the connection between l and j in cases A1, B1, C1? [$l = j$]. How does this simplify the equation?
Complete the table. [M = 1, 2, 3].

What is the connection between l and j in cases A2, B2, C2? [$l = j – 1$] Complete the table [M = 1, 2, 3].

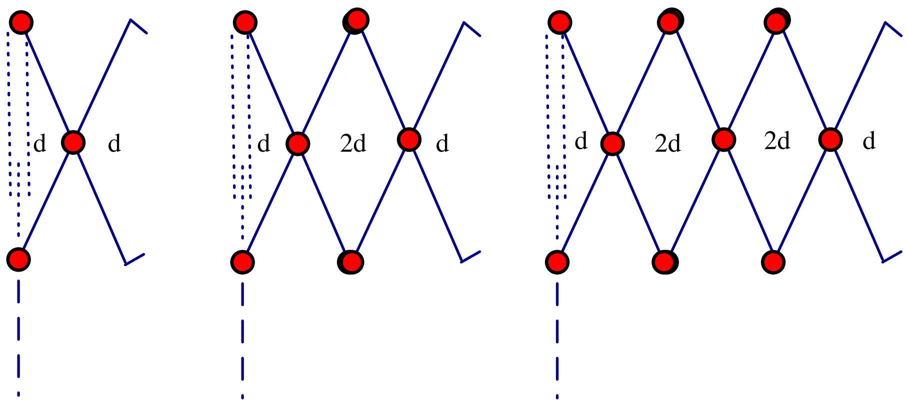
	$l = j$	j	$3l – 2j = j$		$l = j - 1$	j	$3l – 2j = j – 3$
A1	1	1		A2			
B1	2	2		B2			
C1	3	3		C2			

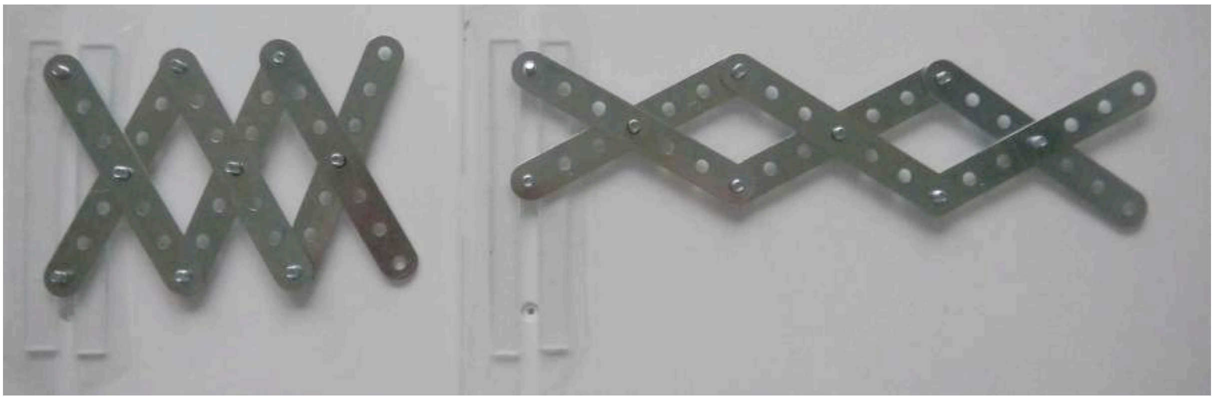
Here's a little non-experiment to test our formula. What happens if, instead of the 3 links in A2, we just have 2? [$3l – 2j = 3.2 – 2.3 = 0$. The result tells us what we already know: the triangle is a rigid shape.]

As we see, the closed loops are no more mobile than the open chains. B2, C2 are rarely used but A2 is common.We shall now take two examples featuring such a closed loop. In the first case the links form a rhombus; in the second, a parallelogram. Both use a slider. As the name implies, this is a joint which doesn't turn but slides. One link slides in another to shorten or lengthen. Because the movement is 1-dimensional, the joint has DOF 1, like a rotational joint. For the purposes of the equation it counts as a link + a joint.

Meccano **E4** *Pupil experiment*

These are called 'lazy tongs' or 'dog tongs' because they were used to restrain vicious dogs in the Middle Ages. There is a slider and the closed loops are rhombuses. Imagine the fixed, reference link as a handle in line with the slider. Add loops to your own model. What is the effect? What is the mobility? [project figure]





[Our formula is $M = 3l - 2j$.

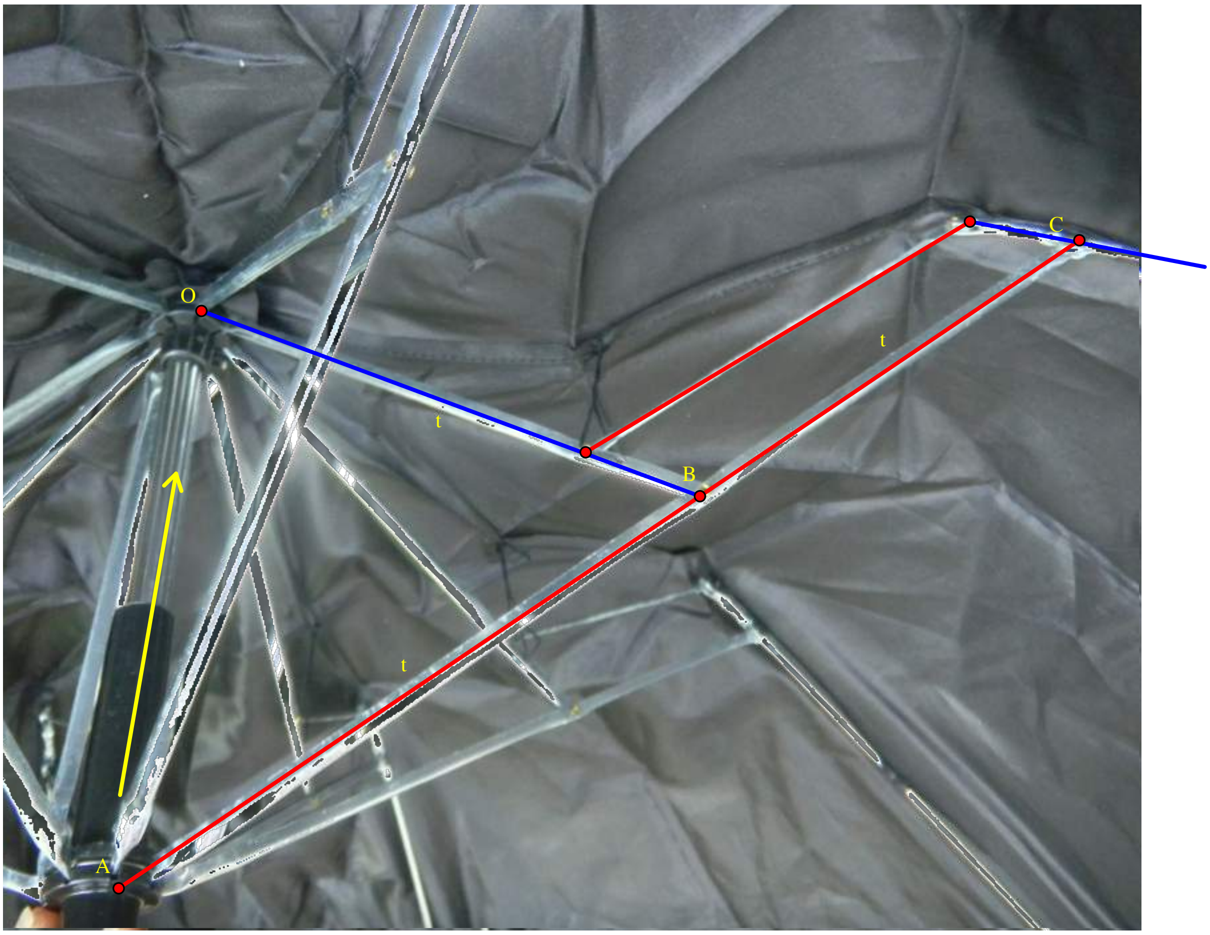
	1st model	2nd model	3rd model
$3l$	3(1 + 2)	3(1 + 4)	3(1 + 6)
$-2j$	-2(1 + 3)	-2(1 + 6)	-2(1 + 9)
	1	1	1

The linkage can only move in 1 dimension. However, the user has complete freedom to choose the direction and the jaws can be extended 2d, 4d, 6d, ...]

Umbrella, graphic

E5 *Teacher demonstration*

There are many different designs of umbrella. This is the simplest. [Demonstrate] Before you work it out, what are you expecting the mobility to be? [Project figure]

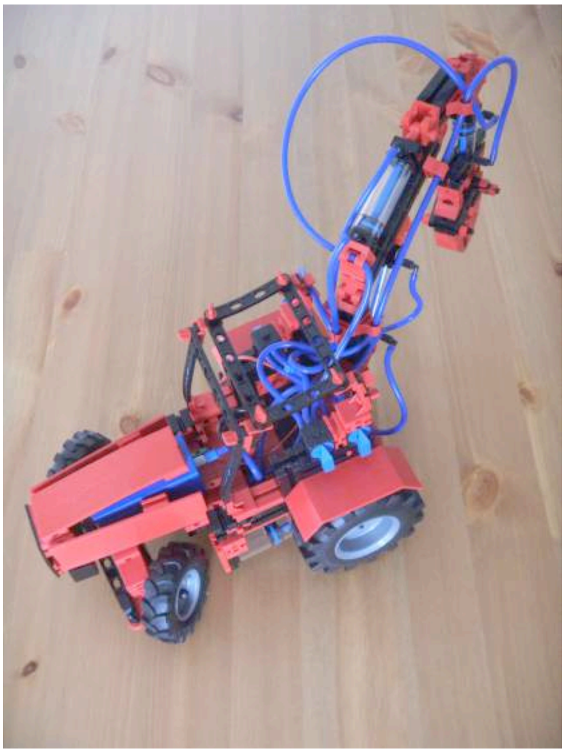


[The calculation is that for the second figure above: 3.5 - 2.7 = 1. Parallel links are coded by colour. B moves on the arc of a circle, centred on O. B is itself the centre of a circle, of which *AC* is the diameter. Angle *AOC* is therefore a right angle and the geometry confirms what we observe: *C* moves out from *O* along a straight line perpendicular to *AO*. It moves in the same way as the lazy tongs, even though here we have a parallelogram not a rhombus.]

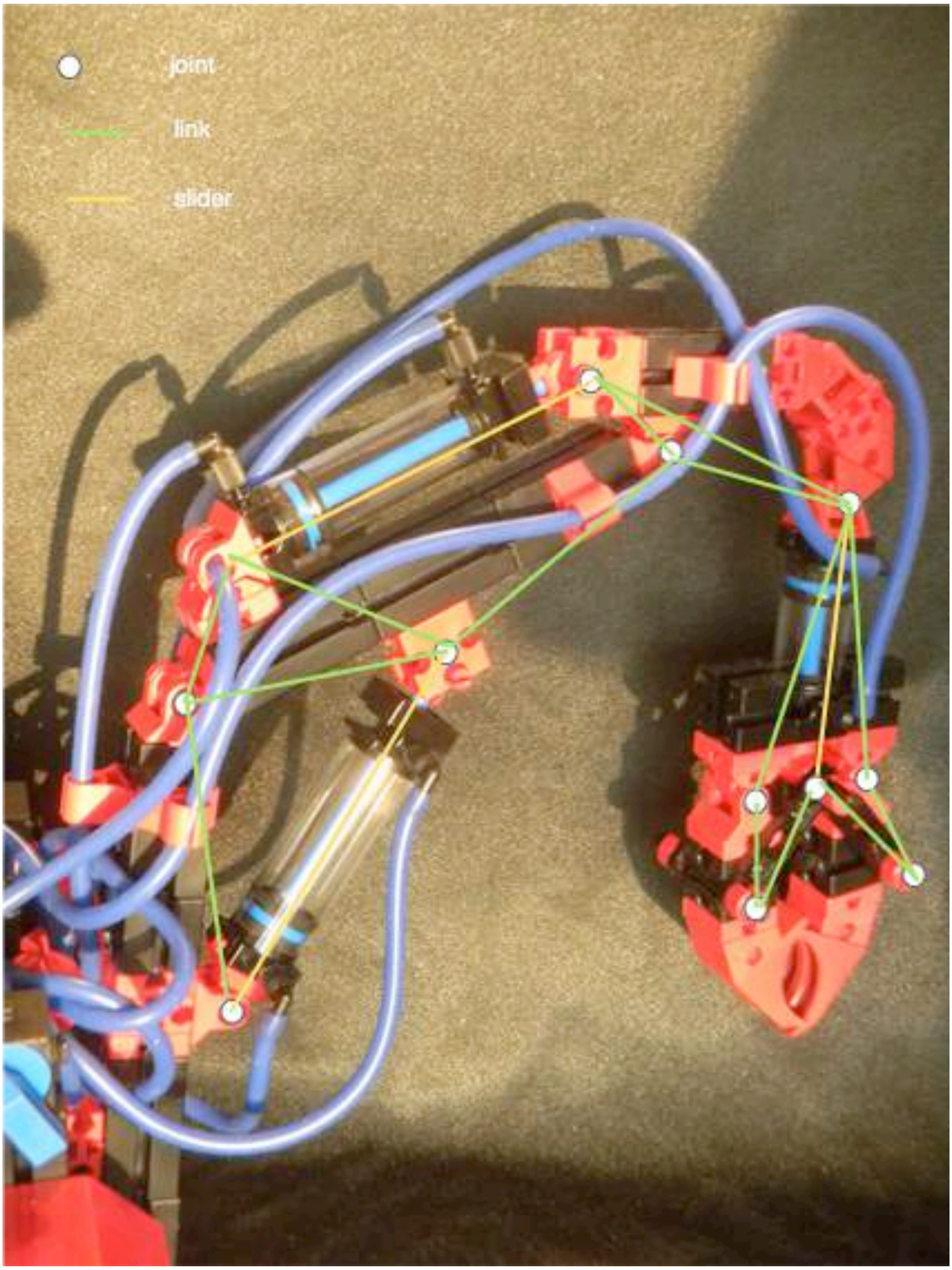
Fischer-technik model grab

E6 *Teacher demonstration*

[Demonstrate the model in action, explain about pneumatic (and by analogy hydraulic) transmission systems, project the labelled picture, discuss the features and get the group to do the arithmetic.]



In fact the grab can move in 3 dimensions: it can rotate in the horizontal plane. But our interest here is the interrelation between joints, links and sliders in the arm itself:



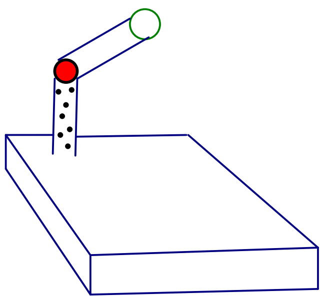
A triangle is a rigid shape so, where three links form a triangle, the piece moves as one and can be considered a single link. On that basis there are 11 joints, 10 links, 3 sliders. Since 1 slider = 1 joint + 1 link, that makes 14 joints, 13 links. M = $3l - 2j$ = 3(13) - 2(14) = 11.

3-dimensional examples

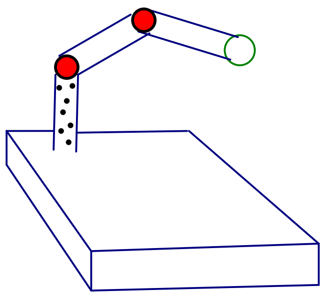
Geomag in base-board **E7** *Pupil experiment*

Build these in sequence [Demonstrate with your own model]:

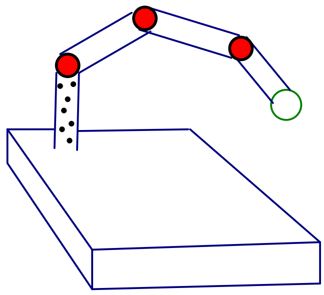
A1



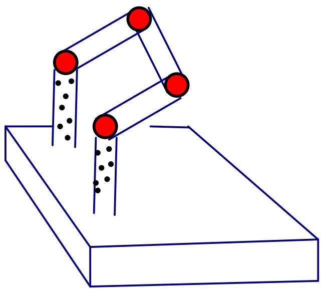
B1



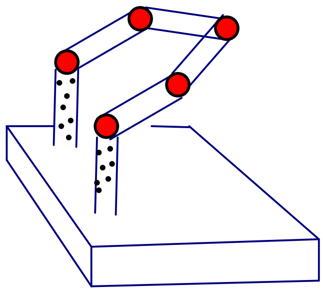
C1



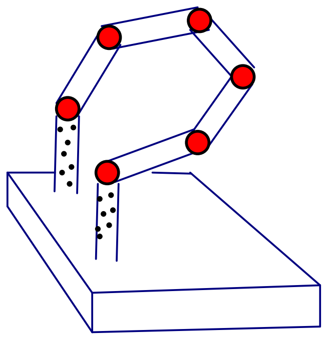
A2



B2



C2



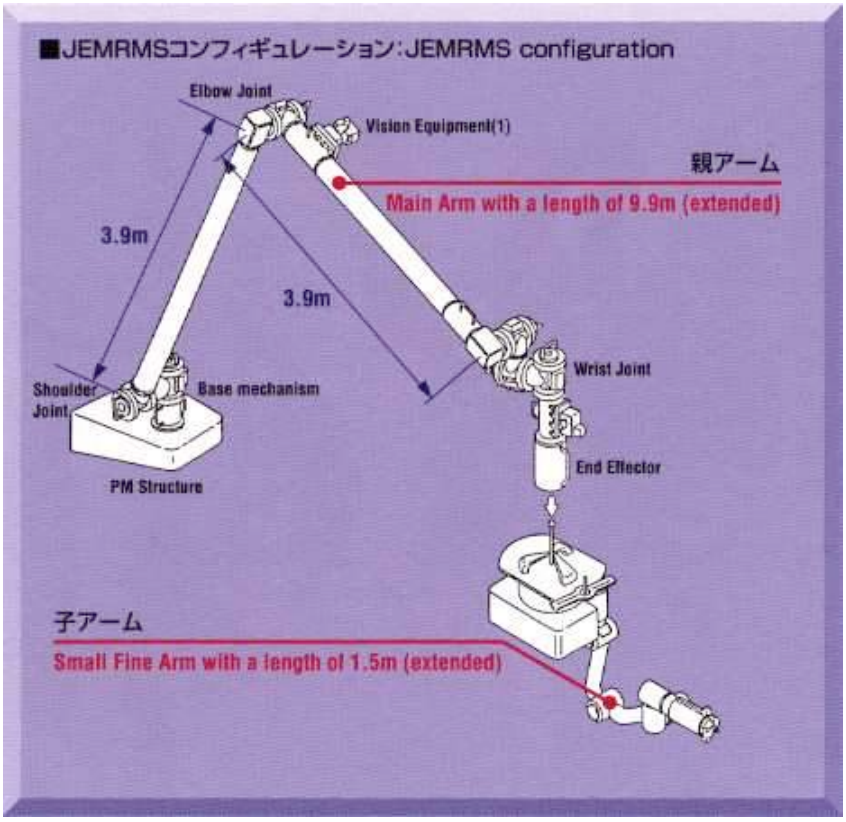
The connection between *l* and *j* in cases A1, B1, C1 is the same as in the 2-D cases. Complete the table and work out the mobilities [M = 3, 6, 9] . The same goes for A2, B2, C2 [M = 6, 9, 12].

	l = *j*	*j*	6*l* – 3*j* = 3*j*		*l* = *j* - 1	*j*	6*l* – 3*j* = 3*j* - 6
A1				A2			
B1				B2			
C1				C2			

The 3D A2, B2, C2 series is unimportant but the 3D A1, B1, C1 series is becoming increasingly important as greater computer power can be harnessed to control these 'robotic arms'. Ask the children how many applications they can think of. [Handling radioactive canisters, performing laser surgery, handling space vehicles in the vicinity of the International Space Station (ISS)]

Show picture locating the
Japanese **E**xperimental **M**odule **R**emote **M**anipulator **S**ystem
on the Kibo module at the ISS

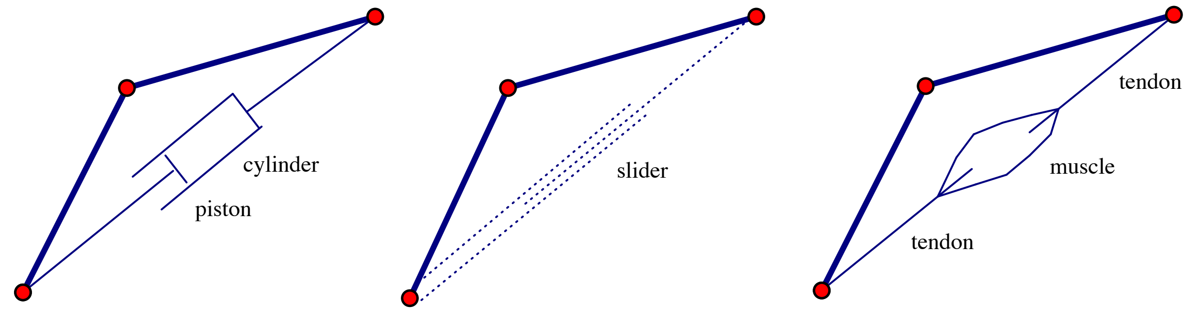
Show second diagram with parts labelled.



Point out how the labelled parts correspond to parts of the human body. Is the human arm the ideal? Discuss with the children. [As a human brain is a more *versatile* thinking machine than any existing computer, a human arm is a more *versatile* handling machine than any existent robot arm.]

The body **E8** *Pupil experiment*

For our last 3-D example we therefore take the human body. Flex your index finger. In terms of the bones, it is a simple chain with two joints and three links. But it only flexes because of loops, completed by the tendons which pull on the bones when muscles shorten. Here are the analogous structures in **E4&5**, **E6**, and here [project figure]. What do I mean by 'analogous'? What is an 'analogy'? Use the examples to help you explain.



However, we can get a good idea of the body's mobility by omitting the muscles and tendons and just considering the bones and joints. As we saw in the case of the lazy tongs, sliders between two adjacent links don't affect the M number. Some of these joints just allow rotations in 2 dimensions, but some are ball joints, allowing movement in 3.
Recall the mobility equation for a simple 2-D chain. If there were j joints, M was just j: we just added 1, the DOF, for each joint. For a 3-D chain M was $3j$: we just added 3, the DOF, for each joint. For a mixture of 2-D and 3-D joints, we just add the appropriate number, 1 or 3, for each. Start at the tip of your index finger and see what number you end up with if you work your way up the arm to the shoulder and all the way down to the waist, including the hips as your last joint. Count the flexing of the spine as 1 ball joint.

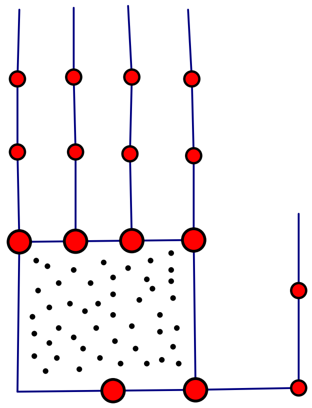
[M = 1 1st finger joint
1 2nd finger joint
3 knuckle joint
3 wrist joint
1 elbow joint
3 shoulder joint
3 spine
3 hip joint

3 + 15 = 18]

As long as the linkage is just a mathematical tree – i.e. it doesn't have any closed loop - we can just add the DOFs for all the joints. Let's check this claim for the case of the hand.

The hand **E9** *Pupil experiment*

Here is my schematic left hand. We'll make the count and do the calculation in two ways: first using the formula, which takes the ideal totals and subtracts the restrictions,
second by just adding the DOF for each joint. We count the hand minus the fingers, shown dotted, as a link.



	DOF	number to subtract from 6	number of joints	
pivot joint			j_p	=
ball joint			j_b	=
link			l	=

M the long way =

M the short way =

The body **E10** *Teacher demonstration*

Accompanying pictures [Project]

This big numbers explain the extraordinary range of positions pianists adopt. Here are 5 jazz and classical pianists [project photos] from the mid- to late-twentieth century. I'll imitate their different techniques:

Sergei Rachmaninov (tall man, long arms, big hands): upright posture, only arms and hands moved
Shura Cherkassky (small man, long arms, average hands): used whole body, had to rise from his seat to reach the ends of the keyboard
Sviatoslav Richter: seat very high, so that forearms sloped downwards
Glenn Gould: seat very low, shoulders hunched, hands had to reach over the edge of the keyboard
Dave Brubeck: fingers used like sticks, hardly any rotation about knuckle joints

Rachmaninov was very tall, with long arms and big hands. He could command the whole keyboard without moving his body.

This picture of Shura cherkassky makes him look bigger than he was. He was so small indeed that he had to rise from his seat to reach the ends of the keyboard.

Sviatoslav Richter used a high stool so that his arms sloped down towards the keys and he maintained a very high hand position.

Glenn Gould was quite the opposite. He used a low stool so that his elbows were level with, or below, the keyboard.

Dave
Brubeck

Dave Brubeck

(As you can see from the photo, this picture of Brubeck's hands was taken when he was older.) Note the long, thin fingers. He kept them fairly straight and shot the finger pads out like bullets.

The extreme mobility of the human body allowed each of these to play in a way which suited his physique.

The machine here is designed to produce sound, so we must connect the pianist with a second linkage: the key-hammer system. The finger pads comprise the interface between the human linkage and the mechanical one.

animated grand piano action

E11 *Teacher demonstration*

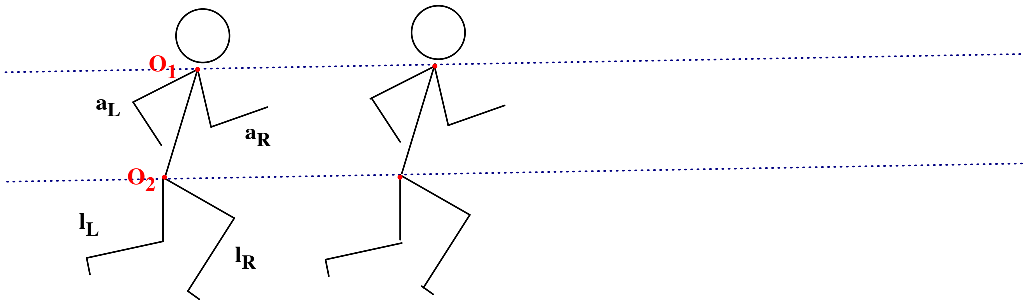
We described the lever, as exemplified by a wheelbarrow, as the simplest linkage. But here, in the action of a modern concert grand piano, is a system of levers which is far from simple.

[Show www.youtube.com/watch?v=01DBtig_Fgw]

We don't need to work out the mobility of the piano action by tracing all the levers. Why not? [We know that it is 1 since all the piano action does is accelerate a hammer towards a string.] Why the action has come to be so complicated is a question needing a whole book to answer – here it is [exhibit *Physics of the Piano* by N. J. Giordano]. Compared with this, the design of a robot arm for the ISS is a simple matter.

You can't get a better feel for the subject of mobility than by trying cartoon animation. A suitable computer package is Anime Studio Pro and its accomapnying tutorials, Learning Anime Studio. You work with an imaginary skeleton in the figure you wish to animate. You can, say, click and drag an upper left leg anticlockwise 55^0 about the hip between seconds 23.0 and 23.4. By combining movements against the clock like this and repeating whole cycles, you achieve locomotion.

Here is a very simple example of a running figure. The head and trunk are rigid. The arms and legs are single pieces which rotate about O_1, O_2 respectively. The angles are measured with respect to the forward horizontal.



time (seconds)	23.000	23.400	23.800	24.200	...
a_L	-100°	-45°	-100°	-45°	...
a_R	-45°	-100°	-45°	-100°	...
l_L	-100°	-45°	-100°	-45°	...
l_R	-45°	-100°	-45°	-100°	...

Of course, it's not just cartoonists who are concerned with locomotion. A bioengineer is designing an artificial knee joint: How can the mobility of the leg match that of the original?

A palaeontologist finds the skeleton of a dinosaur, and, if lucky, fossil footprints: How did it move?

[Project www.newscientist.com/article/dn17621]

28.4.16 draft